3.0 DYNAMICS OF PLANETARY ATMOSPHERES

- Basic equations. Then characteristic force regimes:
- Geostrophic balance, Rossby number on different planets
- Cyclostrophic balance: Venus

Dynamics is the relationship between heating, forces and the winds that they drive.

Differential heating induces pressure gradients in the atmosphere that drive winds. Winds are critical for the climate because they transport heat, e.g., from the tropics to the poles. Winds also carry condensable species (water on Earth and Mars; ammonia on Jupiter; methane on Titan), modify a surface (e.g. Mars), and drive secondary circulations (e.g., ocean currents on Earth).

3.1.1. Equations of motion

[Further info: Ch. 7, Wallace & Hobbs (2006) provides a readable introduction to dynamics. Ch. 4, Andrews (2000) "*An Introduction to Atmospheric Physics*", CUP gives additional info on the material in this section where the equations we discuss are derived in a clear and concise manner. Other useful, more expansive texts are (1) Holton "*An Introduction to Dynamic Meteorology*", Academic. (2) Brown (1991) "*Fluid Mechanics of the Atmosphere*" Academic.]

Fluid dynamical equations are basically various forms of Newton's Second Law:

$$\text{Inertial acceleration} = a_i = \frac{\sum F}{m}$$

ΣF is a sum of forces, m is the mass. Inertial acceleration a_i, is of a parcel of air relative to a coordinate system fixed in space **outside** the planet-atmosphere system. In dynamics, we need to consider the various forces F that cause air to flow. We consider:

- ***real forces***, e.g. gravity, pressure gradient forces, frictional forces
- ***apparent forces*** that arise because we usually use a non-inertial frame of reference that co-rotates with the planet and therefore is accelerated.

Pressure forces (per unit mass)

In hydrostatic balance, gravitational acceleration per unit mass g balances a vertical pressure gradient force per unit mass of $-1/\rho\ (\partial p/\partial z)$. The pressure gradient forces per unit mass in horizontal directions are analogous:

$$\frac{F_{x,PG}}{m} = p_x = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{F_{y,PG}}{m} = p_y = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \text{or total 3D force } \frac{\mathbf{F}_{PG}}{m} = \mathbf{P} = -\frac{1}{\rho}\nabla p$$

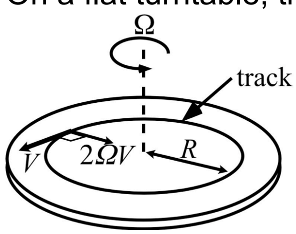
The minus sign indicates that the force acts from high to low pressure.

Coriolis and Centrifugal forces (per unit mass)

A planet rotates with an angular rotation rate of magnitude $\Omega = 2\pi$ (siderial day)$^{-1}$, which is 7.292×10^{-5} rad s^{-1} for Earth. In a co-rotating frame, apparent forces arise for an observer in the accelerated frame. They are:

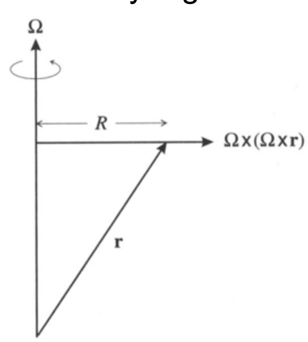
1. The **Coriolis acceleration**, which is perpendicular to both the velocity **V** in the rotating frame and to $\boldsymbol{\Omega}$.

   On a flat turntable, the Coriolis acceleration is shown below:



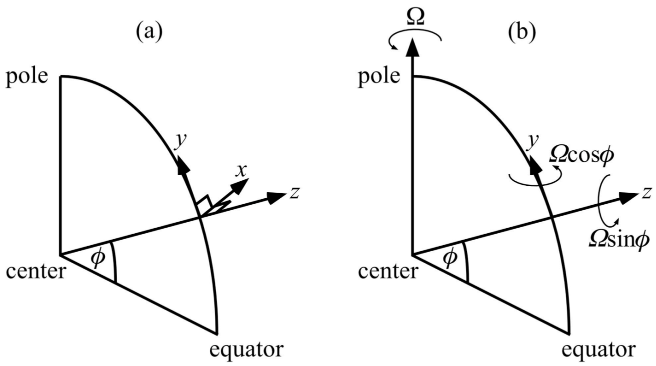
   On a sphere, Coriolis acceleration is $2\Omega V$ in magnitude only at the poles, where $\boldsymbol{\Omega}$, the planet's angular rotation vector, is vertical. Elsewhere, the Coriolis acceleration is scaled according to latitude because $\boldsymbol{\Omega}$ is not perpendicular.

2. The **centrifugal acceleration**, has magnitude $\Omega^2 R$, where R is the perpendicular distance from the point with position vector **r** to the rotation axis, directed perpendicularly away from the rotation axis. Frequently, this is merely regarded as a minor correction to gravity, g.



The spherical coordinate system and approximations for horizontal winds

In dynamics, we use a local Cartesian system on a spherical planet. We define the components of the wind to be a **zonal wind** *u* (in the west-east direction), a **meridional wind** *v* (in the south-north direction), and a **vertical wind** *w* (upwards).



(a) Local Cartesian system at a point on a planet at latitude ϕ. (NB x and z are mislabeled).

(b) Components of the planetary rotation vector about local vertical and meridional axes.

x, *y* and *z* axes point in the horizontal eastward, horizontal northward, and vertical directions, respectively, while the winds in these directions are the **zonal wind** *u* (in the west->east direction), a **meridional wind** *v* (in the south->north direction), and a **vertical wind** *w* (upwards). The $z = 0$ level is sea-level on Earth. On other planets, a mean pressure level or geopotential surface serves as the $z = 0$ surface. We define a wind vector (m s^{-1}) as:

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad \text{3D wind vector}$$

But the vertical wind is generally much smaller than horizontal winds, so often we neglect *w* and define a horizontal wind vector as:

$$\mathbf{v}_h = u\mathbf{i} + v\mathbf{j} \qquad \text{2D horizontal wind vector}$$

And at any latitude ϕ, a planet's angular velocity vector (rad s^{-1}) can be resolved into two orthogonal components along the local vertical and *y*-axis, as shown above:

$$\mathbf{\Omega} = \mathbf{j}(\Omega\cos\phi) + \mathbf{k}(\Omega\sin\phi)$$

Using vectors, the Coriolis acceleration is given by:

$$\frac{-\mathbf{F}_{\text{Coriolis}}}{m} = -2\mathbf{\Omega}\times\mathbf{v} = -2\Omega\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & \cos\phi & \sin\phi\\ u & v & w\end{vmatrix}$$

$$= \mathbf{i}(-2\Omega(w\cos\phi - v\sin\phi)) - \mathbf{j}(2\Omega u\sin\phi) + \mathbf{k}(2\Omega u\cos\phi)$$

Application: if we consider a southward wind only in the northern hemisphere (i.e., negative *v*, positive ϕ, and $u = w = 0$), then the Coriolis acceleration vector would be -**i**($2\Omega v\sin\phi$), which is a horizontal acceleration **toward the west** or negative *x*-direction. This is the origin of terrestrial trade winds in the tropics.

The Coriolis parameter: A useful way to approximate horizontal winds

Let's neglect *w* given that vertical velocities are usually much smaller than horizontal ones.
Also the vertical component term in **k** of the Coriolis acceleration is often much smaller than other vertical accelerations such as gravity.

If we define a new variable,

$$\text{the Coriolis parameter, } f = 2\Omega\sin\phi$$

then we can approximate the Coriolis force per unit mass from above by:

$$\approx \mathbf{i}(2\Omega v\sin\phi) + \mathbf{j}(-2\Omega u\sin\phi)$$

$$= \mathbf{i}(fv) + \mathbf{j}(-fu) = -f\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ u & v & 0 \end{vmatrix} = -f\mathbf{k}\times\mathbf{v}_h$$

Application: Given a typical horizontal wind speed of 10 m s^{-1} = $|\mathbf{v}_h|$ on Earth, the magnitude of Coriolis acceleration in mid-latitudes (ϕ = 45°) is $f|\mathbf{v}_h|$ = 10^{-3} m s^{-2}, typically comparable to a pressure gradient acceleration.

The Navier stokes equation (or "momentum equation")

If we put all the forces per unit mass together, we get a version of the Navier-Stokes equation (Newton's second law for fluid flow) that applies to planetary atmospheres:

$$\frac{d\mathbf{v}}{dt} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure gradient force/kg}} - \underbrace{(2\mathbf{\Omega}\times\mathbf{u})}_{\text{Coriolis force/kg}} - \underbrace{[\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})]}_{\text{centrifugal force/kg}} - g\mathbf{k} + \mathbf{F}_{\text{visc}}$$

Here, $\mathbf{F}_{\text{visc}}$ is the frictional or viscous force per unit mass. The above equation is also called the *momentum equation*.

Often, this equation is expanded into 3 components of acceleration in the *x*, *y* and *z* directions of eastward, northward and upward.
Derivations can be found elsewhere: a vector derivation in Appendix B of Andrews (2000), a geometric derivation in Ch. 2 of Holton (2004), and another derivation in Ch. 4 of Jacobson (2005). The resulting component forms are:

Zonal component $$\frac{du}{dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{x,visc}$$

Meridional component $$\frac{dv}{dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{y,visc}$$

Vertical component $$\frac{dw}{dt} - \frac{u^2 + v^2}{r} - 2\Omega u\cos\phi + g = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_{z,visc}$$

where $r = R_p + z$ where z = height and R_p = radius of the planet. If we neglect motion, the last equation simplifies to the hydrostatic equation. F^x_{visc}, F^y_{visc} and F^z_{visc} are the components of the frictional force

Curvature terms are those in 1/*r* that crop up in the full derivation. Hidden accelerations arise not only from the rotating reference frame but also because two of the frame axes (*x* and *y*) are curved, while the *z* axis tilts as we move on the planetary surface.

Inherent nonlinearity in advective terms, or why dynamics is "interesting"

Even when the curvature terms are neglected, the above eqns. remain nonlinear.

The terms *du*/*dt*, *dv*/*dt* and *dw*/*dt* are ***total derivatives*** (=*substantial derivatives*, *material derivatives* or *advective derivatives*, sometimes written as *D*/*Dt*) and represent the rate of change with respect to time *following* a fluid blob.

This can be contrasted with ∂/∂*t*, the rate of change with respect to time at a *fixed point*. The former is the 'Langrangian' approach, the latter 'Eulerian'. The total derivative expands into local and advective parts:

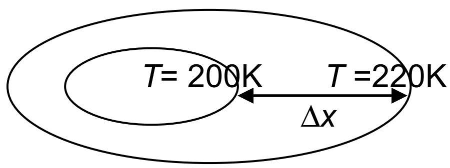
$$\frac{du}{dt} \equiv \underbrace{\frac{\partial u}{\partial t}}_{\text{local derivative}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{advective terms}}$$
$$= \frac{\partial u}{\partial t} + \mathbf{v}\bullet\nabla\mathbf{v}$$

Here, **v** is the 3D wind vector (defined earlier) and we have used vector mathematics, $\mathbf{a}\bullet\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, along with the gradient operator: ∇ = ∂/∂*x* +∂/∂*y* +∂/∂*z*.

Understanding what is meant by advective terms is illustrated below.

Box 2.1 Understanding advective terms

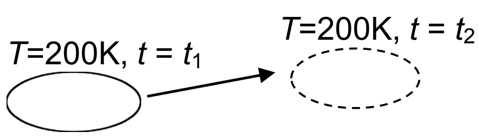
1.Suppose we have a temperature field $T(x,y,z,t)$, for example, which is constant in time but spatially varying, as shown in contours below. As an observer moves from place to place the rate of change of T is



$$\frac{dT}{dt} = \mathbf{u} \bullet \nabla T$$

$$\text{e.g.,} \quad \frac{\Delta T}{\Delta t} = \left(\frac{\Delta x}{\Delta t}\right)\left(\frac{\Delta T}{\Delta x}\right) = \frac{20\ \text{K}}{\Delta t}$$

2. Now suppose contours change position with a velocity **u** but not value. An observer traveling at the same velocity sees no change in T with time, so dT/dt = 0 because $\partial T / \partial t$ is balanced by $\mathbf{u} \bullet \nabla T$.



The nonlinearity of $\mathbf{v} \cdot \nabla \mathbf{v}$ in **v** makes the dynamical behavior of atmospheres "interesting", i.e., difficult to forecast and **chaotic**. However, in uniform flow, the velocity spatial derivatives are zero, which can produce linearized equations.

Simplified equations of motion

For large-scale motions above the surface, we often neglect vertical motion and friction so that eqns simplify to:

Vertical: the hydrostatic equation

Zonal

$$\frac{du}{dt} - \underbrace{\left(f + \frac{u \tan\phi}{r}\right)}_{\text{Coriolis term} \quad \text{Centrifugal term}} v = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Meridional

$$\frac{dv}{dt} + \underbrace{\left(f + \frac{u \tan\phi}{r}\right)}_{\text{Coriolis term} \quad \text{Centrifugal term}} u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

where $f = 2\Omega \sin\phi$ is the Coriolis parameter.

3.1.2. Characteristic Force Regimes

Geostrophic balance and Rossby number

If a planet rotates rapidly (e.g., Earth, Mars, Jupiter, Saturn, Uranus, Neptune), the Coriolis terms *fu* and *fv* can dominate the horizontal accelerations and approximately balance the pressure gradient terms in so-called ***geostrophic balance***.

Then, the zonal (*u*) and meridional (*v*) components of the horizontal wind are:

$$u_g = -\frac{1}{f\rho}\left(\frac{\partial p}{\partial y}\right)_z , \quad v_g = \frac{1}{f\rho}\left(\frac{\partial p}{\partial x}\right)_z$$

Here, the '*z*' subscript indicates that the partial derivative is **along a surface of constant geometric height *z***; the '*g*' subscript indicates geostrophic winds.

(If expressed in terms of variations of geopotential along a surface of constant pressure, these equations become

$$u = -\frac{1}{f}\frac{\partial \Phi}{\partial y}, \qquad v = \frac{1}{f}\frac{\partial \Phi}{\partial x} \quad \text{on a pressure surface}$$

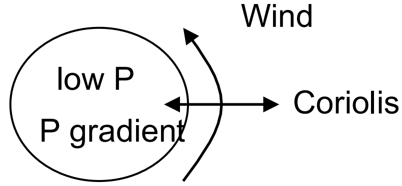
Recall the definitions: geopotential height can be used in place of geometric height, where geopotential is the P.E. per unit mass at height z:

$$\Phi(z) = \int_0^z g\,dz \text{ , geopotential height } = \frac{\Phi(z)}{g_0} = \frac{1}{g_0}\int_0^z g(z,\phi)\,dz, \text{ where } g_0 = \frac{GM}{r_0^2}\text{))}$$

The geostrophic wind has four properties:

1) It's tangential to the pressure gradient, i.e., parallel to isobars. (Equivalently, it's parallel to geopotential contours on a constant pressure surface)
2) In any small latitude range, speed is proportional to the spacing of those contours.
3) The low-pressure region is to the left of the wind vector in the northern hemisphere, giving rise to a '<u>L</u>ow to the <u>L</u>eft' mnemonic known as the *Buy-Ballot* law after an early Dutch meteorologist.
4) the speed implied by isobar spacing is inversely proportional to sine of latitude.

Essentially, the horizontal pressure gradient is balanced by the Coriolis forces associated with the horizontal velocities (i.e. $fu \propto -\partial P/\partial y$). Thus, winds blow around the isobars around *cyclonic* systems as follows:



Wind velocity: "**L**ow to the **L**eft" (N. hemisphere)

The opposite occurs around *anticyclonic* systems, where the pressure gradient is in the opposite direction.

Looking back at the simplified equations of motion, we need to be able to neglect centrifugal curvature terms and the local acceleration terms (dv/dt, du/dt) in order to justify geostrophy,

i.e., the condition:

$$\frac{d(u,v)}{dt}, \frac{u \tan\phi}{r} \ll f$$

We introduce characteristic scales to assess this condition for various planets:

- U for the magnitude of the horizontal wind |**v**|
- L for characteristic length in the derivatives
- f_s for a reference Coriolis parameter f
- τ for a characteristic dynamic timescale

Hence

$$\tau = L/U$$

$$\frac{d(u,v)}{dt} \sim \frac{U}{\tau} \sim \frac{U^2}{L}$$

$$f(u,v) \sim f_s U$$

$$\frac{u \tan\phi}{r}(u,v) \sim \frac{U^2}{r} \le \frac{U^2}{L}$$, since $L \le a$, so we usually neglect this term

Thus, the condition for geostrophic balance in planetary atmospheres is that Coriolis acceleration >> other acceleration terms, or

$$f_s U >> U^2 / L$$

The *Rossby number* (*Ro*) expresses the relative importance of the Coriolis force:

$$Ro = \frac{\text{relative acceleration}}{\text{Coriolis acceleration}} \sim \frac{U^2 / L}{f_s U} = \frac{U}{f_s L}, \quad Ro << 1 \text{ for geostrophy}$$

Here the most obvious limitation is that we are not too close to the equator when $f \rightarrow 0$. In mid-latitudes, we can apply the Rossby number to various planets:

Consider various planets:
Earth: L~1000 km = 10^6 m and U~10 m/s in mid-latitudes. Using $f = 2\Omega\sin\phi$ evaluated at 45°N, we have 2 (7.292×10^{-5} s^{-1})(sin 45) = 10^{-4} s^{-1}. Hence Ro ~ 0.1 and we expect winds to be approximately geostrophic.

Mars: a planet with a similar rotation rate to Earth, has a similar Rossby number and therefore we expect midlatitudes winds to be approximately geostrophic.

Venus: L~1000 km = 10^6 m and U~10 m/s in mid-latitudes. Using $f = 2\Omega\sin\phi$ evaluated at 45°N, we have 2(2.98×10^{-7} s^{-1})(sin 45) = 4.2×10^{-7} s^{-1}. So Ro ~ 10. Evidently, we should *not* expect geostrophic winds on Venus.

Jupiter: U ~ 100 m/s at the boundaries of the bright **zones** and dark **belts**, L~ 100,000 km ~ 10^8 m, and f ~ 1.8×10^{-4} s^{-1}. Thus, Ro ~ 0.01.

Timescale

The fact that geostrophic air flows at right angles to the pressure gradient, instead of down it, might seem peculiar.

However, air that moves in response to a pressure gradient will veer sideways in response to the Coriolis force and take *time* to reach geostrophy. From the definition of Ro, we can write

$Ro \sim 1/f_s\tau$ = (a siderial day)/[$(2\pi)(2\sin\phi)\tau$],

to evaluate the timescale. For Ro < 0.1 at midlatitudes, τ exceeds ~ 1 siderial day. Thus, geostrophy applies to an equilibrium developed by large-scale motion over timescales exceeding a planetary rotation period and therefore inevitably affected by the planetary rotation.

3.3 Thermal Windshear (or thermal wind) equation

The *thermal wind* (or *thermal windshear*) equation relates wind shear—the change of wind speed and direction with height— to horizontal temperature gradients.

A useful conceptual picture is that if the temperature changes along the horizontal, a column of air (if isothermal) would shrink or grow relative to neighboring columns, setting up increasing horizontal pressure gradients at altitude and therefore larger winds higher up. Differentiation of the geostrophic wind equation and substitution of the hydrostatic equation produces the equations governing this behavior.
For the meridional wind, the algebra is as follows:

$$fv_g = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) = \frac{\bar{R}T}{P}\left(\frac{\partial p}{\partial x}\right), \text{ differentiate w.r.t. } z$$

$$\Rightarrow f\frac{\partial v_g}{\partial z} = \bar{R}T\frac{\partial^2}{\partial z \partial x}\left(\ln p\right) \approx \bar{R}T\frac{\partial}{\partial x}\left(\frac{1}{p}\left(\frac{\partial p}{\partial z}\right)\right) = \bar{R}T\left(\frac{g}{\bar{R}T^2}\frac{\partial T}{\partial x}\right), \text{ using } \frac{\partial p}{\partial z} = -g\rho = -\frac{gP}{\bar{R}T}$$

Note: I inserted an approximation sign because I ignored vertical variations in *T*.

Manipulation of the zonal geostrophic wind equation is analogous, giving:

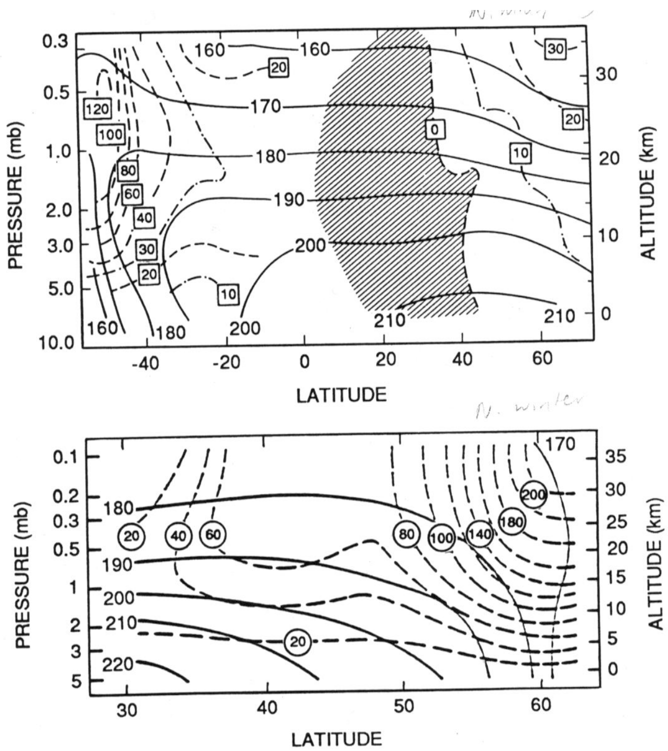
$$f\frac{\partial u_g}{\partial z} \approx -\frac{g}{T}\frac{\partial T}{\partial y}, \qquad f\frac{\partial v_g}{\partial z} \approx \frac{g}{T}\frac{\partial T}{\partial x}$$ **thermal windshear equations**

These thermal wind equations can be used with spacecraft data to compute atmospheric circulation (given that it is difficult to measure winds directly). If we measure the temperature distribution in a planetary atmosphere as a function of latitude and height (or pressure), we can deduce the wind field. Of course, we need a boundary condition to integrate the above: either the measured wind at the surface or the distribution of surface pressures. Often, an assumption is made that the surface wind is small ~0, but this obviously introduces errors.

The thermal wind also links geostrophic flow to the thermodynamics of the atmosphere.

Qu.) Normally, we expect the distribution of incoming solar radiation to set up an equator-to-pole temperature decrease on a planet. So how do we expect the magnitude of zonal winds to vary with altitude (assuming geostrophy applies)?

APPLICATION TO SPACECRAFT DATA: The utility of these equations is that if we measure the temperature distribution in an atmosphere as a function of latitude and height (or pressure) with IR sounding, we can deduce the wind field. There is a snag that we need a boundary condition: the measured wind at the surface or alternatively, the distribution of surface pressures. Often, an assumption is made that the surface wind is small ~0, but this introduces errors.



◄ Atmospheric temperatures (K) in the Martian atmosphere retrieved from Mariner 9 IRIS (Infrared Interferometer Spectrometer).

Zonal (eastward) winds are computed using the thermal wind equation assuming no wind at the surface.
TOP: latitude cross-section for N. mid-Spring (Ls =43-54). Shaded = ill-defined region of westward winds.

BOTTOM: Late northern winter (Ls=347).

3.4 Cyclostrophic Balance: Slowly Rotating Planets

On Earth, we sometimes encounter cases where a strong local pressure gradient force is balanced by centrifugal force known as *cyclostrophic balance*, i.e.,

$$\frac{V^2}{R} = \frac{1}{\rho}\frac{\partial P}{dR}$$

Air moves with tangential speed *V* around a radius *R*. For example, a tornado may have V ~ 50 m/s, R ~100 m, and a correspondingly strong pressure gradient. In such a systems, the effect of the planetary rotation is negligible.

For planets as a whole, we find cases (e.g., Venus, Titan) where there is little planetary rotation, i.e., $\Omega \approx 0$, *f* is small, and geostrophic balance does not apply.

Leovy (1973) first pointed out that on such planets, the large-scale zonal flow should approximate cyclostrophic balance.

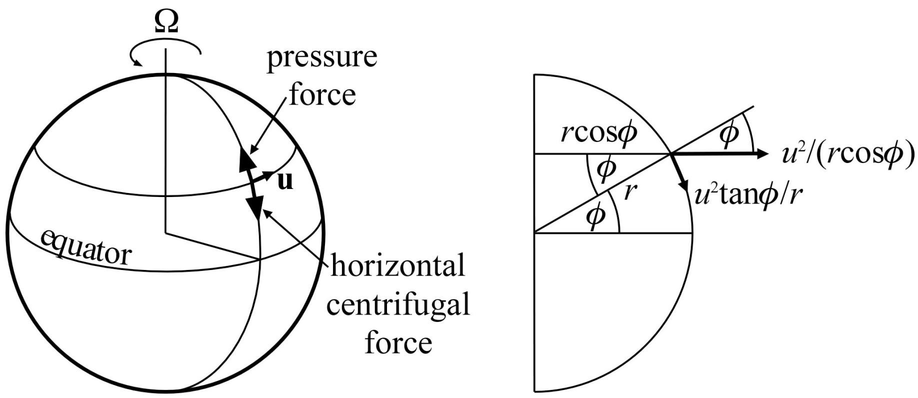
At the equator, the centrifugal force per unit mass is u^2/r and points outwards, merely reducing the effective gravity and so the balance does not apply there. But at latitude ϕ, the centrifugal acceleration due to the zonal wind is $u^2/(r \cos \phi)$, where $r\cos \phi$ is the distance to the rotation axis (see Fig. below). This has a local horizontal component, parallel to the planet's surface, of $(u^2 \tan \phi)/ r$, giving

$$\frac{u^2 \tan\phi}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

(or written as $\frac{u^2 \tan\phi}{a} = -\frac{\partial \Phi}{\partial y}$ in terms of geopotential)

This is just a simplification of the 'simplified equations of motion' neglecting Coriolis and local acceleration terms.

Physically, we have the following picture of cyclostrophic balance where the **centrifugal force balances an equator-to-pole pressure gradient**.



Cyclostrophic thermal windshear equation

Now, we can use u = (angular velocity, ω) × (radius to rotation axis), to define a cyclostrophic thermal windshear equation that relates wind shear to horizontal temperature gradients. So:

$$u = \omega(r\cos\phi)$$

Substituting this in the cyclostrophic balance condition for ($u^2\tan\phi$)/ r balanced by the pressure gradient acceleration, we get:

$$\omega^2 r\sin\phi\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \Rightarrow \omega^2 = -\frac{1}{r\sin\phi\cos\phi}\frac{1}{\rho}\frac{\partial p}{\partial y}$$

As before, we can differentiate w.r.t. z and use the hydrostatic equation to relate wind shear to horizontal temperature gradients:

$$\frac{\partial(\omega^2)}{\partial z} = \frac{-\bar{R}T}{r\sin\phi\cos\phi}\frac{\partial^2}{\partial z\partial y}\left(\partial \ln p\right) \approx \frac{-\bar{R}T}{r\sin\phi\cos\phi}\frac{\partial}{\partial y}\left(\frac{1}{P}\left(\frac{\partial p}{\partial z}\right)\right) = \frac{-\cancel{\bar{R}T}}{r\sin\phi\cos\phi}\left(\frac{g}{\cancel{\bar{R}}T^{\cancel{2}}}\frac{\partial T}{\partial y}\right)$$

The result is a cyclostrophic thermal wind equation:

$$\frac{\partial(\omega^2)}{\partial z} = \frac{-1}{r\sin\phi\cos\phi}\frac{g}{T}\frac{\partial T}{\partial y}$$

APPLICATION: suppose $T = T_{\text{pole}} + \Delta T\cos^2\phi$, which is a symmetric latitudinal temperature distribution, warmest at the equator. In our coordinate system: $\partial y = r\partial\phi$. Noting that the differential of our specified distribution is $\partial T/\partial\phi = -2\cos\phi\,\sin\phi$, we can deduce from the above eq. that over vertical distance Δz the increase in angular velocity will be characterized by:

$$\Delta(\omega^2) \approx \frac{2g\Delta z}{r^2}\frac{\Delta T}{T}$$

Consider Venus.

The middle cloud deck at ~60 km is characterized by winds in **superrotation**, that is in the same sense and exceeding the planet's rotation rate. These move at ~100 m/s and circle the planet in about 4 days. Winds decrease with decreasing height to ~1 m/s near the surface.

On Venus, between 20-60 km, ω^2 increases by ~2.7×10^{-10} s^{-2}, T ~ 400 K for this region, $r = 6.05\times 10^6$ m, and $g = 8.9$ m s^{-2}, which implies that $\Delta T \approx 6$ K. This compares favorably with the measured mean pole-equator temperature gradient on Venus.

In general, deep atmospheres that are relatively warm on the equator on slowly rotating planets will have zonal atmospheric winds in cyclostrophic balance. The approximation works well for the lower and middle atmospheres of Venus and Titan.