Homework Assignment # 7
Due by noon, Friday November 30

Note: Problem 1 is the same as Problem 5 in HW 6. 1(f-j) and 4 are optional.

1. (adapted from Foot 10.6) Bose-Einstein condensate with repulsive interactions
The time-independent Gross-Pitaevskii equation (GPE) for a BEC is

\[ \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + g|\psi|^2 \right\} \psi = \mu \psi \]

Here \( \psi \) is a single particle wavefunction satisfying \( \int d\mathbf{r}|\psi|^2 = 1 \), \( \mu \) is the (constant) chemical potential and \( g = \frac{4\pi\hbar^2}{m} N a \) with \( N \) the number of atoms in the condensate and \( a \) the two-body s-wave scattering length. Note that this is identical to the Schrödinger equation except for the non-linear term which captures all the interatomic interactions as a mean-field effect. The GPE is appropriate for most BEC experiments with dilute atomic gases.

(a) Assume an isotropic harmonic trapping potential with frequency \( \omega \). Use \( \psi = A e^{-r^2/2b^2} \) as a trial wavefunction and derive the expectation value \( E \) of the left-hand side of the GPE. Utilize the shorthand \( a_{ho} = \sqrt{\frac{k}{m\omega}} \).

(b) Specialize to the case of \(^{174}\text{Yb} \) atoms with \( \omega/2\pi = 100 \) Hz, \( a = 5.6 \) nm, and \( N = 10^5 \). Show that the repulsive interactions dominate over the kinetic term. Neglect the kinetic term for the next parts. This is known as the Thomas-Fermi approximation.

(c) Find an expression for the equilibrium radius (Thomas-Fermi radius) in terms of the above parameters and evaluate it for the values given in (b).

(d) What is the chemical potential?

(e) What is the condensate peak density?

(f-j) are optional

(f) Show that the contribution to the energy from the repulsive interactions represents two-fifths of the total.

(g) Find an expression for the total energy in units of trap frequency and units of temperature.

(h) When the trapping potential is switched off suddenly, the potential energy goes to zero and the repulsive interaction between the atoms causes the condensate to expand. After a few milliseconds almost all this energy is converted into kinetic energy. Estimate the velocity at which the atoms fly outwards and the size of the condensate 30 ms after the trap is switched off.

(i) What is the lengthscale \( \xi \) associated with the chemical potential? You can obtain this by considering that a particle of mass \( m \) confined to a size \( l \) will have a quantum “confinement” energy of \( \hbar^2/2ml^2 \). \( \xi \) is known as the healing length and determines the minimum size-scale of density variations in the condensate and the core size of vortex excitations.

(j) What is the particle speed associated with the local mean field energy? This gives the local speed of sound propagation in a BEC.

2. Angular momentum barrier for collisions
Consider the interaction potential between two neutral atoms (each of mass \( m \)) which at long range goes as \( V(r) = -C_6/r^6 \).
(a) Express the height of the angular momentum barrier for \( l \)-wave scattering in terms of \( l, C_6, m \) and other known constants.

(b) What is the interatomic distance at which this barrier is peaked?

(c) For collisions between ground state \(^6\text{Li}\) atoms, \( C_6 = 1389 \text{AU} \). Express the height of the \( p \)-wave barrier as a temperature in \( \mu \text{K} \). Note: AU=atomic units, defined by setting \( e = \hbar = m_e = 1 \).

3. Rotations and Vibrations of a Diatomic Molecule
(a) In a pure rotation spectrum of a diatomic molecule, two adjacent lines are found at 20 cm\(^{-1}\) and 24 cm\(^{-1}\). What values of rotational angular momentum are involved in each line?

(b) Suppose that this molecule has vibrational energy levels given by \((v + \frac{1}{2})\hbar \nu\), with \( \nu_c = 2340 \text{cm}^{-1} \). In the vibration-rotation spectrum of this molecule due to \( v = 0 \rightarrow v = 1 \) transitions, locate all the lines that should be seen between 2330 cm\(^{-1}\) and 2350 cm\(^{-1}\).

4. (Optional Problem) Bose-Einstein condensation from an Optical Lattice
Suppose that a gas of bosons occupy a 3D optical lattice such that the filling at a particular site is either 0 or 1. Let the average number of atoms per site is \( \kappa \) (in this case \( \kappa \leq 1 \)). What is the smallest value of \( \kappa \) for which the adiabatic lowering of lattice depth to zero will result in the formation of a BEC? Hint: Consider the entropy of a partially-filled lattice and that of a Bose gas at the critical temperature. Assume that entropy is conserved during the lattice depth lowering.