1. **Saturated Absorption Laser Spectroscopy**

   This problem guides you through the concepts of saturation spectroscopy. This is one of the techniques to perform Doppler-free spectroscopy, i.e. to extract a narrow line (with the natural linewidth) in a gas with a broad velocity distribution. It nicely illustrates the combination of homogenous and inhomogeneous line broadening. Saturation spectroscopy is frequently used to lock lasers to atomic lines. You should not get into nasty integrals for this problem. The drawn lineshapes should clearly show the basic features, but don’t have to be exact.

   (a) **Homogeneous broadening**

   Consider a dilute gas of density \( n \) composed of atoms with resonant frequency \( \omega_0 \) and linewidth \( \Gamma \). The gas is exposed to monochromatic light of frequency \( \omega_L = \omega_0 + \delta \) and intensity \( I = s I_{\text{sat}} \) where \( I_{\text{sat}} \) is the saturation intensity. Let us ignore the effects of the motion of the atoms, i.e. consider temperature \( T = 0 \). What are the densities of atoms in the ground state \( n_1 \) and the excited state \( n_2 \), including the effect of saturation? What is the cross-section for absorption? The gas is in a box of length \( L \) along the direction of the incoming light. What fraction of the light is absorbed? (This is a small fraction, so don’t worry about the effect of light attenuation on the saturation of the sample).

   (b) **The Bennet hole**

   Now, let’s endow these atoms with a mass \( m \) and a temperature \( T \). Let the incoming light have a wavevector \( k_L \) along the z-axis. What is the population density distribution in the ground state \( n_1(v_z) \) as a function of \( v_z \), the component of velocity in the z-direction? You should find that the light “burns a hole” (known as the Bennet hole) in the distribution of absorbers. What is the position of the hole? How do the width and depth (relative to the population for \( s = 0 \)) vary with saturation parameter \( s \)?

   (c) **Inhomogeneous broadening**

   Consider that we sweep the frequency of the incident laser \( \omega_L \) and measure the (small) absorption of the beam. Determine the fraction of the light absorbed as a function of \( s \) and \( \delta \) and compare with its value at \( s = 0 \) (you don’t need to solve the integral). For high temperatures \( (k_L \bar{v}_z \gg \Gamma) \), what is the width of the absorption line? Does saturation affect the width?

   (d) **Saturation spectroscopy**

   To actually get some benefit from saturating the gas, we introduce a second laser beam.

   i) We can add a weak probe beam (with saturation parameter far less than 1) at frequency \( \omega_p \) with wavevector \( k_p \). What is the absorption of this beam, including the effects of the saturating beam \( (k_L, \omega_L) \)? Again, just write the integral, and take the length of the box along \( k_p \) to be \( L \). Draw the absorption line shape, identifying the position and width of its features.

   ii) Take \( k_p = -k_L \) and \( \omega_p = \omega_L \), i.e. saturating (pump) beam and probe beam have the same frequency and counter-propagate. Draw the population distribution \( n_1(v_z) \) for \( \omega_L \neq \omega_0 \) and \( \omega_L = \omega_0 \). Identifying the depth of the Bennet hole(s), draw the lineshape of absorption of the probe beam (i.e. we scan \( \omega_L \)). What is the width of the central feature (at \( \delta = 0 \))?
2. **Optical Dipole trap**

A laser beam propagating along the $z$–axis has an intensity profile of

$$I(r,z) = \frac{2P}{\pi w(z)^2} \exp\left(-\frac{2r^2}{w(z)^2}\right)$$

where $r^2 = x^2 + y^2$, $w(z) = w_0 \sqrt{1 + (z/z_R)^2}$, $z_R = \frac{\pi w_0^2}{\lambda}$, and $P$ is the power. This laser beam has a power of 10 W, a wavelength of $\lambda = 1.06 \mu m$ and a spot size $w_0 = 20 \mu m$ at the focus.

Note: The above expression for the intensity corresponds to a diffraction-limited Gaussian beam. $w_0$ is known as the waist and $z_R$ is known as the Rayleigh range.

(a) Show that the integral of $I(r, z)$ over any plane of constant $z$ equals the total power of the beam $P$.

(b) For trapped atoms with a thermal energy much lower than the full optical trap depth, the spatial distribution of the atomic ensemble has radial size $\ll w_0$ and axial size $\ll z_R$, and the potential seen by the atoms can be taken to be approximately harmonic in each direction. For this situation, determine the ratio of the rms size of the cloud in the radial and axial directions.

Now specialize to the case of $^6$Li which can regarded as a two state atom with a transition wavelength $\lambda_0 = 671 \text{ nm}$ and natural linewidth $\Gamma = 2\pi \times 6 \text{ MHz}$. For (c) and (d), you may use the rotating wave approximation (RWA).

(c) Calculate the depth of the optical dipole potential for $^6$Li atoms, expressing your answer as an equivalent temperature.

(d) Calculate the radial and axial trapping frequencies experienced by $^6$Li atoms trapped at a temperature which is an order of magnitude lower than the depth.

3. **Zeeman Slower**

For each part of this question, first derive the relevant expressions analytically. Then plug in relevant numbers for sodium to get quantitative values. Assume Na oven temperature $T = 600 \text{ K}$; natural line width $= 2\pi \times 10 \text{ MHz}$; Zeeman splitting $2\pi \times 1.4 \text{ MHz/Gauss}$; wavelength $\lambda = 2\pi/k = 589 \text{ nm}$.

If you want to slow an atomic beam efficiently, you have to compensate for the changing Doppler shift $(k \cdot v)$ during the deceleration. A method of producing a continuous beam of cold atoms is Zeeman slowing. A well collimated beam of atoms is originating from an oven with a temperature $T$. The beam propagates along a distance $L$ with a longitudinal magnetic field $B(x), (0 < x < L)$. A laser beam of intensity $I$ is counter propagating. Its frequency is detuned by $\delta$ ($\delta = \omega - \omega_0$) from the transition frequency at $B = 0$.

(a) Calculate the maximum deceleration $a_{max}$ you can achieve. Assume you could choose arbitrarily large laser intensities.

Assume you want to slow down atoms with speeds lower than the peak (most probable) velocity $v_{peak}$ of the thermal distribution ($v_{peak} = \sqrt{\frac{3k_B T}{m}}$ for a thermal beam) to a stand still using the constant deceleration $fa_{max}$, ($0 < f < 1$). The Zeeman effect shifts the resonance
(b) Calculate the spatial dependence of the magnetic field $B(x)$ and the length $L$ of the slower as a function of $f$.

(c) Assume that the spontaneous emission is isotropic. For atoms that entered the slower with longitudinal speed $v_{\text{peak}}$ and no transverse speed, estimate the rms transverse speed due to passage through the Zeeman slower.

(d) Off-resonant slowing. Assume you want to slow an atom of velocity $v_{\text{peak}}$ with a counter propagating laser beam that is on resonance with the atom at rest. (Use the same $v_{\text{peak}}$ as above, and $I = 5I_{\text{sat}}$.)

i) How long would it take?

ii) How far would the atom travel?

(Hint: Think about integrating the equation of motion)

4. (Foot 9.9) The properties of a magneto-optical trap

(a) Obtain an expression for the damping coefficient $\alpha$ for an atom in two counter-propagating laser beams (each of intensity $I$), taking into account saturation. Determine the minimum damping time of a rubidium atom in the optical molasses technique (with two laser beams).

(b) The force on an atom in a MOT is given by

$$F_{\text{MOT}} = F_{\text{scatt}}^+(\omega - kv - (\omega_0 + \beta z)) - F_{\text{scatt}}^-(\omega + kv - (\omega_0 - \beta z)) \approx -2\frac{\partial F}{\partial\omega}k v + 2\frac{\partial F}{\partial\omega_0} \beta z.$$ 

Assume the worst-case scenario in the calculation of the damping and the restoring force, along a particular direction, i.e. that the radiation force arises from two counter-propagating laser beams (each of intensity $I$) but the saturation of the scattering rate depends on the total intensity $6I$ of all six laser beams. Show that the damping coefficient can be written in the form

$$\alpha \propto \frac{xy}{(1 + y + x^2)^2},$$

where $x = 2\delta/\Gamma$ and $y = 6I/I_{\text{sat}}$. Determine the nature of the motion for a rubidium atom in a MOT with the values of $I$ and $\delta$ that give maximum damping, and a field gradient $0.1\,\text{T/m}$ (in the direction considered). Consider the $^2S_{1/2} \rightarrow ^2P_{3/2}$ transition in $^{87}\text{Rb}$ at 780 nm with $\Gamma = 2\pi \times 6\,\text{MHz}$.

5. (Adapted from Foot 9.11) The equilibrium number of atoms in a MOT

(Note: there is a typo for this problem as written in the textbook)

The steady-state number of atoms that congregate at the center of a MOT is determined by the balance between the loading rate and the loss caused by collisions. To estimate this equilibrium number $N$, we consider the trapping region formed at the intersection of the six laser beams of diameter $D$ as being approximately a cube with sides of length $D$. This trapping region is situated in a cell filled with a low-pressure vapor of number density $N$.

(a) The loading rate can be estimated from the kinetic theory expression $\frac{1}{4}NvAf(v)$ for the rate
at which atoms with speed $v$ hit a surface of area $A$ in a gas; $f(v) = \frac{m}{\sqrt{2\pi k_B T}} \exp\left(-\frac{mv^2}{2k_B T}\right)dv$ is the fraction of atoms with speeds in the range $v$ to $v + dv$. Integrate this rate from $v = 0$ up to the capture velocity $v_c$ to obtain an expression for the rate at which the MOT captures atoms from the vapor. (The integration can be made simple by assuming that $v_c \ll v_p = \sqrt{\frac{k_B T}{m}}$.)

(b) Atoms are lost ('knocked out of the trap') by collisions with fast atoms in the vapor at a rate $\dot{N} = -N\bar{v}\sigma N$, where $\bar{v}$ is the mean velocity in the vapor and $\sigma$ is the collision cross-section. Show that the equilibrium number of atoms in the MOT is independent of the vapor density.

(c) Atoms enter the trapping region over a surface area $A = 6D^2$. A MOT with $D = 2$ cm has $v_c = 25$ m/s for rubidium. Make a reasonable estimate of the cross-section $\sigma$ for collisions between two atoms and hence find the equilibrium number of atoms captured from a low-pressure vapor at room temperature.

(d) (Optional problem) Now consider that there is a density-dependent loss process of the form $\frac{dn}{dt} = -\beta n^2$ where $n$ is the particle density. Find an expression for the equilibrium density in this case and evaluate it for $\beta = 10^{-11}$ cm$^3$/s. Can you think of processes that would cause density-dependent losses in a MOT? Literature reference: W. Ketterle et. al., Phys. Rev. Lett. 70, 2253 (1993), describes a modification of the standard MOT which can get around some of the density-dependent losses.