Technical Notes and Correspondence

Should Model-Based Inverse Inputs Be Used as Feedforward Under Plant Uncertainty?

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***Abstract*—Bounds on the *size* of the plant uncertainties are found such that the use of the inversion-based feedforward input improves the output-tracking performance when compared to the use of feedback alone (i.e., without the feedforward). The output-tracking error is normalized by the size of the desired output and used as a measure of the output tracking performance. The worst-case performance is compared for two cases: 1) with the use of feedback alone and 2) with the addition of the feedforward input. It is shown that inversion-based feedforward controllers can lead to performance improvements at frequencies ω where the uncertainty $\Delta(j\omega)$ in the nominal plant is smaller than the size of the nominal plant $G_0(j\omega)$ divided by its condition number $\kappa_{G_0}(j\omega)$, i.e., $\|\Delta(j\omega)\|_2 < \|G_0(j\omega)\|_2/\kappa_{G_0}(j\omega)$. A modified feedforward input is proposed that only uses the model information in frequency regions where plant uncertainty is *sufficiently small*. The use of this modified inverse with (any) feedback results in improvement of the output tracking performance, when compared to the use of the feedback alone.**

***Index Terms*—Feedforward, nonminimum phase, output-tracking, robust, system-inverse.**

I. Introduction

Given a vector-valued function $Y(j\omega) \in \mathcal{C}^n$ for all $\omega \in \Re$, $\|Y(j\omega)\|_2 := \sqrt{Y^*(j\omega)Y(j\omega)}$, where the superscript * indicates the complex conjugate transpose. The function-norm is defined as $\|Y(\cdot)\|_2 := \left[\int_{-\infty}^{\infty} Y^*(j\omega)Y(j\omega)d\omega\right]^{1/2}$ and $Y(\cdot)$ belongs to $\mathcal{L}_2$ if $\|Y(\cdot)\|_2$ is finite. Given a matrix-valued function $G(j\omega) \in \mathcal{C}^{n\times n}$ for all $\omega \in (-\infty, \infty)$, the induced matrix 2-norm is

$$\|G(j\omega)\|_2 := \sup_{Y\neq 0, Y\in\mathcal{C}^n} \frac{\|G(j\omega)Y\|_2}{\|Y\|_2} = \bar{\sigma}[G(j\omega)]$$

where the $\bar{\sigma}[G(j\omega)]$ represents the maximum singular value of $G(j\omega)$. Furthermore, $\|G(\cdot)\|_\infty := ess\sup_{\omega\in\Re} \bar{\sigma}\,[G(j\omega)]$.

Inversion-based feedforward controllers (e.g., [1]–[5]) have been used for output tracking in a variety of applications, for example, in aircraft and aerospace systems [6], [7], and flexible structures [3], [8], [9]. Recent successes in using noncausal inverses [3]–[5] for systems with nonminimum-phase dynamics has further renewed the interest in inversion-based feedforward controllers. For example, experimental results have shown that such inverses can be used to achieve high-precision output tracking (e.g., [9]). Previous works [10], [11] have also shown that the inverse varies continuously with plant parameters, which implies that the inverse is *robust* to small plant variations. However, anecdotal evidence has also shown that inversion-based feedforward inputs can adversely affect the output-tracking performance in the presence of *large* modeling errors. This raises the question of when to use inversion-based, feedforward controllers (referred to as inverse feedforward) in the presence of plant uncertainties. This note develops bounds on the *size* of acceptable uncertainties for guaranteed performance improvements when using the inverse feedforward for linear, time-invariant systems. Such uncertainty-acceptability bounds are often violated in typical systems; most plants tend to have some frequency regions where plant uncertainty is unacceptably large, usually at high frequencies and near system zeros. To account for such large plant uncertainties in certain frequency regions, the article develops a modified inverse feedforward that only inverts the model in frequency regions where the plant uncertainty is *sufficiently small*.

Output tracking has a long history marked by the development of regulator theory for linear systems by Francis and Wonham [12] and generalized to the nonlinear case by Byrnes and Isidori [13]. These approaches asymptotically track an output from a class of exosystem-generated outputs. Although the nonlinear regulator design is computationally difficult, the linear regulator is easily designed by solving a manageable set of linear equations. However, a problem with the regulator approach is that the exosystem states are often switched to describe the desired output; this leads to transient tracking-errors after the switching instants. Such switching-caused transient errors can be avoided by using inversion-based approaches to output tracking [4], [14]. Thus, it is advantageous to use inversion-based output tracking when precision tracking of a particular output trajectory is required. Inversion was restricted to causal inverses of minimum phase systems in the early works by Silverman and Hirschorn (e.g., [1] and [2]) because the standard inversion approach leads to unbounded inputs in the nonminimum-phase case. Di Benedetto and Lucibello [15] considered the inversion of time-varying, nonminimum-phase systems with a choice of the system's initial conditions. Instead of choosing initial conditions, preactuation was used in [3]–[5], which extend the inversion technique to nonminimum-phase systems.

Inversion-based feedforward controllers (which are model based) cannot correct for tracking errors caused by plant uncertainties [16]. However, uncertainty-caused errors in the inverse-input can be corrected through feedback. For example, feedback can be used to a) first learn the model to reduce plant uncertainty, and then second invert the improved model to reduce errors in the inverse input (i.e., adaptive inversion of the system model; see, e.g., [17]), or b) directly learn the *correct* inverse input that yields perfect output-tracking (i.e., iterative inversion of the system model, see, e.g., [18]). Alternatively, plant uncertainty can be reduced by optimally designing the feedback [16], and then applying the model-based inversion to the resulting closed-loop system. However, errors in computing the inverse of the closed-loop system can still be large if large uncertainties are present, which in turn, result in substantial tracking errors. It might be better to only use the feedback controller without the use of the inverse-feedforward input. This issue of when to use the inverse feedforward motivates the question: should inversion-based feedforward (which is a model-based approach) be used in the presence of plant uncertainties?

We seek to develop conditions under which the performance with the inversion-based feedforward is better than the performance achieved with feedback controllers alone. A related problem is the robust optimization of two-degrees-of-freedom controllers (feedforward and feedback) under plant uncertainty; see, for example, [19] and the references therein. It is noted that robust synthesis of feedforward

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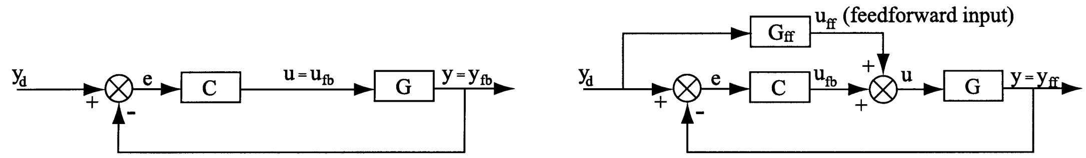

Fig. 1. Block diagram of the system without and with feedforward input ($u_{\rm ff}$).

controllers seeks to achieve the best possible performance over the set of possible uncertainties; in the absence of modeling-error such controllers do not yield (and do not seek to yield) perfect tracking. In contrast, inversion-based approaches seek to achieve perfect tracking in the absence of modeling error; the performance degrades with increase in modeling errors [10], [11]. Another difference is that robust synthesis of feedforward controllers are limited to causal controllers that do not include noncausal inverse-feedforward controllers for nonminimum-phase systems. In contrast, this article also includes noncausal feedforward inputs that are computed offline (online implementation is possible if preview information of the desired output is available [20], [21]).

The article is organized as follows. Tracking errors with and without inverse feedforward are compared in Section II. This section shows that the use of inversion-based feedforward controllers can lead to performance improvements if the uncertainty in the nominal plant is small compared to the size of the nominal plant model. A general result for the multiple-input–multiple-output (MIMO) case is presented. The development of a modified-inversion approach to account for large plant uncertainties is presented in Section III. Conclusions are in Section IV.

II. Tracking Errors With and Without Inverse Feedforward

Consider a linear, time-invariant, finite-dimensional system with the same number of inputs as outputs (square system) and represented by a real rational transfer matrix G. Let the control scheme be as shown in Fig. 1 without a feedforward controller (plot on the left). If the reference input is chosen as a desired output trajectory y_d, then the achieved output $y = y_{fb}$ is given by

$$Y_{fb} := \left[(I+GC)^{-1}\,GC\right]Y_d \tag{1}$$

where the capitalization Y represents the Fourier transform of y and the dependence on $j\omega$ is not written explicitly for ease in notation. With the addition of a feedforward controller $G_{\rm ff}$ as shown in Fig. 1 (plot on the right), the achieved output $Y = Y_{\rm ff}$ with the reference input Y_d is given by

$$Y_{\rm ff} := \left[(I+GC)^{-1}\,G(G_{\rm ff}+C)\right]Y_d. \tag{2}$$

The corresponding tracking error $E_{({\rm ff},\,Y_d)} := Y_d - Y_{\rm ff}$ is given by

$$E_{({\rm ff},\,Y_d)} = \left[(I+GC)^{-1}\,(I-GG_{\rm ff})\right]Y_d. \tag{3}$$

If the plant G is invertible, then the feedforward controller can be chosen as the inverse of the transfer function

$$G_{\rm ff} = G^{-1}. \tag{4}$$

With this inverse feedforward, we obtain exact-output tracking, i.e., by substituting this feedforward control law into (2), we obtain $Y = Y_d$. It is noted that the inversion-based feedforward input achieves exact-output tracking of the desired output in the absence of initial condition errors and external perturbations. However, feedback must still be used (in conjunction with the inverse input) to correct for tracking errors.

In practice, the plant G may not be known exactly due to modeling errors. Therefore, in the following, the feedforward controller will be chosen as the inverse of the nominal plant model G_0 [3]

$$G_{\rm ff} = G_0^{-1} \tag{5}$$

which assumes that the nominal plant is invertible; this will be assumed in the rest of this article.

Assumption 1: The nominal plant G_0 with $G_0(j\omega) \in \mathcal{C}^{n\times n}$ has full-normal rank n [22].

Remark 1: If the nominal plant G_0 is nonminimum phase, then the inverse G_0^{-1} can be accomplished using offline noncausal approaches [3], [4]. Online implementation of the inverse is possible if preview information of the desired output is available [20].

Remark 2: The design of the feedback controller such that the closed-loop system remains stable is not the focus of the current study. Therefore, in the following, it is assumed that the plant and uncertainties are such that the closed-loop system is stable — then, we address the question whether adding the model-based inverse input improves the output-tracking performance.

Assumption 2: The nominal system, the uncertainty, and the controller are such that the nominal and perturbed closed-loop system are stable.

A. Measure to Evaluate the Tracking Performance

With the inverse feedforward found using the nominal plant, the tracking error can be found from (3) as

$$\begin{aligned} E_{({\rm ff},Y_d,\Delta)} &= \left[(I+GC)^{-1}\left(I-GG_0^{-1}\right)\right]Y_d \\ &= \left[(I+GC)^{-1}(G_0-G)G_0^{-1}\right]Y_d \\ &:= \left[(I+GC)^{-1}(\Delta)G_0^{-1}\right]Y_d \end{aligned} \tag{6}$$

where the subscript Δ in the tracking error $E_{({\rm ff},\,Y_d,\,\Delta)}$ indicates the dependence on the particular plant uncertainty $\Delta := G_0 - G$. Similarly, the tracking error without inverse feedforward ($G_{\rm ff} = 0$) can be obtained as

$$E_{(fb,\,Y_d,\,\Delta)} := Y_d - Y_{fb} = [I+GC]^{-1}\,Y_d. \tag{7}$$

These tracking errors depend on the particular desired output trajectory Y_d. This dependence can be removed by normalizing the tracking error by the size of the desired output. The worst-case normalized error is found over all possible desired outputs $Y_d(j\omega)$, and then compared for two cases: 1) with the inverse feedforward and 2) without the inverse feedforward. Formally, the measures used to evaluate the worst-case tracking error with inverse-feedforward ($\hat{E}_{({\rm ff},\,\Delta)}(j\omega)$) and without inverse-feedforward ($\hat{E}_{(fb,\,\Delta)}(j\omega)$) are defined as

$$\begin{aligned} \hat{E}_{({\rm ff},\,\Delta)}(j\omega) &:= \max_{\|Y_d(j\omega)\|_2\neq 0} \frac{\|E_{({\rm ff},\,Y_d,\,\Delta)}(j\omega)\|_2}{\|Y_d(j\omega)\|_2} \\ &= \|\,(I+G(j\omega)C(j\omega))^{-1}\,\Delta(j\omega)G_0^{-1}(j\omega)\|_2 \\ \hat{E}_{(fb,\,\Delta)}(j\omega) &:= \max_{\|Y_d(j\omega)\|_2\neq 0} \frac{\|E_{(fb,\,Y_d,\,\Delta)}(j\omega)\|_2}{\|Y_d(j\omega)\|_2} \\ &= \|\,(I+G(j\omega)C(j\omega))^{-1}\,\|_2. \end{aligned} \tag{8}$$

B. Comparison of Tracking Performance

In the following Lemma, the tracking performance $\hat{E}_{(fb,\,\Delta)}(j\omega)$ without the inversion-based feedforward ($G_{\rm ff}(j\omega) = 0$) is compared with the tracking performance $\hat{E}_{({\rm ff},\,\Delta)}(j\omega)$ with the addition of the inversion-based feedforward controller ($G_{\rm ff}(j\omega) = G_0^{-1}(j\omega)$). We begin with the following condition that requires the invertibility of the nominal plant at a given frequency ω.

Condition 1: The nominal plant $G_0(j\omega)$ is full ranked at ω, i.e., it does not have poles or transmission-zeros at ω. □

Along with Assumption 1, this condition implies that the rank of the plant matrix $G_0(j\omega)$ is equal to the full normal rank n of the nominal system G_0, and that the terms in the matrix $G_0(j\omega)$ are finite at the frequency ω. The next condition specifies a bound on acceptable uncertainties which is used in following Lemmas when comparing the tracking performance with and without the inverse feedforward.

Condition 2: Uncertainty Acceptability: The plant uncertainty satisfies the acceptability condition for inversion at ω if the possible uncertainty is bounded by $\delta(j\omega)$ such that

$$\|\Delta(j\omega)\|_2 \leq \delta(j\omega) \leq \frac{\|G_0(j\omega)\|_2}{\kappa_{G_0}(j\omega)} \tag{9}$$

where $\kappa_{G_0}(j\omega)$ is the condition number [23] of the matrix $G_0(j\omega)$ based on the induced 2-norm

$$\kappa_{G_0}(j\omega) := \|G_0(j\omega)\|_2 \|G_0^{-1}(j\omega)\|_2.$$

□

The following lemma states that the worst-case tracking performance with inverse feedforward is better than (or equal to) the tracking performance without inverse feedforward if the uncertainty satisfies the acceptability Condition 2.

Lemma 1: At a given frequency ω, let the nominal plant G_0 satisfy the invertibility Condition 1 and the uncertainty acceptability Condition 2. Then, for any feedback controller $C(j\omega)$, the output-tracking performance with the inverse feedforward is better than or equal to the performance without the inverse feedforward, i.e.,

$$\hat{E}_{({\rm ff},\,\Delta)}(j\omega) \leq \hat{E}_{(fb,\,\Delta)}(j\omega) \tag{10}$$

where $\hat{E}_{({\rm ff},\,\Delta)}(j\omega)$ and $\hat{E}_{(fb,\,\Delta)}(j\omega)$ are defined in (8).

Proof:

$$\begin{aligned}
\|\hat{E}_{({\rm ff},\,\Delta)}(j\omega)\|_2 =& \| [I + G(j\omega)C(j\omega)]^{-1}\, \Delta(j\omega) G_0^{-1}(j\omega)\|_2 \\
& \text{from } (8) \\
\leq & \| [I + G(j\omega)C(j\omega)]^{-1} \|_2 \\
& \times \|\Delta(j\omega)\|_2 \|G_0^{-1}(j\omega)\|_2 \\
\leq & \| [I + G(j\omega)C(j\omega)]^{-1} \|_2 \\
& \times \|\Delta(j\omega)\|_2 \frac{\kappa_{G_0}(j\omega)}{\|G_0(j\omega)\|_2} \text{ from Condition 2} \\
\leq & \| [I + G(j\omega)C(j\omega)]^{-1} \|_2 \text{ from } (9) \\
\leq & \hat{E}_{(fb,\,\Delta)}(j\omega) \text{ from } (8).
\end{aligned}$$

□

If the size of the plant-uncertainty is allowed to exceed the size of nominal plant, then there are uncertainties for which the output-tracking performance with the inverse feedforward is worse than the tracking performance without the inverse feedforward (irrespective of the choice of the feedback controller C). This is stated formally in the following Lemma.

Lemma 2: Let the nominal plant satisfy Condition 1 at ω. Then, for any feedback controller $C(j\omega)$ and for any scalar $\tilde{\epsilon} > 0$ there exists an uncertainty $\tilde{\Delta}(j\omega)$ satisfying

$$\|G_0(j\omega)\|_2 < \|\tilde{\Delta}(j\omega)\|_2 \leq (1 + \tilde{\epsilon})\|G_0(j\omega)\|_2 \tag{11}$$

such that the tracking performance with the inverse feedforward is worse than the tracking performance without the inverse feedforward, i.e.,

$$\hat{E}_{({\rm ff},\,\tilde{\Delta})}(j\omega) > \hat{E}_{(fb,\,\tilde{\Delta})}(j\omega) \tag{12}$$

where $\hat{E}_{({\rm ff},\,\tilde{\Delta})}(j\omega)$ and $\hat{E}_{(fb,\,\tilde{\Delta})}(j\omega)$ are defined in (8).

Proof: Consider the uncertainty $\tilde{\Delta}(j\omega) = (1+\tilde{\epsilon})G_0(j\omega)$ which satisfies the constraint in (11). Then, from (8)

$$\begin{aligned}
\hat{E}_{({\rm ff},\,\tilde{\Delta})}(j\omega) =& \| [I + G(j\omega)C(j\omega)]^{-1}\, \tilde{\Delta}(j\omega) G_0^{-1}(j\omega)\|_2 \\
=& (1+\tilde{\epsilon}) \| [I + G(j\omega)C(j\omega)]^{-1} \|_2 \\
=& (1+\tilde{\epsilon}) \hat{E}_{(fb,\,\tilde{\Delta})}(j\omega) \text{ from Eq. } (8) \\
>& \hat{E}_{(fb,\,\tilde{\Delta})}(j\omega).
\end{aligned}$$

□

For MIMO systems, a sufficient condition for improvement in output-tracking (worst-case) performance with the use of the inverse feedforward is that the perturbation be smaller than the size of the nominal plant divided by its condition number, i.e., satisfy (9) in Condition 2. If this acceptance bound is violated, then for some feedback controller C, the use of feedforward may make the performance worse; the necessity of Condition 2 is shown in the next Lemma.

Lemma 3: Let the nominal plant G_0 satisfy the invertibility Condition 1 at ω, and have a condition number $\kappa_{G_0}(j\omega)$ greater than one. Then, given an arbitrarily small scalar $\hat{\epsilon} > 0$ there exists a controller $\hat{C}(j\omega)$ and an uncertainty $\hat{\Delta}(j\omega)$ satisfying

$$\frac{1}{\kappa_{G_0}(j\omega)} \|G_0(j\omega)\|_2 \leq \|\hat{\Delta}(j\omega)\|_2 \leq \frac{1+\hat{\epsilon}}{\kappa_{G_0}(j\omega)} \|G_0(j\omega)\|_2 \tag{13}$$

such that the tracking performance with the inverse feedforward is worse than the tracking performance without the inverse feedforward, i.e.,

$$\hat{E}_{({\rm ff},\,\hat{\Delta})}(j\omega) > \hat{E}_{(fb,\,\hat{\Delta})}(j\omega)$$

where $\hat{E}_{({\rm ff},\,\hat{\Delta})}(j\omega)$ and $\hat{E}_{(fb,\,\hat{\Delta})}(j\omega)$ are defined in (8).

Proof: Let the nominal plant model have the following singular value decomposition:

$$G_0(j\omega) = U \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix} V^* \tag{14}$$

where U and V are unitary matrices and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Note that $\sigma_n > 0$ from the invertibility of the nominal plant model at fre-

TABLE I
COMPARISON OF TRACKING PERFORMANCE WITH INVERSE FEEDFORWARD $\hat{E}_{(\mathrm{ff},\,\Delta)}(j\omega)$ AND WITHOUT INVERSE FEEDFORWARD $\hat{E}_{(fb,\,\Delta)}(j\omega)$ FOR DIFFERENT UNCERTAINTY SIZE $\Delta(j\omega)$

Size of Uncertainty	**Comparison of Tracking Performance**
$\Vert\Delta(j\omega)\Vert_2 \le \frac{\Vert G_0(j\omega)\Vert_2}{\kappa_{G_0}(j\omega)}$	For all controllers and any uncertainty $\Delta(j\omega)$, $\hat{E}_{(ff,\,\Delta)}(j\omega) \le \hat{E}_{(fb,\,\Delta)}(j\omega)$.
$\frac{\Vert G_0(j\omega)\Vert_2}{\kappa_{G_0}(j\omega)} < \Vert\Delta(j\omega)\Vert_2$	There exists a controller and an uncertainty $\tilde{\Delta}(j\omega)$ such that $\hat{E}_{(ff,\,\tilde{\Delta})}(j\omega) > \hat{E}_{(fb,\,\tilde{\Delta})}(j\omega)$.
$\Vert G_0(j\omega)\Vert_2 < \Vert\Delta(j\omega)\Vert_2$	For any controller, there exists an uncertainty $\hat{\Delta}(j\omega)$ such that $\hat{E}_{(ff,\,\hat{\Delta})}(j\omega) > \hat{E}_{(fb,\,\hat{\Delta})}(j\omega)$.

quency ω (Condition 1 and Assumption 1). Furthermore, the nominal plant's condition number is

$$\kappa_{G_0}(j\omega) = \frac{\sigma_1}{\sigma_n} \tag{15}$$

and its norm is $\Vert G_0(j\omega)\Vert_2 = \sigma_1$. Consider the following uncertainty $\hat{\Delta}(j\omega)$

$$\hat{\Delta}(j\omega) = \frac{1+\hat{\epsilon}}{\kappa_{G_0}(j\omega)} U \begin{bmatrix} \sigma_n & 0 & \cdots & 0 \\ 0 & \sigma_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_1 \end{bmatrix} V^* \tag{16}$$

which satisfies the uncertainty bound in (13)

$$\Vert\hat{\Delta}(j\omega)\Vert_2 = \frac{(1+\hat{\epsilon})}{\kappa_{G_0}(j\omega)}\sigma_1 = \frac{(1+\hat{\epsilon})}{\kappa_{G_0}(j\omega)}\Vert G_0(j\omega)\Vert_2$$

and consider the controller $\hat{C}(j\omega)$

$$\hat{C}(j\omega) = \mathrm{adj}[G]\gamma(j\omega) \tag{17}$$

where $\mathrm{adj}[G(j\omega)]$ stands for the adjoint of the matrix $G(jw)$ and $\gamma(j\omega)$ is a scalar. This controller $\hat{C}(j\omega)$ input–output decouples the plant G; the controller is not restrictive because substantial freedom is still available in the choice of $\gamma(j\omega)$; however, we do place a constraint on its magnitude. In particular, the controller component $\gamma(j\omega)$ is chosen such that $|\det(G(j\omega))\gamma(j\omega)| < 1$ and the closed-loop system is stable. With this choice of controller, we have

$$\begin{aligned} I + G(j\omega)\hat{C}(j\omega) &= I + G(j\omega)\mathrm{adj}[G]\gamma(j\omega) \\ &= [1 + \det(G(j\omega))\gamma(j\omega)]I \neq 0 \end{aligned} \tag{18}$$

and the output-tracking performance $\hat{E}_{(fb,\,\hat{\Delta})}(j\omega)$ without feedforward input is given by

$$\begin{aligned} &\hat{E}_{(fb,\,\hat{\Delta})}(j\omega) \\ &\quad = \left\Vert \left[I + G(j\omega)\hat{C}(j\omega)\right]^{-1} \right\Vert_2 \\ &\quad = |1 + \det(G(j\omega))\gamma(j\omega)|^{-1}. \end{aligned} \tag{19}$$

Similarly, the output-tracking performance with inverse feedforward $\hat{E}_{(\mathrm{ff},\,\hat{\Delta})}(j\omega)$ can be found from (8) as

$$\begin{aligned} &\hat{E}_{(\mathrm{ff},\,\hat{\Delta})}(j\omega) \\ &\quad = \left\Vert \left[I + G(j\omega)\hat{C}(j\omega)\right]^{-1} \hat{\Delta}(j\omega)G_0^{-1}(j\omega) \right\Vert_2 \\ &\quad = |1 + \det(G(j\omega))\gamma(j\omega)|^{-1} \\ &\qquad \times \Vert\hat{\Delta}(j\omega)G_0^{-1}(j\omega)\Vert_2 \text{ from Eq. } (18) \\ &\quad = |1 + \det(G(j\omega))\gamma(j\omega)|^{-1} \frac{1+\hat{\epsilon}}{\kappa_{G_0}(j\omega)} \\ &\qquad \times \left\Vert U \begin{bmatrix} \frac{\sigma_n}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_{n-1}}{\sigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\sigma_1}{\sigma_n} \end{bmatrix} U^* \right\Vert_2 \\ &\qquad \text{using Eq. } (14) \text{ and Eq. } (16) \\ &\quad = |1 + \det(G(j\omega))\gamma(j\omega)|^{-1} \frac{1+\hat{\epsilon}}{\kappa_{G_0}(j\omega)} \left(\frac{\sigma_1}{\sigma_n}\right) \\ &\quad = |1 + \det(G(j\omega))\gamma(j\omega)|^{-1}(1+\hat{\epsilon}) \text{ from Eq. } (15) \\ &\quad > |1 + \det(G(j\omega))\gamma(j\omega)|^{-1} = \hat{E}_{(fb,\,\hat{\Delta})}(j\omega). \end{aligned}$$

□

Remark 3: When the uncertainty lies in the range

$$\frac{1}{\kappa_{G_0}(j\omega)}\Vert G_0(j\omega)\Vert_2 \le \Vert\Delta(j\omega)\Vert_2 \le \Vert G_0(j\omega)\Vert_2$$

the feedback-controller may be optimally designed to reduce the tracking error caused by modeling uncertainty in the feedforward (see [16]).

Remark 4: The inverse feedforward is not robust at frequencies close to an imaginary-axis transmission zero of the nominal plant. Near imaginary-axis transmission zeros, the size of the nominal plant is small and, hence, the amount of acceptable uncertainty is small. In this sense, hyperbolicity of the nominal plant's zero-dynamics [24] is critical to the robustness of the exact inverse.

The results of the previous three lemmas are summarized in Table I.

C. Use of Inverse Feedforward for Single-Input–Single-Ouput (SISO) Systems

The condition number of the nominal plant's transfer function is always one for SISO systems. Therefore, a necessary and a sufficient condition for improvement in output-tracking performance with the use of the inverse feedforward is that the uncertainty bound δ must be smaller than the size of the nominal plant $\|G_0(j\omega)\|_2$. This is stated as follows.

Theorem 1: Let a SISO system with a nominal plant model G_0 satisfy the invertibility Condition 1 at ω. Then, the tracking error $E_{(\mathrm{ff},\, Y_d,\, \Delta)}(j\omega)$ with inverse feedforward is less than or equal to the tracking error $E_{(fb,\, Y_d,\, \Delta)}(j\omega)$ without inverse feedforward [defined in (8)]

$$E_{(\mathrm{ff},\, Y_d,\, \Delta)}(j\omega) \le E_{(fb,\, Y_d,\, \Delta)}(j\omega) \tag{20}$$

for all uncertainties satisfying $|\Delta(j\omega)| \le \delta(j\omega)$ if and only if the uncertainty bound satisfies Condition 2 (acceptability condition)

$$|\Delta(j\omega)| \le \delta(j\omega) \le |G_0(j\omega)|. \tag{21}$$

Proof: The condition number of the SISO plant model G_0 is always one and, therefore, the necessity and sufficiency of Condition 2 for worst-case performance improvement [(20)] follows from Lemmas 1 and 2. Furthermore, for any desired output $Y_d(j\omega)$, the improvement in output-tracking performance follows by comparing the tracking errors with and without inverse feedforward. Using (6) and (7)

$$\begin{aligned}
|E_{(\mathrm{ff},\, Y_d,\, \Delta)}(j\omega)| &= \left| \frac{\Delta(j\omega) G_0^{-1}(j\omega)}{1+G(j\omega)C(j\omega)} Y_d(j\omega) \right| \\
&= \left| \frac{\Delta(j\omega) G_0^{-1}(j\omega)}{1+G(j\omega)C(j\omega)} \right. \\
&\quad \left. \times\, (1+G(j\omega)C(j\omega)) E_{(fb,\, Y_d,\, \Delta)}(j\omega) \right| \\
&= \frac{|\Delta(j\omega)|}{|G_0(j\omega)|} |E_{(fb,\, Y_d,\, \Delta)}(j\omega)| \\
&\le \frac{|\delta(j\omega)|}{|G_0(j\omega)|} |E_{(fb,\, Y_d,\, \Delta)}(j\omega)| \\
&\le |E_{(fb,\, Y_d,\, \Delta)}(j\omega)|.
\end{aligned}$$

□

Remark 5: If the size of the uncertainty is sufficiently small ($|\Delta(j\omega)| \le |G_0(j\omega)|$), then the inverse feedforward improves the output-tracking performance for each desired output in the SISO case (as opposed to worst-case performance improvement as in the MIMO case). This performance improvement with the use of the inverse input is independent of the particular choice of the feedback controller.

III. Modified Inverse-Feedforward

Results of the previous section show that it is better to use the inverse input with the feedback than not to use it (i.e., only use the feedback) whenever the plant uncertainty is relatively small. However, most experimental systems tend to have relatively large modeling uncertainties in some frequency regions, e.g., at high frequencies. This does not imply that the inverse should not be used at all. Rather, the inverse should be only used in the frequency region where the modeling uncertainty is *small*. This motivates the development of a modified inverse that inverts the system dynamics in frequency regions where the modeling uncertainty is sufficiently *small*. The use of this modified inverse with (any) feedback results in improvement of the output tracking performance (pointwise in frequency as discussed in Section II), when compared to the use of the feedback alone.

We begin with a description of the standard inverse feedforward controller, and then extend it to develop a modified inverse feedforward controller. Issues in the design of the modified inverse are also studied.

Definition 1: The exact inverse $u_{\mathrm{ff,exact}}$ can be described as [1]

$$U_{\mathrm{ff,exact}} = G_0^{-1} Y_d := \hat{G}_{\mathrm{inv}} \hat{Y}_d \tag{22}$$

where $\hat{Y}_d$ is the Fourier transform of $\hat{y}_d$ (which is a linear combination of the desired output and its time derivatives, [1]) and $\hat{G}_{\mathrm{inv}}$ is the reduced-order inverse of the nominal plant model.

Lemma 4: If the nominal system G_0 has hyperbolic internal dynamics, i.e., it has no zeros on the imaginary axis, then the exact inverse input $U_{\mathrm{ff,exact}}$(22) belongs to $\mathcal{L}_2$ if $\hat{Y}_d$ belongs to $\mathcal{L}_2$.

Proof: The poles of the reduced-order inverse are the zeros of the system G_0 [1]. If the nominal system G_0 has hyperbolic internal dynamics, i.e., it has no zeros on the imaginary axis, then the reduced-order inverse $\hat{G}_{\mathrm{inv}}$ is hyperbolic, and it belongs to the set of functions $\mathcal{RL}_\infty$ that are essentially bounded on the imaginary axis, i.e., $\|\hat{G}_{\mathrm{inv}}(\cdot)\|_\infty < \infty$. The lemma follows because a bound on the exact-inverse input $U_{\mathrm{ff,exact}}$ can be obtained as $\|U_{\mathrm{ff,exact}}(\cdot)\|_2 \le \|\hat{G}_{\mathrm{inv}}(\cdot)\|_\infty \|\hat{Y}_d(\cdot)\|_2$. □

A. Modified Inverse

The exact-inverse feedforward controller, $G_{\mathrm{ff}} = G_0^{-1}$, is used to define the modified inverse feedforward controller $G_{\mathrm{ff}\,\Delta}(j\omega)$.

Definition 2: The modified inverse-feedforward controller, $G_{\mathrm{ff}\,\Delta}(j\omega)$, is defined as shown in the equation at the bottom of the page. □

For nonminimum-phase systems, the inverse input is noncausal [3], [4]. We note that the modified feedforward-inverse also tends to be noncausal – even for minimum phase systems – when unacceptably-large plant uncertainty is present over a frequency range. Such large plant uncertainty is common at high frequencies for most models. The noncausality of the modified inverse is shown in the next lemma.

Lemma 5: Let the reduced-order inverse $\hat{G}_{\mathrm{inv}}$ be bounded ($\|\hat{G}_{\mathrm{inv}}(\cdot)\|_\infty < \infty$ as in Lemma 4), and let the plant uncertainty be large in the set $\mathcal{S}$

$$\mathcal{S} := \left\{ \omega \mid \|\Delta(j\omega)\|_2 > \frac{\|G_0(j\omega)\|_2}{\kappa_{G_0}(j\omega)} \right\}. \tag{23}$$

Furthermore, let the output and its time derivatives (Definition 1) $\hat{Y}_d(\cdot)$ belong to $\mathcal{L}_2$. Then

1) the modified inverse-feedforward input $U_{\mathrm{ff}\,\Delta}(\cdot) := G_{\mathrm{ff}\,\Delta} Y_d(\cdot)$ belongs to $\mathcal{L}_2$;
2) the inverse input, which is the inverse Fourier transform of $U_{\mathrm{ff}\,\Delta}$, is noncausal in the time domain if the set $\mathcal{S}$ has a nonzero Lebesgue measure, and the desired output trajectory $\hat{Y}_d(\cdot)$ is nonzero.

Proof: The first part of the Lemma follows from Lemma 4 and Definition 2:

$$\begin{aligned}
\|U_{\mathrm{ff}\,\Delta}(\cdot)\|_2 &= \|G_{\mathrm{ff}\,\Delta}(\cdot) Y_d(\cdot)\|_2 \le \|G_{\mathrm{ff}}(\cdot) Y_d(\cdot)\|_2 \\
&= \|\hat{G}_{\mathrm{inv}}(\cdot)\hat{Y}_d(\cdot)\|_2 \le \|\hat{G}_{\mathrm{inv}}(\cdot)\|_\infty \|\hat{Y}_d(\cdot)\|_2 < \infty.
\end{aligned}$$

By definition, the modified inverse $U_{\mathrm{ff}\,\Delta}(\cdot)$ is zero on the set $\mathcal{S}$. If this set has nonzero Lebesgue measure, then the Paley–Wiener Condition implies that the modified inverse-feedforward input cannot be the Fourier transform of a causal function (see, e.g., [25, Ch. 10]). □

$$G_{\mathrm{ff}\,\Delta}(j\omega) Y_d(j\omega) := \begin{cases} G_0^{-1}(j\omega) Y_d(j\omega) = \hat{G}_{\mathrm{inv}}(j\omega) \hat{Y}_d(j\omega) & \text{if } \|\Delta(j\omega)\|_2 \le \frac{\|G_0(j\omega)\|_2}{\kappa_{G_0}(j\omega)} \\ 0 & \text{otherwise} \end{cases}$$

$$J(u) = \int_{-\infty}^{\infty} \left\{ U^*(j\omega)R(j\omega)U(j\omega) + \Big[Y(j\omega) - Y_d(j\omega)\Big]^* Q(j\omega) \Big[Y(j\omega) - Y_d(j\omega)\Big] \right\} d\omega \tag{24}$$

$$J = \int_{-\infty}^{\infty} \left\{ \begin{array}{l} \left[U - (R + G_0{}^*QG_0)^{-1} G_0^* Q Y_d\right]^* (R + G_0^* Q G_0) \left[U - (R + G_0^* Q G_0)^{-1} G_0^* Q Y_d\right] \\ + Y_d^* [Q - QG_0 (R + G_0^* Q G_0)^{-1} G_0^* Q] Y_d \end{array} \right\} dw.$$

B. Design of the Modified Inverse-Feedforward Controller

The modified inverse-feedforward controller can be considered as a special case of the following optimization problem, which can be used to design the modified inverse. Consider the problem of minimizing the following quadratic performance index (over input U); see (24), as shown at the top of the page, where * denotes the conjugate transpose of matrices with complex elements, $R(j\omega)$ and $Q(j\omega)$ are symmetric, positive-semidefinite, real matrices that represent the weights on the input and the output-tracking error respectively, and Y_d is the desired output trajectory specified by the user. Given a desired output trajectory Y_d, the optimal inversion problem is stated as the minimization of the performance index J over U.

Remark 6: A similar frequency-dependent quadratic performance index has been used in the past (e.g., [26]) for system regulation ($Y_d = 0$), however, an approximate solution to the problem was found to obtain causal control laws. In contrast, we allow noncausal inputs to find the optimal solution; these noncausal solutions can be implemented using preview-based approaches [20], [21].

The solution to the optimal inversion problem is given in the following Lemma adapted from [27].

Lemma 6: Let the nominal plant satisfy Assumption 1 and Condition 1 at ω. Furthermore, let at least one of the matrices $R(j\omega)$ or $Q(j\omega)$ be positive definite almost everywhere (a.e.) in ω. Then, the optimal input trajectory U_{opt}, that minimizes the performance index [J in (24)] for the nominal model (G_0) can be described by (for almost all ω)

$$U_{opt}(j\omega) = [R(j\omega) + G_0^*(j\omega)Q(j\omega)G_0(j\omega)]^{-1} \times G_0^*(j\omega)Q(j\omega)Y_d(j\omega). \tag{25}$$

Proof: In the following, the dependence on $j\omega$ is not explicitly written for compactness. Substituting $Y(j\omega) = G_0(j\omega)U(j\omega)$ into the the performance index given by (24), we obtain the second equation shown at the top of the page. Where $[R(j\omega) + G_0^*(j\omega)Q(j\omega)G_0(j\omega)]$ is invertible almost everywhere in ω because at least one of the matrices, $R(j\omega)$ or $G_0^*(j\omega)Q(j\omega)G_0(j\omega)$, is invertible almost everywhere in ω. Note that the first term (enclosed in square brackets) in the performance index is quadratic and the result follows by setting this quadratic term to zero. □

Lemma 7: The modified inverse-feedforward $G_{\mathrm{ff}\,\Delta}$ in Definition 2 is a particular case of the optimal inverse (25) with the following choices of R and Q:

$$\begin{aligned} R(j\omega) &= 0; \quad Q(j\omega) = I; \quad \text{if } \|\Delta(j\omega)\|_2 \leq \frac{\|G_0(j\omega)\|_2}{\kappa_{G_0}(j\omega)} \\ R(j\omega) &= I; \quad Q(j\omega) = 0; \quad \text{otherwise.} \end{aligned} \tag{26}$$

Proof: If $R(j\omega) = 0$ and $Q(j\omega) = I$, then the performance index is minimized when $U(j\omega) = G_0^{-1}(j\omega)Y_d(j\omega)$. This is the exact-tracking input found by inverting the nominal closed-loop system, $G_{\mathrm{ff}}(j\omega) = G_0^{-1}(j\omega)$. Therefore, exact inversion can be specified at frequencies where the uncertainty is small. At frequencies where the uncertainty is large, the inversion-based feedforward can be turned off by choosing $R(j\omega) = I$ and $Q(j\omega) = 0$ leading to $U(j\omega) = 0$ or $G_{\mathrm{ff}}(j\omega) = 0$. □

Remark 7: The aforementioned modified inverse can be used to tradeoff the exact-output tracking requirement to achieve other goals like reduction of input and vibration control or to meet actuator bandwidth limitations as in [27]. Furthermore, the approach can be used to design output-tracking controllers for nonsquare systems [28].

IV. Conclusion

The article established bounds on the acceptable plant uncertainty for the use of model-based feedforward input. The worst-case output-tracking error (over all desired outputs) was compared a) with the use of feedback alone (i.e., without feedforward input) and b) with the addition of the feedforward input to the feedback. The analysis showed that the addition of inversion-based feedforward can lead to performance improvements (at a given frequency ω) if the uncertainty $\Delta(j\omega)$ in the nominal plant is smaller than the size of the nominal plant $G_0(j\omega)$ divided by its condition number $\kappa_{G_0}(j\omega)$. Additionally, because most systems tend to have large model uncertainties in some frequency regions, a modified feedforward input was proposed that only uses the model information in frequency regions where plant uncertainty is *sufficiently small*. The use of this modified inverse with (any) feedback results in improvement of the output tracking performance, when compared to the use of the feedback alone.

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Rational Multiplier IQCs for Uncertain Time-Delays and LMI Stability Conditions

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***Abstract*—This note describes a set of delay-dependent integral quadratic constraint (IQC) stability conditions for time-delay uncertainty. The IQCs are linearly parameterized in terms of a pair of rational stability multipliers, each active over one of a pair of complementary frequency intervals. Using the finite-frequency positive real lemma, each of these finite-frequency IQC conditions are shown to be equivalent to a frequency-independent linear matrix inequality condition, thereby dispensing with the need for frequency-sweeping.**

***Index Terms*—Integral quadratic constraint (IQC), multiplier, robust control, stability criteria, time-delay system.**

I. INTRODUCTION

The robust stability methodology is useful in dealing with structured uncertainties [1], [2]. In recent years, robust control theory has been reformulated within the framework of integral quadratic constraints (IQCs) [3], which in turn are linked via the Kalman–Yakubovich–Popov (KYP) lemma to linear matrix inequalities (LMIs). A salient feature of the IQC stability results is that they apply directly to complex interconnected systems consisting of any number of different types of IQC bounded uncertainties. Key to minimizing the conservativeness of robustness results based on IQC/LMI stability theory is the discovery of linear parameterizations of the broadest possible classes of IQCs for each type of uncertainty (cf. [4]).

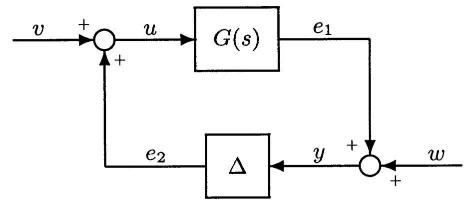

Fig. 1. Basic feedback configuration.

Time-delays have been considered as a type of structured uncertainty for analysis using robust control techniques [5], [6]. Megretski *et al.* [3], Fu *et al.* [7], and Jun *et al.* [6] provided *delay-dependent* results based on IQCs and LMIs. Scorletti [5] and Jun *et al.* [8] expanded these results, determining the broadest available class of IQCs for time delays, linearly parameterized in terms of a positive-real frequency-dependent multiplier matrix. The results of [5] and [8] involve “switching multipliers.” That is, a frequency-dependent multiplier matrix makes a typically nonsmooth change from one complex frequency-dependent multiplier to another multiplier at a specified frequency. Previous IQCs for time-delay such as the ones in [3] were shown to correspond to special cases arising from particular choices of these multipliers. This means that the results of [5] and [8] generally produce tightest, least conservative IQC robustness bounds for systems with uncertain time delays.

However, there is an important difficulty with the results of [5] and [8]. Switching from one frequency-depending multiplier to a constant multiplier results in an irrational multiplier, which means that the standard KYP cannot be applied. Frequency sweeping could be used to bypass the KYP lemma, but no matter how fine the frequency grid, there always remains a small risk that a crucial frequency will be missed, resulting an erroneous prediction of robustness. In this note, we show how to solve this problem by employing the recent *finite frequency strictly positive real lemma* of [9]. Our main result is a finite-dimensional LMI representation of switched rational-multiplier IQCs. The result eliminates the need for, and the risks of, frequency-sweeping in testing the delay-dependent IQC robustness conditions of [5] and [8].

The note is organized as follows. Preliminary background is covered in Section II and the problem formulation is in Section III. Our main result is given in Section IV. Finally, conclusions are in Section V.

II. PRELIMINARIES

This section briefly covers preliminary results such as multiplier IQCs for time-delay by M. Jun *et al.* [8] and finite frequency positive real condition by T. Iwasaki *et al.* [9]. Notation used in the note is standard. $\mathbb{R}$ ($\mathbb{R}_+$) denotes the set of all (positive) real numbers and $\mathbb{C}$ denotes the set of all complex numbers. $A(s)^*$ means para-Hermitian conjugate, that is, $A(-s)^T$. A^{-*} is abbreviation of $(A^*)^{-1}$. I_q denotes $q \times q$ identity matrix. $\mathrm{herm}(m)$ and $\mathrm{skew}(m)$ are Hermitian and skew part of m, that is, $(1/2)(m + m^*)$ and $(1/2)(m - m^*)$, respectively. $\Re e(\cdot)$ ($\Im m(\cdot)$) denotes the real (imaginary) part of $(\cdot)$. $\hat{x}(j\omega)$ means Fourier transform of the signal $x(t)$.

We consider the feedback system in Fig. 1 where G and Δ are bounded causal operators on $\mathcal{L}_{2e}^m[0, \infty)$ and $\mathcal{L}_{2e}^l[0, \infty)$, respectively.

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