Spectral Analysis for Univariate Time Series: Solutions to Embedded Exercises Version 1.0

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Appendix

Answers to Embedded Exercises

Here we give solutions to all exercises embedded within the main parts of Chapters 1 to 11 (as noted in the Preface, solutions to the exercises at the ends of these chapters are available to instructors upon contacting us). We believe readers can develop a basic understanding of spectral analysis by making an honest effort to do these exercises, so we encourage tackling them before looking at our solutions. Working these exercises independently opens up the possibility of finding better solutions, which we would appreciate learning about (our email addresses are `dbpercival@gmail.com` and `atwalden86@gmail.com`).

Answer to Exercise [8] First, note that, since $E\{A_j\} = E\{B_j\} = 0$ for all j,

$$E\{X_t\} = \mu + \sum_{j=0}^{\lfloor N/2 \rfloor} E\{A_j\} \cos(2\pi f_j t) + E\{B_j\} \sin(2\pi f_j t) = \mu.$$

Next, note that

$$\begin{aligned}
&E\{(X_{t+\tau} - \mu)(X_t - \mu)\} \\
&= E\Bigg\{ \Bigg(\sum_{j=1}^{\lfloor N/2 \rfloor} A_j \cos(2\pi f_j [t+\tau]) + B_j \sin(2\pi f_j [t+\tau]) \Bigg) \\
&\qquad \times \Bigg(\sum_{k=1}^{\lfloor N/2 \rfloor} A_k \cos(2\pi f_k t) + B_k \sin(2\pi f_k t) \Bigg) \Bigg\} \\
&= \sum_{j=1}^{\lfloor N/2 \rfloor} \sum_{k=1}^{\lfloor N/2 \rfloor} \Big(E\{A_j \cos(2\pi f_j [t+\tau]) A_k \cos(2\pi f_k t)\} \\
&\qquad + E\{A_j \cos(2\pi f_j [t+\tau]) B_k \sin(2\pi f_k t)\} \\
&\qquad + E\{B_j \sin(2\pi f_j [t+\tau]) A_k \cos(2\pi f_k t)\} \\
&\qquad + E\{B_j \sin(2\pi f_j [t+\tau]) B_k \sin(2\pi f_k t)\} \Big).
\end{aligned}$$

Because the A_j and B_j RVs are all mutually uncorrelated; because $E\{A_j^2\} = E\{B_j^2\} = \sigma_j^2$; and because of the trigonometric identity $\cos(x+y)\cos(x) + \sin(x+y)\sin(x) =$

$\cos(y)$, we have

$$
\begin{aligned}
&E\{(X_{t+\tau}-\mu)(X_t-\mu)\}\\
&=\sum_{j=1}^{\lfloor N/2\rfloor} E\{A_j^2\}\cos(2\pi f_j[t+\tau])\cos(2\pi f_j t)+E\{B_j^2\}\sin(2\pi f_j[t+\tau])\sin(2\pi f_j t)\\
&=\sum_{j=1}^{\lfloor N/2\rfloor} \sigma_j^2\Big[\cos(2\pi f_j[t+\tau])\cos(2\pi f_j t)+\sin(2\pi f_j[t+\tau])\sin(2\pi f_j t)\Big]\\
&=\sum_{j=1}^{\lfloor N/2\rfloor} \sigma_j^2\cos(2\pi f_j\tau),
\end{aligned}
$$

as required. Letting $\tau = 0$ in the above yields

$$E\{(X_t-\mu)^2\}=\sigma^2=\sum_{j=1}^{\lfloor N/2\rfloor}\sigma_j^2,$$

from which we obtain

$$\rho_\tau \stackrel{\text{def}}{=} \frac{E\{(X_{t+\tau}-\mu)(X_t-\mu)\}}{\sigma^2}=\frac{\sum_{j=1}^{\lfloor N/2\rfloor}\sigma_j^2\cos(2\pi f_j\tau)}{\sum_{j=1}^{\lfloor N/2\rfloor}\sigma_j^2},$$

as required.

Answer to Exercise [11] Multiplying both sides of Equation (8a) by $\cos(2\pi f_j t)$ and summing over t yields

$$
\begin{aligned}
\sum_{t=0}^{N-1} X_t\cos(2\pi f_j t) &= \mu\sum_{t=0}^{N-1}\cos(2\pi f_j t)+\sum_{t=0}^{N-1}\sum_{k=1}^{\lfloor N/2\rfloor} A_k\cos(2\pi f_k t)\cos(2\pi f_j t)\\
&\quad+\sum_{t=0}^{N-1}\sum_{k=1}^{\lfloor N/2\rfloor} B_k\sin(2\pi f_k t)\cos(2\pi f_j t)\\
&=\sum_{k=1}^{\lfloor N/2\rfloor} A_k\sum_{t=0}^{N-1}\cos(2\pi f_k t)\cos(2\pi f_j t)\\
&\quad+\sum_{k=1}^{\lfloor N/2\rfloor} B_k\sum_{t=0}^{N-1}\sin(2\pi f_k t)\cos(2\pi f_j t),
\end{aligned}
$$

since it follows from Exercise [1.2d] that $\sum_{t=0}^{N-1}\cos(2\pi f_j t)=0$. For $1\le j<N/2$, Exercise [1.3c] gives the following "orthogonality relationships":

$$\sum_{t=0}^{N-1}\cos(2\pi f_k t)\cos(2\pi f_j t)=\begin{cases}0, & \text{if } k\neq j;\\ N/2, & \text{if } k=j,\end{cases} \tag{A–2}$$

and

$$\sum_{t=0}^{N-1}\sin(2\pi f_k t)\cos(2\pi f_j t)=0 \quad \text{for all } j \text{ and } k.$$

We thus have

$$\sum_{t=0}^{N-1} X_t \cos(2\pi f_j t) = A_j \frac{N}{2},$$

from which the first desired result readily follows. If N is even and $j = N/2$, the only difference is that $N/2$ is replaced by N in Equation (A–2) (again, see Exercise [1.3c]). We now have

$$\sum_{t=0}^{N-1} X_t \cos(2\pi f_{N/2} t) = A_{N/2} N,$$

from which the second desired result follows.

Answer to Exercise [12] Summation of both sides of Equation (8a) yields

$$\begin{aligned}\sum_{t=0}^{N-1} X_t &= \sum_{t=0}^{N-1} \left(\mu + \sum_{j=1}^{\lfloor N/2 \rfloor} \left[A_j \cos(2\pi f_j t) + B_j \sin(2\pi f_j t) \right] \right) \\ &= N\mu + \sum_{j=1}^{\lfloor N/2 \rfloor} A_j \sum_{t=0}^{N-1} \cos(2\pi f_j t) + \sum_{j=1}^{\lfloor N/2 \rfloor} B_j \sum_{t=0}^{N-1} \sin(2\pi f_j t).\end{aligned}$$

Exercise [1.2d] tells us that for all j

$$\sum_{t=0}^{N-1} \cos(2\pi f_j t) = 0 \text{ and } \sum_{t=0}^{N-1} \sin(2\pi f_j t) = 0,$$

thus yielding $\sum_{t=0}^{N-1} X_t = N\mu$, from which the desired result follows immediately.

Answer to Exercise [29] Without loss of generality, we can assume $\mu = 0$ (if not, we just need to replace $X_{1,t}$ and $X_{2,t}$ with, respectively, $X_{1,t} - \mu_1$ and $X_{2,t} - \mu_2$ in what follows). We have

$$\begin{aligned}\operatorname{cov}\{Z_{t+\tau}, Z_t\} &= E\{Z_{t+\tau} Z_t^*\} \\ &= E\{(X_{1,t+\tau} + \mathrm{i}X_{2,t+\tau})(X_{1,t} + \mathrm{i}X_{2,t})^*\} \\ &= E\{(X_{1,t+\tau} + \mathrm{i}X_{2,t+\tau})(X_{1,t} - \mathrm{i}X_{2,t})\} \\ &= E\{X_{1,t+\tau} X_{1,t}\} + E\{X_{2,t+\tau} X_{2,t}\} \\ &\quad + \mathrm{i}E\{X_{2,t+\tau} X_{1,t}\} - \mathrm{i}E\{X_{1,t+\tau} X_{2,t}\}.\end{aligned}$$

Now, for any integers t_1, t_2 and τ', the joint second-order moments of X_{1,t_1}, X_{2,t_1}, X_{1,t_2} and X_{2,t_2} exist, are finite and are equal to the corresponding joint moments of $X_{1,t_1+\tau'}$, $X_{2,t_1+\tau'}$, $X_{1,t_2+\tau'}$ and $X_{2,t_2+\tau'}$. Letting $t_1 = t + \tau$, $t_2 = t$ and $\tau' = -t$ says that the joint moments of $X_{1,t+\tau}$, $X_{2,t+\tau}$, $X_{1,t}$ and $X_{2,t}$ are equal to the corresponding joint moments of $X_{1,\tau}$, $X_{2,\tau}$, $X_{1,0}$ and $X_{2,0}$. Hence

$$E\{Z_{t+\tau} Z_t^*\} = E\{X_{1,\tau} X_{1,0}\} + E\{X_{2,\tau} X_{2,0}\} + \mathrm{i}E\{X_{2,\tau} X_{1,0}\} - \mathrm{i}E\{X_{1,\tau} X_{2,0}\} \stackrel{\text{def}}{=} s_\tau,$$

a sequence that depends upon τ but not t. Now

$$s_{-\tau} = E\{X_{1,-\tau} X_{1,0}\} + E\{X_{2,-\tau} X_{2,0}\} + \mathrm{i}E\{X_{2,-\tau} X_{1,0}\} - \mathrm{i}E\{X_{1,-\tau} X_{2,0}\}.$$

Adding τ to each time index leaves the expectations unchanged, so we have (using the elementary fact that $E\{XY\} = E\{YX\}$)

$$\begin{aligned}
s_{-\tau} &= E\{X_{1,0}X_{1,\tau}\} + E\{X_{2,0}X_{2,\tau}\} + \mathrm{i}E\{X_{2,0}X_{1,\tau}\} - \mathrm{i}E\{X_{1,0}X_{2,\tau}\} \\
&= E\{X_{1,\tau}X_{1,0}\} + E\{X_{2,\tau}X_{2,0}\} + \mathrm{i}E\{X_{1,\tau}X_{2,0}\} - \mathrm{i}E\{X_{2,\tau}X_{1,0}\} \\
&= \left(E\{X_{1,\tau}X_{1,0}\} + E\{X_{2,\tau}X_{2,0}\} - \mathrm{i}E\{X_{1,\tau}X_{2,0}\} + \mathrm{i}E\{X_{2,\tau}X_{1,0}\}\right)^* = s_\tau^*.
\end{aligned}$$

Finally, note that

$$\begin{aligned}
E\{Z_{t+\tau}Z_t\} &= E\{(X_{1,t+\tau} + \mathrm{i}X_{2,t+\tau})(X_{1,t} + \mathrm{i}X_{2,t})\} \\
&= E\{X_{1,t+\tau}X_{1,t}\} - E\{X_{2,t+\tau}X_{2,t}\} + \mathrm{i}E\{X_{2,t+\tau}X_{1,t}\} + \mathrm{i}E\{X_{1,t+\tau}X_{2,t}\}.
\end{aligned}$$

Using the same argument as before, we see that

$$E\{Z_{t+\tau}Z_t\} = E\{X_{1,\tau}X_{1,0}\} - E\{X_{2,\tau}X_{2,0}\} + \mathrm{i}E\{X_{2,\tau}X_{1,0}\} + \mathrm{i}E\{X_{1,\tau}X_{2,0}\},$$

which depends upon τ but not t, as required.

Answer to Exercise [32] Letting $Z_t = X_{0,t} + \mathrm{i}X_{1,t}$ as per Equation (29b) and assuming (without loss of generality) that $E\{Z_t\} = 0$ and hence $E\{X_{0,t}\} = E\{X_{1,t}\} = 0$, we have

$$\sigma^2 = \mathrm{var}\,\{Z_t\} = E\{|Z_t|^2\} = E\{X_{0,t}^2 + X_{1,t}^2\} = \mathrm{var}\,\{X_{0,t}\} + \mathrm{var}\,\{X_{1,t}\}.$$

Since $\{Z_t\}$ is proper, its autorelation sequence at lag $\tau = 0$ must be zero; i.e., we have $r_0 = E\{Z_t^2\} = 0$. Since $Z_t^2 = X_{0,t}^2 + 2\mathrm{i}X_{0,t}X_{1,t} - X_{1,t}^2$, we find that

$$E\{X_{0,t}^2\} - E\{X_{1,t}^2\} = 0 \ \text{ and } \ E\{X_{0,t}X_{1,t}\} = 0;$$

i.e.,

$$\mathrm{var}\,\{X_{0,t}\} = \mathrm{var}\,\{X_{1,t}\} = \frac{\sigma^2}{2} \ \text{ and } \ \mathrm{cov}\,\{X_{0,t}, X_{1,t}\} = 0,$$

as claimed. (Note that we haven't made use of the fact that $\{Z_t\}$ is white noise – *any* proper complex-valued stationary process $\{Z_t'\}$ is such that the real and imaginary parts of Z_t' are uncorrelated and have the same variance.)

Answer to Exercise [36] First, we note that

$$E\{X_t\} = \mu + \sum_{l=1}^{L} E\{A_l\}\cos(2\pi f_l t) + E\{B_l\}\sin(2\pi f_l t) = \mu$$

since $E\{A_l\} = E\{B_l\} = 0$ for all l. Next note that

$$\begin{aligned}
\mathrm{cov}\,\{X_{t+\tau}, X_t\} &= \mathrm{cov}\left\{\sum_{l=1}^{L} A_l\cos(2\pi f_l[t+\tau]) + B_l\sin(2\pi f_l[t+\tau]),\right. \\
&\qquad\left.\sum_{l'=1}^{L} A_{l'}\cos(2\pi f_{l'}t) + B_{l'}\sin(2\pi f_{l'}t)\right\} \\
&= \sum_{l=1}^{L}\sum_{l'=1}^{L} \mathrm{cov}\,\{A_l\cos(2\pi f_l[t+\tau]) + B_l\sin(2\pi f_l[t+\tau]), \\
&\qquad A_{l'}\cos(2\pi f_{l'}t) + B_{l'}\sin(2\pi f_{l'}t)\},
\end{aligned}$$

where the above manipulations can be justified by results stated in Exercise [2.1]. Now

$$\begin{aligned}
&\operatorname{cov}\{A_l\cos(2\pi f_l[t+\tau]) + B_l\sin(2\pi f_l[t+\tau]), A_{l'}\cos(2\pi f_{l'}t) + B_{l'}\sin(2\pi f_{l'}t)\}\\
&\quad= \cos(2\pi f_l[t+\tau])\cos(2\pi f_{l'}t)\operatorname{cov}\{A_l, A_{l'}\}\\
&\qquad+ \cos(2\pi f_l[t+\tau])\sin(2\pi f_{l'}t)\operatorname{cov}\{A_l, B_{l'}\}\\
&\qquad+ \sin(2\pi f_l[t+\tau])\cos(2\pi f_{l'}t)\operatorname{cov}\{B_l, A_{l'}\}\\
&\qquad+ \sin(2\pi f_l[t+\tau])\sin(2\pi f_{l'}t)\operatorname{cov}\{B_l, B_{l'}\}\\
&\quad= \cos(2\pi f_l[t+\tau])\cos(2\pi f_{l'}t)\operatorname{cov}\{A_l, A_{l'}\}\\
&\qquad+ \sin(2\pi f_l[t+\tau])\sin(2\pi f_{l'}t)\operatorname{cov}\{B_l, B_{l'}\},
\end{aligned}$$

because $\operatorname{cov}\{A_l, B_{l'}\} = \operatorname{cov}\{B_l, A_{l'}\} = 0$ for all l and l'. We also have $\operatorname{cov}\{A_l, A_{l'}\} = \operatorname{cov}\{B_l, B_{l'}\} = 0$ when $l \neq l'$ and $\operatorname{cov}\{A_l, A_{l'}\} = \operatorname{cov}\{B_l, B_{l'}\} = \sigma_l^2$ when $l = l'$. Hence

$$\begin{aligned}
\operatorname{cov}\{X_{t+\tau}, X_t\} &= \sum_{l=1}^{L}\sigma_l^2\left[\cos(2\pi f_l[t+\tau])\cos(2\pi f_l t) + \sin(2\pi f_l[t+\tau])\sin(2\pi f_l t)\right]\\
&= \sum_{l=1}^{L}\sigma_l^2\cos(2\pi f_l\tau),
\end{aligned}$$

where we have made use of the trigonometric identity $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$. The above is a function of τ and not t. Hence the harmonic process of Equation (35c) is a stationary process, with an ACVS $\{s_\tau\}$ whose τth element is given by the above expression.

Answer to Exercise [37] As noted below Equation (35d), the trigonometric identity $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ allows us to put that equation into the form of Equation (35c), in which $A_l = D_l\cos(\phi_l)$ and $B_l = -D_l\sin(\phi_l)$. In order to show that the resulting $\{X_t\}$ is a harmonic process, we need to argue that A_l and B_l are zero mean uncorrelated RVs such that $\operatorname{var}\{A_l\} = \operatorname{var}\{B_l\}$ for all l. To show that they have zero mean, note that, since the PDF for ϕ_l is given by $f_{\phi_l}(u) = 1/2\pi$ for $u \in (-\pi, \pi]$,

$$E\{A_l\} = D_l E\{\cos(\phi_l)\} = D_l\int_{-\pi}^{\pi}\cos(u)\frac{1}{2\pi}\,\mathrm{d}u = 0$$

and that a similar argument yields $E\{B_l\} = 0$. We can make use of the indefinite integral $\int\cos^2(x)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2x)}{4}$ to show that

$$\operatorname{var}\{A_l\} = E\{A_l^2\} = D_l^2E\{\cos^2(\phi_l)\} = D_l^2\int_{-\pi}^{\pi}\cos^2(u)\frac{1}{2\pi}\,\mathrm{d}u = \frac{D_l^2}{2};$$

similarly the indefinite integral $\int\sin^2(x)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2x)}{4}$ yields $\operatorname{var}\{B_l\} = D^2/2$ also. The fact that the ϕ_l's are independent RVs implies that $\operatorname{cov}\{A_l, A_{l'}\}$, $\operatorname{cov}\{B_l, B_{l'}\}$ and $\operatorname{cov}\{A_l, B_{l'}\}$ are all zero when $l \neq l'$. The desired result thus follows if we can show that $\operatorname{cov}\{A_l, B_l\} = 0$ for all l. Now

$$\operatorname{cov}\{A_l, B_l\} = E\{A_lB_l\} = D_l^2E\{\cos(\phi_l)\sin(\phi_l)\} = D_l^2\int_{-\pi}^{\pi}\cos(u)\sin(u)\frac{1}{2\pi}\,\mathrm{d}u = 0,$$

where we have made use of the indefinite integral $\int\cos(x)\sin(x)\,\mathrm{d}x = \sin^2(x)/2$. Finally the ACVS of Equation (37b) follows from Equation (36a) by noting that the common variance for A_l and B_l is given by $\sigma_l^2 = D_l^2/2$.

Answer to Exercise [49a] If we take both sides of Equation (49c), multiply them by $\exp(-\mathrm{i}2\pi f_m t)$ and integrate from $-T/2$ to $T/2$, we get

$$\begin{aligned}\int_{-T/2}^{T/2} g_p(t)\mathrm{e}^{-\mathrm{i}2\pi f_m t}\,\mathrm{d}t &= \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} G_n \mathrm{e}^{\mathrm{i}2\pi(f_n - f_m)t}\,\mathrm{d}t \\ &= \sum_{n=-\infty}^{\infty} G_n \int_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}2\pi(f_n - f_m)t}\,\mathrm{d}t.\end{aligned} \tag{A–6a}$$

Since $f_n - f_m = (n-m)/T$, the change of variable $u = t/T$ shows that

$$\begin{aligned}\int_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}2\pi(f_n - f_m)t}\,\mathrm{d}t &= T\int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}2\pi(n-m)u}\,\mathrm{d}u \\ &= T\int_{-1/2}^{1/2} \cos(2\pi[n-m]u)\,\mathrm{d}u + \mathrm{i}T\int_{-1/2}^{1/2} \sin(2\pi[n-m]u)\,\mathrm{d}u.\end{aligned}$$

If $m \neq n$, both integrations are over over one or more complete cycles of the sinusoids and are hence equal to zero; if $m = n$, the integrals for the real and imaginary parts are, respectively, one and zero. Hence

$$\int_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}2\pi(f_n - f_m)t}\,\mathrm{d}t = \begin{cases} 0, & m \neq n; \\ T, & m = n. \end{cases} \tag{A–6b}$$

Equation (A–6a) now reduces to

$$\int_{-T/2}^{T/2} g_p(t)\mathrm{e}^{-\mathrm{i}2\pi f_m t}\,\mathrm{d}t = TG_m,$$

from which (49b) follows immediately.

Answer to Exercise [49b] Recalling that $|z|^2 = zz^*$ for any complex-valued variable, we have

$$\begin{aligned}\int_{-T/2}^{T/2} |g_p(t)|^2\,\mathrm{d}t = \int_{-T/2}^{T/2} g_p(t)g_p^*(t)\,\mathrm{d}t &= \int_{-T/2}^{T/2} g_p(t)\left(\sum_{n=-\infty}^{\infty} G_n^* \mathrm{e}^{\mathrm{i}2\pi f_n t}\right)\mathrm{d}t \\ &= T\sum_{n=-\infty}^{\infty} G_n^* \left(\frac{1}{T}\int_{-T/2}^{T/2} g_p(t)\mathrm{e}^{\mathrm{i}2\pi f_n t}\,\mathrm{d}t\right) \\ &= T\sum_{n=-\infty}^{\infty} G_n^* G_n = T\sum_{n=-\infty}^{\infty} |G_n|^2.\end{aligned}$$

Answer to Exercise [55] Recalling that $|g(t)|^2 = g(t)g^*(t)$ and making use of Equation (53), we have

$$\begin{aligned}\int_{-\infty}^{\infty} |g(t)|^2\,\mathrm{d}t &= \int_{-\infty}^{\infty} g(t)\left(\int_{-\infty}^{\infty} G(f)\mathrm{e}^{\mathrm{i}2\pi ft}\,\mathrm{d}f\right)^*\mathrm{d}t \\ &= \int_{-\infty}^{\infty} G^*(f)\left(\int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}2\pi ft}\,\mathrm{d}t\right)\mathrm{d}f \\ &= \int_{-\infty}^{\infty} G^*(f)G(f)\,\mathrm{d}f = \int_{-\infty}^{\infty} |G(f)|^2\,\mathrm{d}f.\end{aligned}$$

Answer to Exercise [67] If we substitute Equation (67a) into the left-hand side of (67b), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u)h(t-u)\mathrm{e}^{-\mathrm{i}2\pi ft}\,\mathrm{d}u\,\mathrm{d}t = \int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{i}2\pi fu}\int_{-\infty}^{\infty} h(t-u)\mathrm{e}^{-\mathrm{i}2\pi f(t-u)}\,\mathrm{d}t\,\mathrm{d}u.$$

Let $y = t-u$ to get

$$\int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{i}2\pi fu}\,\mathrm{d}u\int_{-\infty}^{\infty} h(y)\mathrm{e}^{-\mathrm{i}2\pi fy}\,\mathrm{d}y = G(f)H(f),$$

as required.

Answer to Exercise [69] We have

$$\begin{aligned} g*h(t) &= \int_{-\infty}^{\infty} g(u)h(t-u)\,\mathrm{d}u \\ &= \frac{1}{(2\pi\sigma^2)^{1/2}}\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\sigma^2)}\left[\sum_{l=1}^{L} A_l\cos\left(2\pi f_l[t-u]+\phi_l\right)\right]\mathrm{d}u \\ &= \frac{1}{(2\pi\sigma^2)^{1/2}}\sum_{l=1}^{L} A_l\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\sigma^2)}\cos\left(2\pi f_l[t-u]+\phi_l\right)\,\mathrm{d}u. \end{aligned}$$

Using the trigonometric identity, we have

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\sigma^2)}\cos\left(2\pi f_l[t-u]+\phi_l\right)\mathrm{d}u &= \cos\left(2\pi f_l t+\phi_l\right)\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\sigma^2)}\cos\left(2\pi f_l u\right)\mathrm{d}u \\ &\quad + \sin\left(2\pi f_l t+\phi_l\right)\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\sigma^2)}\sin\left(2\pi f_l u\right)\mathrm{d}u \\ &= (2\pi\sigma^2)^{1/2}\mathrm{e}^{-(\sigma 2\pi f_l)^2/2}\cos\left(2\pi f_l t+\phi_l\right) \end{aligned}$$

because, letting $v = u/(2\pi\sigma^2)^{1/2}$ and making use of the integral expressions,

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\sigma^2)}\cos\left(2\pi f_l u\right)\mathrm{d}u &= (2\pi\sigma^2)^{1/2}\int_{-\infty}^{\infty}\mathrm{e}^{-\pi v^2}\cos\left(2\pi f_l[2\pi\sigma^2]^{1/2}v\right)\mathrm{d}v \\ &= (2\pi\sigma^2)^{1/2}\mathrm{e}^{-\pi\cdot f_l^2 2\pi\sigma^2} = (2\pi\sigma^2)^{1/2}\mathrm{e}^{-(\sigma 2\pi f_l)^2/2}, \end{aligned}$$

while

$$\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\sigma^2)}\sin\left(2\pi f_l u\right)\mathrm{d}u = (2\pi\sigma^2)^{1/2}\int_{-\infty}^{\infty}\mathrm{e}^{-\pi v^2}\sin\left(2\pi f_l[2\pi\sigma^2]^{1/2}v\right)\mathrm{d}v = 0.$$

The stated result now follows immediately.

Answer to Exercise [71] We have

$$r*h(t) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} h(u)\,\mathrm{d}u = \frac{1}{2\delta}\sum_{l=1}^{L} A_l\int_{t-\delta}^{t+\delta}\cos\left(2\pi f_l u+\phi_l\right)\mathrm{d}u. \qquad \text{(A–7)}$$

The integral is the real part of

$$\begin{aligned} \int_{t-\delta}^{t+\delta}\mathrm{e}^{\mathrm{i}(2\pi f_l u+\phi_l)}\,\mathrm{d}u = \mathrm{e}^{\mathrm{i}\phi_l}\int_{t-\delta}^{t+\delta}\mathrm{e}^{\mathrm{i}2\pi f_l u}\,\mathrm{d}u &= \mathrm{e}^{\mathrm{i}\phi_l}\left(\frac{\mathrm{e}^{\mathrm{i}2\pi f_l(t+\delta)}}{\mathrm{i}2\pi f_l} - \frac{\mathrm{e}^{\mathrm{i}2\pi f_l(t-\delta)}}{\mathrm{i}2\pi f_l}\right) \\ &= \mathrm{e}^{\mathrm{i}(2\pi f_l t+\phi_l)}\left(\frac{\mathrm{e}^{\mathrm{i}2\pi f_l\delta}-\mathrm{e}^{-\mathrm{i}2\pi f_l\delta}}{\mathrm{i}2\pi f_l}\right) = \mathrm{e}^{\mathrm{i}(2\pi f_l t+\phi_l)}\frac{\sin\left(2\pi f_l\delta\right)}{\pi f_l}, \end{aligned}$$

so we have

$$\int_{t-\delta}^{t+\delta}\cos\left(2\pi f_l u+\phi_l\right)\mathrm{d}u = \cos\left(2\pi f_l t+\phi_l\right)\frac{\sin\left(2\pi f_l\delta\right)}{\pi f_l}.$$

Substitution of the above into (A–7) leads to the desired result.

Answer to Exercise [73a] It follows from Equation (72c) that

$$g \star g^*(0) = \int_{-\infty}^{\infty} g(u)g^*(u)\,\mathrm{d}u = \int_{-\infty}^{\infty} |g(t)|^2\,\mathrm{d}t,$$

so we can establish the desired result by showing that

$$\int_{-\infty}^{\infty} g \star g^*(t)\,\mathrm{d}t = \left|\int_{-\infty}^{\infty} g(t)\,\mathrm{d}t\right|^2.$$

Note that, if $a(\cdot) \longleftrightarrow A(\cdot)$, it follows from the relationship

$$A(f) = \int_{-\infty}^{\infty} a(t)\mathrm{e}^{-\mathrm{i}2\pi ft}\,\mathrm{d}t \ \text{ that } \ A(0) = \int_{-\infty}^{\infty} a(t)\,\mathrm{d}t.$$

Thus $\int g \star g^*(t)\,\mathrm{d}t$ is the Fourier transform of $g \star g^*(\cdot)$ evaluated at $f = 0$; however, Equation (73a) says that this transform is given by $|G(\cdot)|^2$, and hence we have

$$\int_{-\infty}^{\infty} g \star g^*(t)\,\mathrm{d}t = |G(0)|^2 = \left|\int_{-\infty}^{\infty} g(t)\,\mathrm{d}t\right|^2,$$

as required. Alternatively we can integrate both side of Equation (72c) with respect to t to obtain (letting $t' = u + t$)

$$\begin{aligned}\int_{-\infty}^{\infty} g \star g^*(t)\,\mathrm{d}t &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u+t)g^*(u)\,\mathrm{d}u\,\mathrm{d}t = \int_{-\infty}^{\infty} g^*(u)\int_{-\infty}^{\infty} g(u+t)\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} g^*(u)\int_{-\infty}^{\infty} g(t')\,\mathrm{d}t'\,\mathrm{d}u = \int_{-\infty}^{\infty} g(t')\,\mathrm{d}t'\int_{-\infty}^{\infty} g^*(u)\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty} g(t')\,\mathrm{d}t'\left(\int_{-\infty}^{\infty} g(u)\,\mathrm{d}u\right)^* = \left|\int_{-\infty}^{\infty} g(t')\,\mathrm{d}t'\right|^2,\end{aligned}$$

as required.

Answer to Exercise [73b] From Exercise [73a], we have, because a PDF must integrate to unity,

$$\text{width}_{\mathrm{a}}\left\{g_\sigma(\cdot)\right\} = \frac{\left|\int_{-\infty}^{\infty} g_\sigma(t)\,\mathrm{d}t\right|^2}{\int_{-\infty}^{\infty} g_\sigma^2(t)\,\mathrm{d}t} = \frac{1}{\int_{-\infty}^{\infty} g_\sigma^2(t)\,\mathrm{d}t}.$$

Now

$$g_\sigma^2(t) = \frac{1}{2\pi\sigma^2}\mathrm{e}^{-t^2/\sigma^2} = \frac{(\pi\sigma^2)^{1/2}}{2\pi\sigma^2}\cdot\frac{1}{(\pi\sigma^2)^{1/2}}\mathrm{e}^{-t^2/\sigma^2} = \frac{1}{2\pi^{1/2}\sigma}g_{\sigma/\sqrt{2}}(t),$$

and hence

$$\int_{-\infty}^{\infty} g_\sigma^2(t)\,\mathrm{d}t = \frac{1}{2\pi^{1/2}\sigma},$$

which leads to the stated result.

Answer to Exercise [76] Using the inverse Fourier transform of Equation (75a) (with $\Delta_\text{t} = 1$), we have

$$G_{p,m}(f) \stackrel{\text{def}}{=} \sum_{t=-m}^{m} g_t \mathrm{e}^{-\mathrm{i}2\pi f t} = \sum_{t=-m}^{m} \left(\int_{-1/2}^{1/2} G_p(f')\mathrm{e}^{\mathrm{i}2\pi f' t}\,\mathrm{d}f' \right) \mathrm{e}^{-\mathrm{i}2\pi f t}$$
$$= \int_{-1/2}^{1/2} G_p(f') \left(\sum_{t=-m}^{m} \mathrm{e}^{\mathrm{i}2\pi (f'-f)t} \right) \mathrm{d}f'$$
$$= (2m+1) \int_{-1/2}^{1/2} G_p(f') \mathcal{D}_{2m+1}(f-f')\,\mathrm{d}f'$$

since

$$\sum_{t=-m}^{m} \mathrm{e}^{\mathrm{i}2\pi f t} = (2m+1)\mathcal{D}_{2m+1}(f)$$

from Exercise [1.2e].

Answer to Exercise [79] Using the inverse Fourier transform of Equation (75a) (with $\Delta_\text{t} = 1$), we have

$$G_{p,m}^{(C)}(f) = \sum_{t=-m}^{m} \left(1 - \frac{|t|}{m}\right) \left(\int_{-1/2}^{1/2} G_p(f')\mathrm{e}^{\mathrm{i}2\pi f' t}\,\mathrm{d}f' \right) \mathrm{e}^{-\mathrm{i}2\pi f t}$$
$$= \int_{-1/2}^{1/2} G_p(f') \sum_{t=-m}^{m} \left(1 - \frac{|t|}{m}\right) \mathrm{e}^{-\mathrm{i}2\pi (f-f')t}\,\mathrm{d}f'$$
$$= m \int_{-1/2}^{1/2} G_p(f') \mathcal{D}_m^2(f-f')\,\mathrm{d}f',$$

where we have made use of the fact that

$$\sum_{t=-m}^{m} \left(1 - \frac{|t|}{m}\right) \mathrm{e}^{-\mathrm{i}2\pi f t} = m\mathcal{D}_m^2(f).$$

This equation can be verified by filling in the details in the following line of thought: (a) the sequence defined by $1-|t|/m$ is essentially the autocorrelation of a rectangular sequence of length m with itself; (b) the Fourier transform of a rectangular sequence is essentially Dirichlet's kernel; and (c) an appropriate version of the autocorrelation theorem thus says that the Fourier transform of $\{1-|t|/m\}$ is the square of Dirichlet's kernel. Here are the details. Let

$$r_t = \begin{cases} 1/\sqrt{m}, & t = 0, 1, \ldots, m-1; \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier transform of $\{r_t\}$ is

$$\sum_{t=-\infty}^{\infty} r_t \mathrm{e}^{-\mathrm{i}2\pi f t} = \frac{1}{\sqrt{m}} \sum_{t=0}^{m-1} \mathrm{e}^{-\mathrm{i}2\pi f t}.$$

From Exercise [1.2c], we have

$$\sum_{t=0}^{m-1} \mathrm{e}^{-\mathrm{i}2\pi f t} = m\mathrm{e}^{-\mathrm{i}(m-1)\pi f} \mathcal{D}_m(f)$$

and hence

$$\frac{1}{\sqrt{m}} \sum_{t=0}^{m-1} \mathrm{e}^{-\mathrm{i}2\pi f t} = \sqrt{m}\mathrm{e}^{-\mathrm{i}(m-1)\pi f} \mathcal{D}_m(f).$$

It follows from the discussion following Equation (100a) that the Fourier transform of the autocorrelation $\{r\star r_t\}$ is the function given by $|\sqrt{m}\mathrm{e}^{-\mathrm{i}(m-1)\pi f}\mathcal{D}_m(f)|^2 = m\mathcal{D}_m^2(f)$. The desired result follows if we can establish that

$$r \star r_t = \begin{cases} 1 - \frac{|t|}{m}, & |t| \le m-1; \\ 0 & \text{otherwise.} \end{cases}$$

From Equation (99f), we have

$$r \star r_t = \sum_{u=-\infty}^{\infty} r_{u+t} r_u = \frac{1}{\sqrt{m}} \sum_{u=0}^{m-1} r_{u+t}.$$

If $|t| \ge m$, then $r_{u+t} = 0$ when $u = 0, \ldots, m-1$ since then either $u + t < 0$ always or $u + t > m - 1$ always; if $|t| < m$, then

$$\sum_{u=0}^{m-1} r_{u+t} = \frac{I_t}{\sqrt{m}}, \text{ where } I_t \stackrel{\text{def}}{=} \{\# \text{ of } u \text{ such that } 0 \le u + t \le m - 1\}.$$

If t is nonnegative, then $I_t = m - t = m - |t|$; if t is negative, then $I_t = m - |t|$ also. Hence $\{r \star r_t\}$ has the required form.

Answer to Exercise [84] It follows from Equation (74a) that

$$\begin{aligned} g(t) &= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \left(\Delta_{\mathrm{t}} \sum_{t'=-\infty}^{\infty} g_{t'} \mathrm{e}^{-\mathrm{i}2\pi f t' \Delta_{\mathrm{t}}} \right) \mathrm{e}^{\mathrm{i}2\pi f t} \, \mathrm{d}f \\ &= \sum_{t'=-\infty}^{\infty} g_{t'} \left(\Delta_{\mathrm{t}} \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathrm{e}^{\mathrm{i}2\pi f (t - t' \Delta_{\mathrm{t}})} \, \mathrm{d}f \right). \end{aligned}$$

Recalling that $f_{\mathcal{N}} = 1/(2\,\Delta_{\mathrm{t}})$, we have

$$\begin{aligned} \Delta_{\mathrm{t}} \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathrm{e}^{\mathrm{i}2\pi f a} \, \mathrm{d}f &= \Delta_{\mathrm{t}} \left(\frac{\mathrm{e}^{\mathrm{i}2\pi f_{\mathcal{N}} a}}{\mathrm{i}2\pi a} - \frac{\mathrm{e}^{-\mathrm{i}2\pi f_{\mathcal{N}} a}}{\mathrm{i}2\pi a} \right) \\ &= \frac{\mathrm{e}^{\mathrm{i}2\pi f_{\mathcal{N}} a} - \mathrm{e}^{-\mathrm{i}2\pi f_{\mathcal{N}} a}}{\mathrm{i}4\pi f_{\mathcal{N}} a} = \frac{\sin(2\pi f_{\mathcal{N}} a)}{2\pi f_{\mathcal{N}} a} = \mathrm{sinc}(2 f_{\mathcal{N}} a), \end{aligned}$$

from which the desired result follows immediately by letting $a = t - t'\,\Delta_{\mathrm{t}}$.

Answer to Exercise [92] If we take both sides of Equation (91b), multiply them by $\exp(\mathrm{i}2\pi n t'/N)$ and sum with respect to n, we get

$$\sum_{n=0}^{N-1} G_n \mathrm{e}^{\mathrm{i}2\pi n t'/N} = \Delta_{\mathrm{t}} \sum_{n=0}^{N-1} \sum_{t=0}^{N-1} g_t \mathrm{e}^{\mathrm{i}2\pi n (t'-t)/N} = \Delta_{\mathrm{t}} \sum_{t=0}^{N-1} g_t \sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi n (t'-t)/N}. \quad \text{(A–10)}$$

It follows from Exercise [1.2] that

$$\sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi n k/N} = \begin{cases} N, & \text{if } k = mN \text{ for integer } m; \\ 0, & \text{otherwise.} \end{cases}$$

The only time that $t' - t$ is an integer multiple of N is when $t' = t$, so Equation (A–10) reduces to

$$\sum_{n=0}^{N-1} G_n \mathrm{e}^{\mathrm{i}2\pi n t'/N} = g_{t'} N \,\Delta_{\mathrm{t}},$$

from which the desired result follows.

Answer to Exercise [93] If we insert the Fourier representation for $\{g_t\}$ given by Equation (75a) into Equation (91b) and make use of the definition $f_n = n/(N\,\Delta_t)$, we obtain

$$\begin{aligned} G_n &= \Delta_t \sum_{t=0}^{N-1} \left(\int_{-f_\mathcal{N}}^{f_\mathcal{N}} G_p(f) e^{i2\pi f t dt}\, df \right) e^{-i2\pi f_n t\,\Delta_t} \\ &= \Delta_t \int_{-f_\mathcal{N}}^{f_\mathcal{N}} G_p(f) \sum_{t=0}^{N-1} e^{i2\pi t\,(f-f_n)\,\Delta_t}\, df. \end{aligned}$$

If we appeal to Exercise [1.2c], the above becomes

$$G_n = \Delta_t \int_{-f_\mathcal{N}}^{f_\mathcal{N}} G_p(f) e^{-i(N-1)\pi(f_n - f)\,\Delta_t} N\mathcal{D}_N\left([f_n - f]\,\Delta_t\right) df,$$

where $\mathcal{D}_N(\cdot)$ is Dirichlet's kernel (note that this is an even function). The desired result now follows from the definition of $P(\cdot)$.

Answer to Exercise [122] We have

$$\begin{aligned} X_t = X(t_0 + t\,\Delta_t) &= \int_{-\infty}^{\infty} e^{i2\pi f(t_0 + t\,\Delta_t)}\, dZ_{X(t)}(f) \\ &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)/(2\,\Delta_t)}^{(2k+1)/(2\,\Delta_t)} e^{i2\pi f t\,\Delta_t} e^{i2\pi f t_0}\, dZ_{X(t)}(f) \\ &= \sum_{k=-\infty}^{\infty} \int_{-1/(2\,\Delta_t)}^{1/(2\,\Delta_t)} e^{i2\pi(f + \frac{k}{\Delta_t})t\,\Delta_t} e^{i2\pi(f+\frac{k}{\Delta_t})t_0}\, dZ_{X(t)}(f + \tfrac{k}{\Delta_t}) \\ &= \int_{-f_\mathcal{N}}^{f_\mathcal{N}} e^{i2\pi f t\,\Delta_t} \sum_{k=-\infty}^{\infty} e^{i2\pi(f+\frac{k}{\Delta_t})t_0}\, dZ_{X(t)}(f + \tfrac{k}{\Delta_t}) \\ &= \int_{-f_\mathcal{N}}^{f_\mathcal{N}} e^{i2\pi f t\,\Delta_t}\, dZ(f). \end{aligned}$$

Answer to Exercise [148] We have

$$\begin{aligned} \sigma_Y^2 &= \int_{-\infty}^{\infty} |G(f)|^2\, dS_X^{(I)}(f) \\ &= \int_{-f'-df'}^{-f'} dS_X^{(I)}(f) + \int_{f'}^{f'+df'} dS_X^{(I)}(f) \\ &= S_X^{(I)}(-f') - S_X^{(I)}(-f' - df') + S_X^{(I)}(f' + df') - S_X^{(I)}(f') \\ &= 2\left[S_X^{(I)}(f' + df') - S_X^{(I)}(f') \right], \end{aligned}$$

where the last part follows because $S_X^{(I)}(-f) + S_X^{(I)}(f) = \sigma_X^2$ for all f.

Answer to Exercise [150] We have

$$\sum_{u=-K}^{K} g_u[\alpha + \beta(t-u)] = (\alpha + \beta t) \sum_{u=-K}^{K} g_u - \beta \sum_{u=-K}^{K} g_u u = \alpha + \beta t$$

because of the assumption that $\sum_{u=-K}^{K} g_u = 1$ and because

$$\sum_{u=-K}^{K} g_u u = \sum_{u=1}^{K} g_u u + \sum_{u=-K}^{-1} g_u u = \sum_{u=1}^{K} g_u u - \sum_{u=1}^{K} g_u u = 0$$

under the assumption $g_{-u} = g_u$.

Answer to Exercise [165] It follows from Exercise [2.1e] that

$$\operatorname{var}\{\overline{X}\} = \operatorname{cov}\left\{\frac{1}{N}\sum_{u=0}^{N-1} X_u, \frac{1}{N}\sum_{t=0}^{N-1} X_t\right\} = \frac{1}{N^2}\sum_{u=0}^{N-1}\sum_{t=0}^{N-1} \operatorname{cov}\{X_u, X_t\}$$
$$= \frac{1}{N^2}\sum_{u=0}^{N-1}\sum_{t=0}^{N-1} s_{u-t}.$$

Consider an $N \times N$ matrix whose (u,t)th element is given by s_{u-t}. The above double summation is just the sum of all the elements in the matrix. Since the matrix is Toeplitz (see Equation (29a)), we can also sum its elements by summing along all its diagonals. The main diagonal has N copies of s_0; the first sub- and super-diagonals each have $N-1$ copies of s_1; the second sub- and super-diagonals each have $N-2$ copies of s_2; and so forth, leading to the conclusion that

$$\operatorname{var}\{\overline{X}\} = \frac{1}{N^2}\sum_{\tau=-(N-1)}^{N-1} (N - |\tau|)\, s_\tau = \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\left(1 - \frac{|\tau|}{N}\right) s_\tau,$$

as required.

Answer to Exercise [170] Define $X_t = 0$ for $t < 0$ and $t \geq N$. The autocorrelation of the infinite sequence $\{X_t\}$ is

$$X \star X_\tau = \Delta_{\mathrm{t}} \sum_{t=-\infty}^{\infty} X_{t+\tau} X_t = N \hat{s}_\tau^{(\mathrm{P})} \Delta_{\mathrm{t}}$$

(see Equations (99f) and (170b), and recall that $\hat{s}_\tau^{(\mathrm{P})} \stackrel{\text{def}}{=} 0$ for $|\tau| \geq N$). The Fourier transform of an autocorrelation is the squared modulus of the Fourier transform of the sequence being autocorrelated:

$$\Delta_{\mathrm{t}} \sum_{\tau=-\infty}^{\infty} X \star X_\tau \mathrm{e}^{-\mathrm{i}2\pi f \tau \, \Delta_{\mathrm{t}}} = \left| \Delta_{\mathrm{t}} \sum_{t=-\infty}^{\infty} X_t \mathrm{e}^{-\mathrm{i}2\pi f t \, \Delta_{\mathrm{t}}} \right|^2$$

(see Equations (100a) and (74a)). Substituting $X \star X_\tau = N\hat{s}_\tau^{(\mathrm{P})}\Delta_{\mathrm{t}}$ into the above and adjusting the limits of the summations to range over the nonzero portions of $\{\hat{s}_\tau^{(\mathrm{P})}\}$ and $\{X_t\}$ yield

$$\Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(\mathrm{P})} \mathrm{e}^{-\mathrm{i}2\pi f \tau \, \Delta_{\mathrm{t}}} = \frac{\Delta_{\mathrm{t}}}{N} \left| \sum_{t=0}^{N-1} X_t \mathrm{e}^{-\mathrm{i}2\pi f t \, \Delta_{\mathrm{t}}} \right|^2,$$

which is the required result.

Here is a second proof. Using the definition for $\hat{s}_\tau^{(\mathrm{P})}$ in Equation (170b), we have

$$\Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(\mathrm{P})} \mathrm{e}^{-\mathrm{i}2\pi f \tau \, \Delta_{\mathrm{t}}} = \frac{\Delta_{\mathrm{t}}}{N} \sum_{\tau=-(N-1)}^{N-1} \sum_{t=0}^{N-|\tau|-1} X_{t+|\tau|} X_t \mathrm{e}^{-\mathrm{i}2\pi f \tau \, \Delta_{\mathrm{t}}} \quad \text{(A–12a)}$$
$$= \frac{\Delta_{\mathrm{t}}}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} X_j X_k \mathrm{e}^{-\mathrm{i}2\pi f (k-j) \, \Delta_{\mathrm{t}}} \quad \text{(A–12b)}$$
$$= \frac{\Delta_{\mathrm{t}}}{N} \left| \sum_{t=0}^{N-1} X_t \mathrm{e}^{-\mathrm{i}2\pi f t \, \Delta_{\mathrm{t}}} \right|^2$$

after a change of variables in the double summation, which we can justify as follows. Consider the $N \times N$ matrix whose (j,k)th element is $X_j X_k \exp\left[-\mathrm{i}2\pi f(k-j)\,\Delta_\mathrm{t}\right]$. The double summation in Equation (A–12b) is just the summation of all the elements of this matrix. The inner summation adds up the elements of the jth row, while the outer summation ranges over all rows. The double summation in Equation (A–12a) again sums up all the elements of the matrix, but now by diagonals indexed by $\tau = k - j$: here the inner summation adds up the elements of the τth diagonal (note that there are $N - |\tau|$ elements in that diagonal), while the outer summation ranges over all $2N - 1$ diagonals.

Answer to Exercise [174] Consider the "rectangular" sequence $\{r_t : t \in \mathbb{Z}\}$ defined as

$$r_t = \begin{cases} 1, & \text{when } 0 \le t \le N-1 \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

With the sampling interval taken to be Δ_t, Equation (74a) and Exercise [1.2c] say that the Fourier transform of $\{r_t\}$ is given by $\Delta_\mathrm{t} N \mathrm{e}^{-\mathrm{i}(N-1)\pi}\mathcal{D}_N(f\,\Delta_\mathrm{t})$. Note that the autocorrelation of $\{r_t\}$ is given by

$$r \star r_\tau \stackrel{\text{def}}{=} \Delta_\mathrm{t} \sum_{u=-\infty}^{\infty} r_{u+\tau} r_u = \begin{cases} \Delta_\mathrm{t}(N - |\tau|), & \text{when } |\tau| < N \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

(see Equation (99f) and cf. Figure 69). Equation (100a) says that the Fourier transform of $\{r \star r_\tau\}$ is given by

$$\left|\Delta_\mathrm{t} N \mathrm{e}^{-\mathrm{i}(N-1)\pi}\mathcal{D}_N(f\,\Delta_\mathrm{t})\right|^2 = \Delta_\mathrm{t}^2 N^2 \mathcal{D}_N^2(f\,\Delta_\mathrm{t}) = \Delta_\mathrm{t} N \mathcal{F}(f).$$

Since $\{s_\tau\} \longleftrightarrow S(\cdot)$, Equations (74a), (99e) and (99d) tell us that the Fourier transform of $\{r \star r_\tau s_\tau\}$ is

$$\Delta_\mathrm{t} \sum_{\tau=-\infty}^{\infty} r \star r_\tau s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_\mathrm{t}} = \Delta_\mathrm{t} N \int_{-f_\mathcal{N}}^{f_\mathcal{N}} \mathcal{F}(f') S(f - f')\,\mathrm{d}f'.$$

Since

$$\frac{1}{N} \sum_{\tau=-\infty}^{\infty} r \star r_\tau s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_\mathrm{t}} = \Delta_\mathrm{t} \sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_\mathrm{t}},$$

we obtain the desired result:

$$\Delta_\mathrm{t} \sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_\mathrm{t}} = \int_{-f_\mathcal{N}}^{f_\mathcal{N}} \mathcal{F}(f') S(f - f')\,\mathrm{d}f'.$$

(When $\Delta_\mathrm{t} = 1$, the above is an immediate consequence of Exercise [4.5b] once we note that $|J(f)|^2 = \hat{S}^{(\mathrm{P})}(f)$ and $N\mathcal{D}_N^2(f - f') = \mathcal{F}(f - f')$.)

Answer to Exercise [179] The first part of the exercise follows from the second by letting $N_1 = 0$, so we only consider the more general case. Define

$$X_t' = \begin{cases} 0, & 0 \le t \le N_1 - 1, \\ X_{t-N_1}, & N_1 \le t \le N + N_1 - 1 \text{ and} \\ 0, & N + N_1 \le t \le N + N_1 + N_2 - 1 = N' - 1. \end{cases}$$

From Equation (91b), the DFT of $\{X'_t\}$ is given by

$$\mathcal{X}'_k = \Delta_{\rm t} \sum_{t=0}^{N'-1} X'_t \mathrm{e}^{-\mathrm{i}2\pi kt/N'} = \Delta_{\rm t} \sum_{t=N_1}^{N+N_1-1} X_{t-N_1} \mathrm{e}^{-\mathrm{i}2\pi f'_k t\,\Delta_{\rm t}}, \quad k = 0, 1, \ldots, N'-1,$$

from which it follows that

$$\begin{aligned}
\frac{|\mathcal{X}'_k|^2}{N\,\Delta_{\rm t}} &= \frac{\Delta_{\rm t}}{N} \left| \sum_{t=N_1}^{N+N_1-1} X_{t-N_1} \mathrm{e}^{-\mathrm{i}2\pi f'_k t\,\Delta_{\rm t}} \right|^2 \\
&= \frac{\Delta_{\rm t}}{N} \left| \sum_{t=0}^{N-1} X_t \mathrm{e}^{-\mathrm{i}2\pi f'_k (t+N_1)\,\Delta_{\rm t}} \right|^2 \\
&= \frac{\Delta_{\rm t}}{N} \left| \mathrm{e}^{-\mathrm{i}2\pi k N_1/N} \sum_{t=0}^{N-1} X_t \mathrm{e}^{-\mathrm{i}2\pi f'_k t\,\Delta_{\rm t}} \right|^2 \\
&= \frac{\Delta_{\rm t}}{N} \left| \mathrm{e}^{-\mathrm{i}2\pi k N_1/N} \right|^2 \left| \sum_{t=0}^{N-1} X_t \mathrm{e}^{-\mathrm{i}2\pi f'_k t\,\Delta_{\rm t}} \right|^2 = \frac{\Delta_{\rm t}}{N} \left| \sum_{t=0}^{N-1} X_t \mathrm{e}^{-\mathrm{i}2\pi f'_k t\,\Delta_{\rm t}} \right|^2 = \hat{S}^{(\mathrm{P})}(f'_k)
\end{aligned}$$

since $|\exp(\mathrm{i}x)|^2 = 1$ for all real-valued x.

Answer to Exercise [186] Plugging the spectral representation

$$X_t = \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathrm{e}^{\mathrm{i}2\pi f' t\,\Delta_{\rm t}}\,\mathrm{d}Z(f'),$$

into the definition for $J(f)$ and making use of the definition of $H(\cdot)$ yields

$$\begin{aligned}
J(f) &= \Delta_{\rm t}^{1/2} \sum_{t=0}^{N-1} h_t \left(\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathrm{e}^{\mathrm{i}2\pi f' t\,\Delta_{\rm t}}\,\mathrm{d}Z(f') \right) \mathrm{e}^{-\mathrm{i}2\pi f t\,\Delta_{\rm t}} \\
&= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \Delta_{\rm t}^{1/2} \sum_{t=0}^{N-1} h_t \mathrm{e}^{-\mathrm{i}2\pi (f-f') t\,\Delta_{\rm t}}\,\mathrm{d}Z(f') \\
&= \frac{1}{\Delta_{\rm t}^{1/2}} \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} H(f-f')\,\mathrm{d}Z(f'),
\end{aligned}$$

as required.

Answer to Exercise [188] Integration of both sides of Equation (186g) yields

$$\begin{aligned}
\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} E\{\hat{S}^{(\mathrm{D})}(f)\}\,\mathrm{d}f &= \Delta_{\rm t} \sum_{\tau=-(N-1)}^{N-1} \left(s_\tau \sum_{t=0}^{N-|\tau|-1} h_{t+|\tau|} h_t \right) \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_{\rm t}}\,\mathrm{d}f \\
&= s_0 \sum_{t=0}^{N-1} h_t^2
\end{aligned}$$

as required because

$$\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_{\rm t}}\,\mathrm{d}f = \begin{cases} 2f_{\mathcal{N}} = 1/\Delta_{\rm t}, & \tau = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Answer to Exercise [192] Equation (186e) and the facts that both $\mathcal{H}(\cdot)$ and $S(\cdot)$ are even and periodic with a period of $2f_{\mathcal{N}}$, and that $\mathcal{H}(\cdot)$ integrates to unity over $[-f_{\mathcal{N}}, f_{\mathcal{N}}]$, allow us to write

$$\begin{aligned} b(f) \stackrel{\text{def}}{=} E\{\hat{S}^{(\mathrm{D})}(f)\} - S(f) &= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{H}(\phi - f) S(\phi)\,\mathrm{d}\phi - \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{H}(\phi) S(f)\,\mathrm{d}\phi \\ &= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{H}(\phi) S(f+\phi)\,\mathrm{d}\phi - \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{H}(\phi) S(f)\,\mathrm{d}\phi \\ &= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{H}(\phi)\,[S(f+\phi) - S(f)]\,\mathrm{d}\phi \\ &\approx \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{H}(\phi) \left[\phi S'(f) + \frac{\phi^2}{2} S''(f)\right] \mathrm{d}\phi \\ &= \frac{S''(f)}{2} \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \phi^2 \mathcal{H}(\phi)\,\mathrm{d}\phi, \end{aligned}$$

as required (the integral involving $\phi\mathcal{H}(\phi)$ vanishes because this defines an odd function).

Answer to Exercise [212] Recall that, if $\{a_t\} \longleftrightarrow A_p(\cdot)$ and $\{b_t\} \longleftrightarrow B_p(\cdot)$, the two-sequence version of Parseval's theorem says that

$$\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} A_p(u) B_p^*(u)\,\mathrm{d}u = \Delta_{\mathrm{t}} \sum_{t=-\infty}^{\infty} a_t b_t^*$$

(see Equation (99a)). Let $A_p(u) = H(u)$ so that $a_t = h_t$, and let $B_p^*(u) = H(\eta - u)$. Since $\{h_t\} \longleftrightarrow H(\cdot)$ and since $\{h_t\}$ is real-valued, it follows that

$$B_p(u) = H^*(\eta - u) = \Delta_{\mathrm{t}} \sum_{t=0}^{N-1} h_t \mathrm{e}^{\mathrm{i}2\pi(\eta-u)t\Delta_{\mathrm{t}}} = \Delta_{\mathrm{t}} \sum_{t=0}^{N-1} \left(h_t \mathrm{e}^{\mathrm{i}2\pi\eta t\Delta_{\mathrm{t}}}\right) \mathrm{e}^{-\mathrm{i}2\pi u t\Delta_{\mathrm{t}}}.$$

Hence $b_t = h_t \exp(\mathrm{i}2\pi\eta t\Delta_{\mathrm{t}})$, and use of Parseval's theorem leads to

$$\left| \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} H(u) H(\eta - u)\,\mathrm{d}u \right|^2 = \Delta_{\mathrm{t}}^2 \left| \sum_{t=-\infty}^{\infty} h_t^2 \mathrm{e}^{-\mathrm{i}2\pi\eta t\Delta_{\mathrm{t}}} \right|^2,$$

from which the required result follows.

Answer to Exercise [215] Note first that

$$\sum_{j=0}^{J-1} |\theta_j| = \sum_{j=0}^{J-1} |\phi|^j = \frac{1 - |\phi|^J}{1 - |\phi|} \stackrel{\text{def}}{=} S(|\phi|),$$

where we have made use of Equation (17a). For $0 < |\phi| < 1$, we have

$$\frac{\mathrm{d}S(|\phi|)}{\mathrm{d}|\phi|} = \sum_{j=1}^{J-1} j|\phi|^{j-1} = \frac{-J(1 - |\phi|)|\phi|^{J-1} + (1 - |\phi|^J)}{(1 - |\phi|)^2}.$$

Letting $J \to \infty$ yields

$$\sum_{j=1}^{\infty} j|\phi|^{j-1} = \frac{1}{(1 - |\phi|)^2}, \text{ from which we obtain } \sum_{j=1}^{\infty} j|\phi|^j = \frac{|\phi|}{(1 - |\phi|)^2},$$

as required (the case $|\phi| = 0$ holds trivially).

Answer to Exercise [217] The DFT of the reflection-extended series is given by

$$\begin{aligned}\mathcal{X}_k &= \sum_{t=0}^{N-1} X_t e^{-i2\pi kt/(2N)} + \sum_{t=N}^{2N-1} X_{2N-t-1} e^{-i2\pi kt/(2N)}\\ &= \sum_{t=0}^{N-1} X_t e^{-i\pi kt/N} + \sum_{t=N}^{2N-1} X_{2N-t-1} e^{-i\pi kt/N}.\end{aligned}$$

Letting $u = 2N - t - 1$ so that $t = 2N - u - 1$, we can write

$$\sum_{t=N}^{2N-1} X_{2N-t-1} e^{-i\pi kt/N} = \sum_{u=0}^{N-1} X_u e^{-i\pi k(2N-u-1)/N} = \sum_{u=0}^{N-1} X_u e^{i\pi k(u+1)/N}.$$

Hence

$$\begin{aligned}\mathcal{X}_k &= \sum_{t=0}^{N-1} X_t \left(e^{-i\pi kt/N} + e^{i\pi k(t+1)/N}\right)\\ &= \sum_{t=0}^{N-1} X_t \left(e^{-i\pi kt/N} + e^{i\pi kt/N} e^{i\pi k/N}\right)\\ &= e^{i\pi k/(2N)} \sum_{t=0}^{N-1} X_t \left(e^{-i\pi kt/N} e^{-i\pi k/(2N)} + e^{i\pi kt/N} e^{i\pi k/(2N)}\right)\\ &= 2e^{i\pi k/(2N)} \sum_{t=0}^{N-1} X_t \cos\left(\frac{\pi k[2t+1]}{2N}\right).\end{aligned}$$

The periodogram is thus given by

$$\tilde{S}^{(\mathrm{DCT})}(\tilde{f}_k) = \frac{|\mathcal{X}_k|^2}{2N} = \frac{2}{N}\left(\sum_{t=0}^{N-1} X_t \cos\left(\frac{\pi k[2t+1]}{2N}\right)\right)^2,$$

as required. When $k = N$, the cosine term above reduces to $\cos(\pi[2t+1]/2) = 0$ for all t, and hence $\tilde{S}^{(\mathrm{DCT})}(\tilde{f}_N) = \tilde{S}^{(\mathrm{DCT})}(1/2) = 0$.

Answer to Exercise [218] It follows from Exercises [2.1e] and [1.2b] that

$$\begin{aligned}\sigma_k^2 &= \operatorname{cov}\{\mathcal{C}_k, \mathcal{C}_k\}\\ &= \operatorname{cov}\left\{\left(\frac{2-\delta_k}{N}\right)^{\frac{1}{2}} \sum_{t=0}^{N-1} X_t \cos\left(\frac{\pi k[2t+1]}{2N}\right), \left(\frac{2-\delta_k}{N}\right)^{\frac{1}{2}} \sum_{u=0}^{N-1} X_u \cos\left(\frac{\pi k[2u+1]}{2N}\right)\right\}\\ &= \frac{2-\delta_k}{N} \sum_{t=0}^{N-1}\sum_{u=0}^{N-1} s_{t-u} \cos\left(\frac{\pi k[2t+1]}{2N}\right) \cos\left(\frac{\pi k[2u+1]}{2N}\right)\\ &= \frac{2-\delta_k}{N} \sum_{t=0}^{N-1}\sum_{u=0}^{N-1} s_{t-u} \left(\frac{e^{\frac{i\pi k[2t+1]}{2N}} + e^{-\frac{i\pi k[2t+1]}{2N}}}{2}\right)\left(\frac{e^{-\frac{i\pi k[2u+1]}{2N}} + e^{\frac{i\pi k[2u+1]}{2N}}}{2}\right)\\ &= \frac{2-\delta_k}{4N} \sum_{t=0}^{N-1}\sum_{u=0}^{N-1} s_{t-u} \left(e^{\frac{i\pi k[t-u]}{N}} + e^{\frac{i\pi k[t+u+1]}{N}} + e^{-\frac{i\pi k[t+u+1]}{N}} + e^{-\frac{i\pi k[t-u]}{N}}\right)\\ &= \frac{2-\delta_k}{4N}\left(A_k + B_k + B_k^* + A_k^*\right),\end{aligned}$$

where

$$A_k \stackrel{\text{def}}{=} \sum_{t=0}^{N-1}\sum_{u=0}^{N-1} s_{t-u}\mathrm{e}^{\mathrm{i}\pi k[t-u]/N} \text{ and } B_k \stackrel{\text{def}}{=} \sum_{t=0}^{N-1}\sum_{u=0}^{N-1} s_{t-u}\mathrm{e}^{\mathrm{i}\pi k[t+u+1]/N}.$$

Using a change of variables similar to that used in the solution to Exercise [170], we have

$$A_k = \sum_{\tau=-(N-1)}^{N-1} (N-|\tau|)\, s_\tau \mathrm{e}^{\mathrm{i}\pi k\tau/N} = A_k^*$$

for $k = 0, 1, \ldots, N-1$. A similar change of variables yields

$$\begin{aligned} B_k &= s_0 \sum_{l=0}^{N-1} \mathrm{e}^{\mathrm{i}\pi k(2l+1)/N} + 2\sum_{\tau=1}^{N-1} s_\tau \sum_{l=0}^{N-\tau-1} \mathrm{e}^{\mathrm{i}\pi k(2l+\tau+1)/N} \\ &= s_0\mathrm{e}^{\mathrm{i}\pi k/N}\sum_{l=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi kl/N} + 2\sum_{\tau=1}^{N-1} s_\tau \mathrm{e}^{\mathrm{i}\pi k(\tau+1)/N}\sum_{l=0}^{N-\tau-1} \mathrm{e}^{\mathrm{i}2\pi kl/N}. \end{aligned}$$

When $k = 0$ we have

$$B_0 = Ns_0 + 2\sum_{\tau=1}^{N-1}(N-\tau)s_\tau = \sum_{\tau=-(N-1)}^{N-1}(N-|\tau|)s_\tau.$$

Letting $z = \exp(\mathrm{i}2\pi k/N)$ and noting that $z \neq 1$ for $k = 1, \ldots, N-1$, it follows from Equation (17a) that, for $\tau = 0, 1, \ldots, N-1$,

$$\sum_{l=0}^{N-\tau-1} \mathrm{e}^{\mathrm{i}2\pi kl/N} = \sum_{l=0}^{N-\tau-1} z^l = \frac{1-z^{N-\tau}}{1-z} = \frac{1-\mathrm{e}^{\mathrm{i}2\pi k(N-\tau)/N}}{1-\mathrm{e}^{\mathrm{i}2\pi k/N}} = \frac{1-\mathrm{e}^{-\mathrm{i}2\pi\tau k/N}}{1-\mathrm{e}^{\mathrm{i}2\pi k/N}}.$$

Since the above is equal to zero when $\tau = 0$, we have

$$\begin{aligned} B_k = 2\sum_{\tau=1}^{N-1} s_\tau \mathrm{e}^{\mathrm{i}\pi k(\tau+1)/N}\frac{1-\mathrm{e}^{-\mathrm{i}2\pi\tau k/N}}{1-\mathrm{e}^{\mathrm{i}2\pi k/N}} &= 2\sum_{\tau=1}^{N-1} s_\tau \frac{\mathrm{e}^{\mathrm{i}\pi\tau k/N}-\mathrm{e}^{-\mathrm{i}\pi\tau k/N}}{\mathrm{e}^{-\mathrm{i}\pi k/N}-\mathrm{e}^{\mathrm{i}\pi k/N}} \\ &= -\frac{2}{\sin(\pi k/N)}\sum_{\tau=1}^{N-1} s_\tau \sin(\pi\tau k/N) = B_k^*. \end{aligned}$$

Hence

$$\sigma_0^2 = \sum_{\tau=-(N-1)}^{N-1}\left(1-\frac{|\tau|}{N}\right)s_\tau,$$

while, for $k = 1, \ldots, N-1$,

$$\sigma_k^2 = \sum_{\tau=-(N-1)}^{N-1}\left(1-\frac{|\tau|}{N}\right)s_\tau \mathrm{e}^{\mathrm{i}\pi k\tau/N} - \frac{2}{N\sin(\pi k/N)}\sum_{\tau=1}^{N-1} s_\tau \sin(\pi\tau k/N).$$

Comparison of the above with Equation (174a) yields the required result.

Answer to Exercise [247] Using the right-hand side of (188a) to substitute for $\hat{S}^{(\mathrm{D})}(\phi)$ in (247a), we have

$$\begin{aligned}\hat{S}^{(\mathrm{LW})}(f) &= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} V_m(f-\phi)\left(\Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi\phi\tau\,\Delta_{\mathrm{t}}}\right) \mathrm{d}\phi \\ &= \Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \left(\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} V_m(f-\phi)\mathrm{e}^{\mathrm{i}2\pi(f-\phi)\tau\,\Delta_{\mathrm{t}}}\,\mathrm{d}\phi\right) \hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_{\mathrm{t}}} \\ &= \Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} v_{m,\tau}\hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_{\mathrm{t}}}\end{aligned}$$

as required, where

$$v_{m,\tau} \stackrel{\mathrm{def}}{=} \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} V_m(f-\phi)\mathrm{e}^{\mathrm{i}2\pi(f-\phi)\tau\,\Delta_{\mathrm{t}}}\,\mathrm{d}\phi = \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} V_m(\phi)\mathrm{e}^{\mathrm{i}2\pi\phi\tau\,\Delta_{\mathrm{t}}}\,\mathrm{d}\phi$$

(the second integral follows from the first since $V_m(\cdot)$ is symmetric and $2f_{\mathcal{N}}$ periodic).

Answer to Exercise [249a] Since $\{\hat{s}_\tau^{(\mathrm{D})}\} \longleftrightarrow \hat{S}^{(\mathrm{D})}(\cdot)$ (see Equation (188a)), and since $f'_{k-j} = f'_k - f'_j$, we can use Equation (246b) to see that

$$\begin{aligned}\hat{S}^{(\mathrm{DS})}(f'_k) &= \sum_{j=-M}^{M} g_j \left(\Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi f'_{k-j}\tau\,\Delta_{\mathrm{t}}}\right) \\ &= \Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \left(\sum_{j=-M}^{M} g_j \mathrm{e}^{\mathrm{i}2\pi f'_j\tau\,\Delta_{\mathrm{t}}}\right) \hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi f'_k\tau\,\Delta_{\mathrm{t}}} \\ &= \Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} v_{g,\tau}\hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi f'_k\tau\,\Delta_{\mathrm{t}}},\end{aligned}$$

as required.

Answer to Exercise [249b] Since $\{w_{m,\tau}\} \longleftrightarrow W_m(\cdot)$, Equations (100g) and (101b) in conjunction with Equation (247e) yield

$$\{w_{m,\tau} : \tau = -(N-1), \ldots, N-1\} \longleftrightarrow \{W_m(f'_j) : j = -(N-1), \ldots, N-1\}$$

(cf. the argument leading to Equation (171e)). The inverse DFT (Equation (101a)) gives

$$w_{m,\tau} = \frac{1}{(2N-1)\,\Delta_{\mathrm{t}}} \sum_{j=-(N-1)}^{N-1} W_m(f'_j)\mathrm{e}^{\mathrm{i}2\pi f'_j\tau\,\Delta_{\mathrm{t}}}, \quad \tau = -(N-1), \ldots, N-1.$$

Hence we can write

$$\begin{aligned}
\hat{S}^{(\mathrm{LW})}(f'_k) &= \Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} w_{m,\tau}\, \hat{s}^{(\mathrm{D})}_{\tau} \mathrm{e}^{-\mathrm{i}2\pi f'_k \tau\,\Delta_{\mathrm{t}}} \\
&= \Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \left(\frac{1}{(2N-1)\,\Delta_{\mathrm{t}}} \sum_{j=-(N-1)}^{N-1} W_m(f'_j) \mathrm{e}^{\mathrm{i}2\pi f'_j \tau\,\Delta_{\mathrm{t}}} \right) \hat{s}^{(\mathrm{D})}_{\tau} \mathrm{e}^{-\mathrm{i}2\pi f'_k \tau\,\Delta_{\mathrm{t}}} \\
&= \frac{1}{(2N-1)\,\Delta_{\mathrm{t}}} \sum_{j=-(N-1)}^{N-1} W_m(f'_j) \left(\Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} \hat{s}^{(\mathrm{D})}_{\tau} \mathrm{e}^{-\mathrm{i}2\pi (f'_k - f'_j)\tau\,\Delta_{\mathrm{t}}} \right) \\
&= \frac{1}{(2N-1)\,\Delta_{\mathrm{t}}} \sum_{j=-(N-1)}^{N-1} W_m(f'_j) \hat{S}^{(\mathrm{D})}(f'_{k-j}) \ = \sum_{j=-(N-1)}^{N-1} g_j \hat{S}^{(\mathrm{D})}(f'_{k-j}),
\end{aligned}$$

where g_j is defined as in Equation (249d).

Answer to Exercise [255] Using Equation (186e) to substitute for $E\{\hat{S}^{(\mathrm{D})}(\phi)\}$ in Equation (255a) yields (with f'' in the third line below defined to be $\phi - f'$),

$$\begin{aligned}
E\{\hat{S}^{(\mathrm{LW})}(f)\} &= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} W_m(f-\phi) \left(\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{H}(\phi - f') S(f')\,\mathrm{d}f' \right) \mathrm{d}\phi \\
&= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \left(\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} W_m(f-\phi) \mathcal{H}(\phi - f')\,\mathrm{d}\phi \right) S(f')\,\mathrm{d}f' \\
&= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \left(\int_{-f_{\mathcal{N}}-f'}^{f_{\mathcal{N}}-f'} W_m(f-f'-f'') \mathcal{H}(f'')\,\mathrm{d}f'' \right) S(f')\,\mathrm{d}f' \\
&= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \left(\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} W_m(f-f'-f'') \mathcal{H}(f'')\,\mathrm{d}f'' \right) S(f')\,\mathrm{d}f' \\
&= \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{U}_m(f-f') S(f')\,\mathrm{d}f',
\end{aligned}$$

where $\mathcal{U}_m(\cdot)$ is defined in Equation (255c). The above uses the fact that $W_m(\cdot)$ and $\mathcal{H}(\cdot)$ are both periodic functions with period $2f_{\mathcal{N}}$.

Answer to Exercise [256] Equation (255d) says that

$$\{w_{m,\tau}\, h \star h_{\tau}\} \longleftrightarrow \mathcal{U}_m(\cdot)\,\Delta_{\mathrm{t}},$$

and hence

$$w_{m,\tau}\, h \star h_{\tau} = \Delta_{\mathrm{t}} \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{U}_m(f) \mathrm{e}^{\mathrm{i}2\pi f \tau\,\Delta_{\mathrm{t}}}\,\mathrm{d}f.$$

Setting $\tau = 0$ in the above yields

$$\int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{U}_m(f)\,\mathrm{d}f = 1$$

since $w_{m,0} = 1$ and $h \star h_0 = \Delta_{\mathrm{t}}$. An application of Parseval's theorem (Equation (75b)) yields

$$\Delta_{\mathrm{t}}^2 \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} \mathcal{U}_m^2(f)\,\mathrm{d}f = \Delta_{\mathrm{t}} \sum_{\tau=-(N-1)}^{N-1} w_{m,\tau}^2 (h \star h_{\tau})^2,$$

from which the required result follows.

Answer to Exercise [257] It follows from Equations (256a) and (99f) that

$$B_{\mathcal{U}} = \frac{1}{\Delta_{\rm t} \sum_{\tau=-(N-1)}^{N-1} w_{m,\tau}^2 \left(\sum_{t=0}^{N-1} h_{t+\tau} h_t \right)^2}.$$

Equation (251e) states that

$$B_W = \frac{1}{\Delta_{\rm t} \sum_{\tau=-(N-1)}^{N-1} w_{m,\tau}^2},$$

so $B_{\mathcal{U}} \geq B_W$ holds if we can show the following to be true:

$$\sum_{\tau=-(N-1)}^{N-1} w_{m,\tau}^2 \left(\sum_{t=0}^{N-1} h_{t+\tau} h_t \right)^2 \leq \sum_{\tau=-(N-1)}^{N-1} w_{m,\tau}^2.$$

Since $w_{m,\tau}^2$ is nonnegative, the above holds if the multiplier of $w_{m,\tau}^2$ on the left-hand side is always less than or equal to unity. Recalling that $h_t \stackrel{\rm def}{=} 0$ for $t < 0$ or $t \geq N$ and that $\sum_t h_t^2 = 1$, the Cauchy inequality of Equation (257a) with $a_t = h_{t+\tau}$ and $b_t = h_t$ tells us that

$$\left(\sum_{t=0}^{N-1} h_{t+\tau} h_t \right)^2 \leq \sum_{t=0}^{N-1} h_{t+\tau}^2 \sum_{t=0}^{N-1} h_t^2 \leq \left(\sum_{t=0}^{N-1} h_t^2 \right)^2 = 1,$$

as required.

Answer to Exercise [263] Following the hint, Equations (213b) and (213a) say, upon setting $\Delta_{\rm t} = 1$, that

$$R(0) = \sum_{\tau=-(N-1)}^{N-1} r_\tau = \sum_{\tau=-(N-1)}^{N-1} \sum_{t=0}^{N-|\tau|-1} h_{t+|\tau|}^2 h_t^2;$$

however, Equation (212a) gives

$$R(0) = \left| \sum_{t=0}^{N-1} h_t^2 \right|^2 = 1,$$

as required.

Alternatively, consider an $N \times N$ matrix whose (j,k)th element is $h_j^2 h_k^2$. If we sum its elements across rows and down columns, we get

$$\sum_{j=0}^{N-1} \sum_{k=0}^{N-1} h_j^2 h_k^2 = \left(\sum_{j=0}^{N-1} h_j^2 \right) \left(\sum_{k=0}^{N-1} h_k^2 \right) = 1;$$

on the other hand, if we sum its elements along diagonals, we get the desired result.

Answer to Exercise [277] Without loss of generality, set $\Delta_t = 1$. It follows from Equation (248a) that

$$\hat{S}^{(\mathrm{LW})}(f) = \sum_{\tau=-(N-1)}^{N-1} \mathrm{e}^{-\tau^2/m^2} \hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi f\tau},$$

where $\{\hat{s}_\tau^{(\mathrm{D})}\} \longleftrightarrow \hat{S}^{(\mathrm{D})}(\cdot)$. Since $\hat{s}_\tau^{(\mathrm{D})} = 0$ for all $|\tau| \geq N$, we have

$$\hat{S}^{(\mathrm{LW})}(f) = \sum_{\tau=-\infty}^{\infty} \mathrm{e}^{-\tau^2/m^2} \hat{s}_\tau^{(\mathrm{D})} \mathrm{e}^{-\mathrm{i}2\pi f\tau} = \int_{-1/2}^{1/2} V_m(f-\phi)\hat{S}^{(\mathrm{D})}(\phi)\,\mathrm{d}\phi,$$

where $V_m(\cdot)$ is the Fourier transform of the infinite sequence $\{\exp(-\tau^2/m^2), \tau \in \mathbb{Z}\}$. Since necessarily $\hat{S}^{(\mathrm{D})}(f) \geq 0$ for all f, it follows that $\hat{S}^{(\mathrm{LW})}(f) \geq 0$ also if we can show that $V_m(f) \geq 0$ for all f. Equations (55b) and (55c) imply that the Fourier transform of $g(t) = \exp(-t^2/m^2)$, $t \in \mathbb{R}$, is $G(f) = (m^2\pi)^{1/2} \exp(-m^2\pi^2 f^2)$, $f \in \mathbb{R}$. Regarding the infinite sequence $\{\exp(-\tau^2/m^2)\}$ as arising from sampling $g(\cdot)$ at $t \in \mathbb{Z}$, Equation (82a) says that $V_m(\cdot)$ can be expressed in terms of $G(\cdot)$ as

$$V_m(f) = \sum_{k=-\infty}^{\infty} G(f+k) = (m^2\pi)^{1/2} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-m^2\pi^2(f+k)^2} > 0 \text{ for all } |f| \leq 1/2.$$

The fact that $V_m(\cdot)$ is periodic with a period of unity says that $V_m(f) > 0$ for all f, thus establishing the desired result.

Answer to Exercise [298] An application of Exercise [2.1e] says that

$$\operatorname{var}\{\hat{s}_0^{(\mathrm{P})}\} = \operatorname{cov}\left\{\sum_{t=0}^{N-1} \frac{1}{N} X_t^2, \sum_{u=0}^{N-1} \frac{1}{N} X_u^2\right\} = \frac{1}{N^2} \sum_{t=0}^{N-1}\sum_{u=0}^{N-1} \operatorname{cov}\{X_t^2, X_u^2\}.$$

By letting $Z_0 = Z_1 = X_t$ and $Z_2 = Z_3 = X_u$ in Equation (30), we have

$$\operatorname{cov}\{X_t^2, X_u^2\} = 2\left(\operatorname{cov}\{X_t, X_u\}\right)^2 = 2s_{t-u}^2.$$

Hence

$$\operatorname{var}\{\hat{s}_0^{(\mathrm{P})}\} = \frac{2}{N^2} \sum_{t=0}^{N-1}\sum_{u=0}^{N-1} s_{t-u}^2 = \frac{2}{N} \sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau^2,$$

where the last equality follows from the same argument used in the solution to Exercise [165].

Answer to Exercise [300] For clarity, let $\hat{B}_{T,\delta_{\mathrm{BP}}}$ denote the right-hand side of Equation (300b). When $\delta_{\mathrm{BP}} = 1$, the right-hand side of Equation (300c) is equal to $5\hat{B}_{T,1}/3 - 1/(N\,\Delta_t)$, and hence the result from Walden and White (1990) says that $E\{\tilde{B}_T\} \approx B_T$. When $\delta_{\mathrm{BP}} = 0$, we have

$$\tilde{B}_T = \frac{5\hat{B}_{T,0}}{6} - \frac{1}{N\,\Delta_t} = \frac{5\hat{B}_{T,1}}{3} - \frac{1}{N\,\Delta_t}$$

since $\hat{B}_{T,0} = 2\hat{B}_{T,1}$, and hence $E\{\tilde{B}_T\} \approx B_T$ again.

Answer to Exercise [303] Using the approximations for $E\{\hat{c}'^{(\mathrm{D})}_\tau\}$ and $E\{(\hat{c}'^{(\mathrm{D})}_\tau)^2\}$ stated in Equation (303d) and recalling that $w_{m,0} = 1$ always, we have

$$\begin{aligned}
I_m &= \Delta_\mathrm{t} \sum_{\tau=-(N_L-1)}^{N_U-1} \left[w^2_{m,\tau} E\{(\hat{c}'^{(\mathrm{D})}_\tau)^2\} - 2 w_{m,\tau} c_\tau E\{\hat{c}'^{(\mathrm{D})}_\tau\} + c^2_\tau \right] \\
&\approx \Delta_\mathrm{t} \sum_\tau \left[w^2_{m,\tau} \left(c^2_\tau - 2 c_\tau \frac{\lambda\gamma\delta_\tau}{\Delta_\mathrm{t}} + \frac{\lambda^2\gamma^2\delta^2_\tau}{\Delta^2_\mathrm{t}} + \frac{\lambda^2\pi^2\bigl(1 + \delta_\tau + \delta_{\tau-\frac{N'}{2}}\bigr)}{6N'\,\Delta^2_\mathrm{t}} \right) \right. \\
&\qquad\qquad \left. - 2 w_{m,\tau} c_\tau \left(c_\tau - \frac{\lambda\gamma\delta_\tau}{\Delta_\mathrm{t}} \right) + c^2_\tau \right] \\
&= \frac{\lambda^2\gamma^2}{\Delta_\mathrm{t}} + \Delta_\mathrm{t} \sum_\tau \left[w^2_{m,\tau} \left(c^2_\tau + \frac{\lambda^2\pi^2\bigl(1 + \delta_\tau + \delta_{\tau-\frac{N'}{2}}\bigr)}{6N'\,\Delta^2_\mathrm{t}} \right) - 2 w_{m,\tau} c^2_\tau + c^2_\tau \right] \\
&= \frac{\lambda^2\gamma^2}{\Delta_\mathrm{t}} + \Delta_\mathrm{t} \sum_\tau c^2_\tau (1 - w_{m,\tau})^2 + \frac{\lambda^2\pi^2}{6N'\,\Delta_\mathrm{t}} \sum_\tau w^2_{m,\tau} \bigl(1 + \delta_\tau + \delta_{\tau-\frac{N'}{2}}\bigr),
\end{aligned}$$

as required.

Answer to Exercise [307] Replacing $\hat{S}^{(\mathrm{D})}(\cdot)$ with $\hat{S}^{(\mathrm{P})}(\cdot)$, g_j with $g_{m,j}$, N' with N and M with $N - \lfloor N/2 \rfloor - 1$ in Equation (246b) yields

$$\hat{S}^{(\mathrm{DS})}(f_k) = \sum_{j=-N+\lfloor N/2 \rfloor+1}^{N-\lfloor N/2 \rfloor-1} g_{m,j} \hat{S}^{(\mathrm{P})}(f_{k-j}).$$

When N is odd, the lower limit becomes $j = -N + \frac{N-1}{2} + 1 = -\lfloor N/2 \rfloor$, so the right-hand side of the above is in agreement with the right-hand side of Equation (307), as required; on the other hand, when N is even, we have $j = -N + \frac{N}{2} + 1 = -\lfloor N/2 \rfloor + 1$, so we again have the required agreement since the assumption that $g_{m,-N/2} = 0$ allows us to adjust the lower limit of right-hand side of Equation (307) to be $j = -\lfloor N/2 \rfloor + 1$.

Answer to Exercise [326] Take Y'_t to be $X_t - X_{t-1}$ (the time differences of the atomic clock), substitute Y' for Y in Equations (143d) and (143b), and, in Equation (143e), set $g_0 = 1$, $g_1 = -1$ and $g_u = 0$ otherwise. These three equations then tell us that

$$\begin{aligned}
S_{Y'}(f) = \bigl|1 - \mathrm{e}^{-\mathrm{i}2\pi f\,\Delta_\mathrm{t}}\bigr|^2 S_X(f) &= \bigl|\mathrm{e}^{-\mathrm{i}\pi f\,\Delta_\mathrm{t}} \left(\mathrm{e}^{\mathrm{i}\pi f\,\Delta_\mathrm{t}} - \mathrm{e}^{-\mathrm{i}\pi f\,\Delta_\mathrm{t}}\right)\bigr|^2 S_X(f) \\
&= 4 \sin^2(\pi f\,\Delta_\mathrm{t})\, S_X(f),
\end{aligned}$$

where we have made use of $|\mathrm{e}^{-\mathrm{i}x}|^2 = 1$ and $(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x})/(2\mathrm{i}) = \sin(x)$ (Equation (17b)). The integral of any SDF is equal to the process variance (Equation (113e)), which implies that, for any real-valued constant C, the process $\{CY'_t\}$ has $C^2 S_{Y'}(\cdot)$ as its SDF since $\mathrm{var}\,\{CY'_t\} = C^2\,\mathrm{var}\,\{Y'_t\}$. If we let $C = 1/\Delta_\mathrm{t}$, then $CY'_t = Y_t$ (the fractional frequency deviates), which gives the desired result.

Answer to Exercise [375a] Using the definition for $\hat{s}^{(\mathrm{D})}_\tau$, we have

$$\begin{aligned}
\sum_{\tau=-(N-1)}^{N-1} w_{m,\tau} \hat{s}^{(\mathrm{D})}_\tau \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_\mathrm{t}} &= \sum_{\tau=-(N-1)}^{N-1} w_{m,\tau} \sum_{t=0}^{N-|\tau|-1} h_{t+|\tau|} X_{t+|\tau|} h_t X_t \mathrm{e}^{-\mathrm{i}2\pi f\tau\,\Delta_\mathrm{t}} \\
&= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} w_{m,j-k} h_j X_j h_k X_k \mathrm{e}^{-\mathrm{i}2\pi f(j-k)\,\Delta_\mathrm{t}}
\end{aligned}$$

after a change of variables in the double summation, which we can justify by considering an $N \times N$ matrix whose (j,k)th element is $w_{j-k,\tau} h_j X_j h_k X_k \exp\left[-\mathrm{i}2\pi f(j-k)\,\Delta_{\mathrm{t}}\right]$ and by applying the same argument used to justify the change in variables in going from the double summation in Equation (A–12a) to the one in Equation (A–12b).

Answer to Exercise [375b] We have

$$\boldsymbol{Q} = \boldsymbol{H}_N \boldsymbol{D}_N \boldsymbol{H}_N^T = \left[\,\boldsymbol{h}_0 \quad \boldsymbol{h}_1 \quad \dots \quad \boldsymbol{h}_{N-1}\,\right] \begin{bmatrix} d_0 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{N-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_0^T \\ \boldsymbol{h}_1^T \\ \vdots \\ \boldsymbol{h}_{N-1}^T \end{bmatrix}$$
$$= \left[d_0\,\boldsymbol{h}_0 \quad d_1 \boldsymbol{h}_1 \quad \dots \quad d_{N-1}\boldsymbol{h}_{N-1}\,\right] \begin{bmatrix} \boldsymbol{h}_0^T \\ \boldsymbol{h}_1^T \\ \vdots \\ \boldsymbol{h}_{N-1}^T \end{bmatrix} = \sum_{k=0}^{K-1} d_k \boldsymbol{h}_k \boldsymbol{h}_k^T$$

since $d_k = 0$ for all $k \geq K$; on the other hand, merely replacing all instances of $\boldsymbol{h}_{N-1}$ and d_{N-1} in the above with $\boldsymbol{h}_{K-1}$ and d_{K-1} shows that we also have

$$\boldsymbol{H}\boldsymbol{D}_K \boldsymbol{H}^T = \sum_{k=0}^{K-1} d_k \boldsymbol{h}_k \boldsymbol{h}_k^T, \tag{A–23}$$

and hence $\boldsymbol{Q} = \boldsymbol{H}\boldsymbol{D}_K \boldsymbol{H}^T$.

Answer to Exercise [376] Making use of Equation (A–23), we have

$$\hat{S}^{(\mathrm{Q})}(f) = \Delta_{\mathrm{t}}\, \boldsymbol{Z}^H \boldsymbol{H}\boldsymbol{D}_K \boldsymbol{H}^T \boldsymbol{Z} = \Delta_{\mathrm{t}}\, \boldsymbol{Z}^H \left(\sum_{k=0}^{K-1} d_k \boldsymbol{h}_k \boldsymbol{h}_k^T\right) \boldsymbol{Z} = \Delta_{\mathrm{t}} \sum_{k=0}^{K-1} d_k \boldsymbol{Z}^H \boldsymbol{h}_k \boldsymbol{h}_k^T \boldsymbol{Z}$$
$$= \Delta_{\mathrm{t}} \sum_{k=0}^{K-1} d_k (\boldsymbol{h}_k^T \boldsymbol{Z})^* \boldsymbol{h}_k^T \boldsymbol{Z} = \Delta_{\mathrm{t}} \sum_{k=0}^{K-1} d_k \left|\boldsymbol{h}_k^T \boldsymbol{Z}^*\right|^2 = \Delta_{\mathrm{t}} \sum_{k=0}^{K-1} d_k \left|\sum_{t=0}^{N-1} h_{k,t} Z_t^*\right|^2$$
$$= \Delta_{\mathrm{t}} \sum_{k=0}^{K-1} d_k \left|\sum_{t=0}^{N-1} h_{k,t} X_t \mathrm{e}^{-\mathrm{i}2\pi f t\,\Delta_{\mathrm{t}}}\right|^2,$$

as required.

Answer to Exercise [380] Since $K = 1$, $d_0 = 1$, $\boldsymbol{Q} = \boldsymbol{H}\boldsymbol{D}_N \boldsymbol{H}^T$ in general and $\boldsymbol{Q} = \boldsymbol{h}_0 \boldsymbol{h}_0^T$ in this specific case, we have

$$\beta^{(\mathrm{B})}\{\hat{S}^{(\mathrm{Q})}(\cdot)\} = \sum_{k=0}^{K-1} d_k - \operatorname{tr}\{\boldsymbol{D}_N \boldsymbol{H}^T \boldsymbol{\Sigma}^{(\mathrm{BL})} \boldsymbol{H}\} = 1 - \operatorname{tr}\{\boldsymbol{H}\boldsymbol{D}_N \boldsymbol{H}^T \boldsymbol{\Sigma}^{(\mathrm{BL})}\}$$
$$= 1 - \operatorname{tr}\{\boldsymbol{Q}\boldsymbol{\Sigma}^{(\mathrm{BL})}\} = 1 - \operatorname{tr}\{\boldsymbol{h}_0 \boldsymbol{h}_0^T \boldsymbol{\Sigma}^{(\mathrm{BL})}\} = 1 - \operatorname{tr}\{\boldsymbol{h}_0^T \boldsymbol{\Sigma}^{(\mathrm{BL})} \boldsymbol{h}_0\}.$$

Because $\boldsymbol{h}_0$ is an eigenvector of $\boldsymbol{\Sigma}^{(\mathrm{BL})}$ corresponding to the eigenvalue $\lambda_0(N,W)$, we have $\boldsymbol{\Sigma}^{(\mathrm{BL})}\boldsymbol{h}_0 = \lambda_0(N,W)\boldsymbol{h}_0$. Hence

$$\beta^{(\mathrm{B})}\{\hat{S}^{(\mathrm{Q})}(\cdot)\} = 1 - \lambda_0(N,W)\operatorname{tr}\{\boldsymbol{h}_0^T \boldsymbol{h}_0\} = 1 - \lambda_0(N,W)$$

because $\boldsymbol{h}_0^T \boldsymbol{h}_0 = \sum_{t=0}^{N-1} h_{0,t}^2 = 1$, and the trace of a scalar is just the scalar itself.

Answer to Exercise [392] We can regard the single-taper bias measure $\beta^2_{\mathcal{H}}$ of Equation (391a) as a special case of the measure $\beta^2_{\widetilde{\mathcal{H}}}$ of Equation (391c) with K set to unity. When $K = 1$, we have $\boldsymbol{D}_K = 1$ and $\boldsymbol{H} = \boldsymbol{h}_k$, so Equation (391c) says that

$$\beta^2_{\mathcal{H}_k} = \frac{12}{\Delta_{\rm t}^2}\,\mathrm{tr}\,\{\boldsymbol{h}_k^T\boldsymbol{\Sigma}^{(\mathrm{PL})}\boldsymbol{h}_k\} = \frac{12}{\Delta_{\rm t}^2}\boldsymbol{h}_k^T\boldsymbol{\Sigma}^{(\mathrm{PL})}\boldsymbol{h}_k$$

since the trace of a scalar is just the scalar itself. The desired result follows since

$$\boldsymbol{h}_k^T\boldsymbol{\Sigma}^{(\mathrm{PL})}\boldsymbol{h}_k = \tilde{\lambda}_k\boldsymbol{h}_k^T\boldsymbol{h}_k = \tilde{\lambda}_k,$$

where we have used the fact that $\boldsymbol{\Sigma}^{(\mathrm{PL})}\boldsymbol{h}_k = \tilde{\lambda}_k\boldsymbol{h}_k$ because $\boldsymbol{h}_k$ is the eigenvector of $\boldsymbol{\Sigma}^{(\mathrm{PL})}$ associated with eigenvalue $\tilde{\lambda}_k$.

Answer to Exercise [465] The orthogonality principle of Equation (454a) states that the prediction error $\overrightarrow{\epsilon}_k(k)$ is uncorrelated with all RVs utilized in the prediction of X_k, which, because $j < k$, includes $X_0, \ldots, X_j$. Thus, an appeal to Exercise [2.1e] says that

$$\begin{aligned}\mathrm{cov}\,\{\overrightarrow{\epsilon}_j(j), \overrightarrow{\epsilon}_k(k)\} &= \mathrm{cov}\left\{X_j - \sum_{l=1}^{j}\phi_{j,l}X_{j-l}, \overrightarrow{\epsilon}_k(k)\right\}\\ &= \mathrm{cov}\,\{X_j, \overrightarrow{\epsilon}_k(k)\} - \sum_{l=1}^{j}\phi_{j,l}\,\mathrm{cov}\,\{X_{j-l}, \overrightarrow{\epsilon}_k(k)\} = 0,\end{aligned}$$

as claimed.

Answer to Exercise [524] It follows from Equations (523b), (523a) and (516a) along with the facts $E\{\hat{A}_l\} = A_l$ and $E\{\hat{B}_l\} = B_l$ that

$$\begin{aligned}E\{\hat{S}^{(\mathrm{P})}(f_l)\} &= \frac{N\,\Delta_{\rm t}}{4}E\{\hat{D}_l^2\} = \frac{N\,\Delta_{\rm t}}{4}\left(E\{\hat{A}_l^2\} + E\{\hat{B}_l^2\}\right)\\ &= \frac{N\,\Delta_{\rm t}}{4}\left(\mathrm{var}\,\{\hat{A}_l\} + \left[E\{\hat{A}_l\}\right]^2 + \mathrm{var}\,\{\hat{B}_l\} + \left[E\{\hat{B}_l\}\right]^2\right)\\ &= \frac{N\,\Delta_{\rm t}}{4}\left(\frac{2\sigma_\epsilon^2}{N} + A_l^2 + \frac{2\sigma_\epsilon^2}{N} + B_l^2\right) = \frac{N\,\Delta_{\rm t}}{4}D_l^2 + \sigma_\epsilon^2\,\Delta_{\rm t},\end{aligned}$$

as claimed.

Answer to Exercise [553] Letting $\omega = 2\pi f\,\Delta_{\rm t}$ for convenience and making use of Equation (17b), we can write

$$x_t = D\cos(\omega t + \phi) = 2D\left(\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}\phi}\right).$$

Hence

$$x_t + x_{t-2} = 2D\left(\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}\phi} + \mathrm{e}^{\mathrm{i}\omega(t-2)}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\omega(t-2)}\mathrm{e}^{-\mathrm{i}\phi}\right).$$

On the other hand,

$$\begin{aligned}2\cos(\omega)x_{t-1} &= \left(\mathrm{e}^{\mathrm{i}\omega} + \mathrm{e}^{-\mathrm{i}\omega}\right)2D\left(\mathrm{e}^{\mathrm{i}\omega(t-1)}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\omega(t-1)}\mathrm{e}^{-\mathrm{i}\phi}\right)\\ &= 2D\left(\mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\omega(t-2)}\mathrm{e}^{-\mathrm{i}\phi} + \mathrm{e}^{\mathrm{i}\omega(t-2)}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}\phi}\right) = x_t + x_{t-2},\end{aligned}$$

from which Equation (553a) follows immediately.

Answer to Exercise [562] Letting $\omega = 2\pi f\,\Delta_{\mathrm{t}}$ for convenience, we have

$$\frac{1}{N}\sum_{t=0}^{N-\tau-1} Z_{t+\tau}Z_t^* = \frac{1}{N}\sum_{t=0}^{N-\tau-1} \mathrm{e}^{\mathrm{i}(\omega[t+\tau]+\phi)}\mathrm{e}^{-\mathrm{i}(\omega t+\phi)} = \frac{1}{N}\sum_{t=0}^{N-\tau-1}\mathrm{e}^{\mathrm{i}\omega\tau} = \frac{N-\tau}{N}\mathrm{e}^{\mathrm{i}\omega\tau},$$

as claimed.

Answer to Exercise [597] Using Equations (597b) and (596a) with t set to 2 yields

$$\phi_{1,1} = \frac{\phi_{2,1}+\phi_{2,2}\phi_{2,1}}{1-\phi_{2,2}^2} = \frac{\frac{3}{4}-\frac{1}{2}\cdot\frac{3}{4}}{1-\frac{1}{4}} = \tfrac{1}{2} \text{ and } \sigma_1^2 = \frac{\sigma_2^2}{1-\phi_{2,2}^2} = \frac{1}{1-\frac{1}{4}} = \tfrac{4}{3}.$$

A second use of Equation (596a) with t now set to 1 yields

$$\sigma_0^2 = \frac{\sigma_1^2}{1-\phi_{1,1}^2} = \frac{\frac{4}{3}}{1-\frac{1}{4}} = \tfrac{16}{9}.$$

Given realizations of $Z_0, Z_1, \ldots, Z_{N-1}$, we can generate realizations of the prediction errors $\overrightarrow{\epsilon}_0(0)$, $\overrightarrow{\epsilon}_1(1)$, $\epsilon_2, \ldots, \epsilon_{N-1}$ based on $\sigma_0 Z_0, \sigma_1 Z_1, \sigma_2 Z_2, \ldots, \sigma_2 Z_{N-1}$. Making the following substitutions into Equation (597a) with p set to 2 yields the procedure stated in the exercise: $\overrightarrow{\epsilon}_0(0) = \sigma_0 Z_0 = \frac{4}{3}Z_0$, $\phi_{1,1} = \frac{1}{2}$, $\overrightarrow{\epsilon}_1(1) = \sigma_1 Z_1 = \frac{2}{\sqrt{3}}Z_1$, $\phi_{2,1} = \frac{3}{4}$, $\phi_{2,2} = -\frac{1}{2}$ and $\epsilon_t = \sigma_2 Z_t = Z_t$ for $t = 2, \ldots, N-1$.

Answer to Exercise [598] We have

$$\begin{aligned} X_t &= \sum_{j=0}^{q}\vartheta_{q,j}\left(\sum_{k=1}^{p}\phi_{p,k}Y_{t-j-k} + \epsilon_{t-j}\right) \\ &= \sum_{k=1}^{p}\phi_{p,k}\left(\sum_{j=0}^{q}\vartheta_{q,j}Y_{t-j-k}\right) + \sum_{j=0}^{q}\vartheta_{q,j}\epsilon_{t-j} = \sum_{k=1}^{p}\phi_{p,k}X_{t-k} + \sum_{j=0}^{q}\vartheta_{q,j}\epsilon_{t-j}, \end{aligned}$$

as claimed.

Answer to Exercise [601] Taking (601b) to be a contiguous segment of length $2N$ from an infinite periodic sequence with a period of $2N$, and recalling that $s_{-\tau} = s_\tau$, Equation (100g) tells us that we can write

$$\begin{aligned} S_k &= s_N\mathrm{e}^{-\mathrm{i}2\pi\tilde{f}_k N} + \sum_{\tau=-(N-1)}^{N-1} s_\tau \mathrm{e}^{-\mathrm{i}2\pi\tilde{f}_k\tau} \qquad \text{(A–25)} \\ &= s_N\mathrm{e}^{-\mathrm{i}\pi k} + s_0 + \sum_{\tau=1}^{N-1} s_\tau\left(\mathrm{e}^{-\mathrm{i}\pi k\tau/N} + \mathrm{e}^{\mathrm{i}\pi k\tau/N}\right) \\ &= s_N(-1)^k + s_0 + 2\sum_{\tau=1}^{N-1} s_\tau\cos(\pi k\tau/N) \end{aligned}$$

since $\mathrm{e}^{-\mathrm{i}x} + \mathrm{e}^{\mathrm{i}\pi x} = 2\cos(x)$ (see Equation (17b)) – hence S_k is real-valued, as claimed.

Answer to Exercise [602] For convenience, let $\tilde{\omega}_k = 2\pi \tilde{f}_k$, and consider

$$\begin{aligned} Y_{\Re,t} &= \sum_{k=0}^{2N-1} \Re\left\{\mathcal{Y}_k \mathrm{e}^{-\mathrm{i}\tilde{\omega}_k t}\right\} = \sum_{k=0}^{2N-1} \left(\frac{S_k}{2N}\right)^{1/2} \Re\left\{[Z_{2k} + \mathrm{i}Z_{2k+1}][\cos(\tilde{\omega}_k t) - \mathrm{i}\sin(\tilde{\omega}_k t)]\right\} \\ &= \sum_{k=0}^{2N-1} \left(\frac{S_k}{2N}\right)^{1/2} [Z_{2k}\cos(\tilde{\omega}_k t) + Z_{2k+1}\sin(\tilde{\omega}_k t)], \end{aligned}$$

which is a special case of the harmonic process of Equation (35c). Hence $\{Y_{\Re,t}\}$ is a zero mean Gaussian stationary process with an ACVS dictated by Equation (36a), namely,

$$s_{\Re,\tau} \stackrel{\text{def}}{=} \frac{1}{2N} \sum_{k=0}^{2N-1} S_k \cos(\tilde{\omega}_k \tau). \tag{A–26a}$$

Similarly, we have

$$Y_{\Im,t} = \sum_{k=0}^{2N-1} \left(\frac{S_k}{2N}\right)^{1/2} [Z_{2k+1}\cos(\tilde{\omega}_k t) - Z_{2k}\sin(\tilde{\omega}_k t)],$$

which has properties similar to $\{Y_{\Re,t}\}$ including an ACVS $\{s_{\Im,\tau}\}$ identical to $\{s_{\Re,\tau}\}$. Now

$$\operatorname{cov}\{Y_{\Re,t+\tau}, Y_{\Im,t}\} = E\{Y_{\Re,t+\tau} Y_{\Im,t}\} = \frac{1}{2N} \sum_{k=0}^{2N-1} \sum_{l=0}^{2N-1} \sqrt{S_k S_l} E\{A_{k,l}\},$$

where

$$A_{k,l} \stackrel{\text{def}}{=} [Z_{2k}\cos(\tilde{\omega}_k[t+\tau]) + Z_{2k+1}\sin(\tilde{\omega}_k[t+\tau])]\,[Z_{2l+1}\cos(\tilde{\omega}_l t) - Z_{2l}\sin(\tilde{\omega}_l t)].$$

Since $E\{Z_m Z_n\} = 0$ for all $m \neq n$, it follows that $E\{A_{k,l}\} = 0$ when $l \neq k$, and hence

$$\operatorname{cov}\{Y_{\Re,t+\tau}, Y_{\Im,t}\} = \frac{1}{2N} \sum_{k=0}^{2N-1} S_k E\{A_{k,k}\},$$

Now

$$\begin{aligned} E\{A_{k,k}\} &= E\Big\{ [Z_{2k}\cos(\tilde{\omega}_k[t+\tau]) + Z_{2k+1}\sin(\tilde{\omega}_k[t+\tau])] \\ &\qquad \times [Z_{2k+1}\cos(\tilde{\omega}_k t) - Z_{2k}\sin(\tilde{\omega}_k t)] \Big\} \\ &= \sin(\tilde{\omega}_k[t+\tau])\cos(\tilde{\omega}_k t) - \cos(\tilde{\omega}_k[t+\tau])\sin(\tilde{\omega}_k t). \end{aligned}$$

Use of $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ along with $\sin^2(x) + \cos^2(x) = 1$ yields $E\{A_{k,k}\} = \sin(\tilde{\omega}_k \tau)$, and hence

$$\operatorname{cov}\{Y_{\Re,t+\tau}, Y_{\Im,t}\} = \frac{1}{2N} \sum_{k=0}^{2N-1} S_k \sin(\tilde{\omega}_k \tau), \tag{A–26b}$$

which depends on τ, but not t. Next note that, since $\{S_k\}$ is the DFT of the sequence displayed in Equation (601b), the associated inverse DFT says that

$$\frac{1}{2N} \sum_{k=0}^{2N-1} S_k \mathrm{e}^{\mathrm{i}\tilde{\omega}_k \tau} = \begin{cases} s_\tau, & \tau = 0, 1, \ldots, N; \\ s_{2N-\tau}, & \tau = N+1, N+2, \ldots, 2N-1 \end{cases}$$

(see Equation (92a)). Since $S_k e^{i\tilde{\omega}_k\tau} = S_k[\cos(\tilde{\omega}_k\tau) + i\sin(\tilde{\omega}_k\tau)]$ and since the inverse DFT above is real-valued, we can conclude from Equations (A–26a) and (A–26b) that

$$s_{\Re,\tau} = s_{\Im,\tau} = \begin{cases} s_\tau, & \tau = 0, 1, \ldots, N; \\ s_{2N-\tau}, & \tau = N+1, N+2, \ldots, 2N-1, \end{cases}$$

and

$$\operatorname{cov}\{Y_{\Re,t+\tau}, Y_{\Im,t}\} = 0,$$

as required (the above also shows that the complex-valued periodic process $\{Y_t\}$ of Equation (602) is stationary – see the discussion surrounding Equation (29b)). Note that $s_{\Re,N+1} = s_{\Im,N+1} = s_{N-1}$, and hence the ACVSs for $\{Y_{\Re,t}\}$ and $\{Y_{\Im,t}\}$ need not be in agreement with the ACVS for $\{X_t\}$ at lag $N+1$ (the same holds at lags $\tau > N+1$).

Answer to Exercise [605] Under the assumptions that $s_\tau = 0$ for $|\tau| \geq N$ and that $\Delta_t = 1$, the claim that $S_k = S(\tilde{f}_k)$ follows from Equations (A–25) and (111e). Equation (605a) follows from an argument analogous to the one leading to Equation (181).

Answer to Exercise [632a] Since

$$\left(\sum_{j=0}^{\infty} \psi_j^2\right)^2 - \sum_{j=0}^{\infty} \psi_j^4 = 2\sum_{j=0}^{\infty}\sum_{i>j}^{\infty} \psi_i^2\psi_j^2,$$

it follows from Equation (632b) that

$$\delta_2(X_t) = \frac{\sum_{j=0}^{\infty}\psi_j^4}{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2}\delta_2(\epsilon_t) + 3\frac{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2 - \sum_{j=0}^{\infty}\psi_j^4}{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2},$$

which leads to the first required result, namely,

$$\delta_2(X_t) - 3 = \frac{\sum_{j=0}^{\infty}\psi_j^4}{\left(\sum_{j=0}^{\infty}\psi_j^2\right)^2}[\delta_2(\epsilon_t) - 3].$$

We have

$$\frac{\kappa_4(X_t)}{\sigma^4} = \frac{E\{X_t^4\} - 3\sigma^4}{\sigma^4} = \frac{E\{X_t^4\}}{\sigma^4} - 3 = \delta_2(X_t) - 3 \text{ and, likewise, } \frac{\kappa_4(\epsilon_t)}{\sigma_\epsilon^4} = \delta_2(\epsilon_t) - 3,$$

from which the second required result follows.

Answer to Exercise [632b] With $\vartheta_{1,1}$ and $\phi_{1,1}$ taken to be ϑ and ϕ, use of Equation (600d) for an ARMA(1,1) process gives $\psi_0 = 1$ and $\psi_j = \phi^{j-1}(\phi + \vartheta)$, $j \geq 1$. Assume $\phi \neq 0$ (the proofs of all three equations are trivial for the special case $\phi = 0$). First note that

$$\sum_{j=0}^{\infty}\psi_j^k = 1 + \sum_{j=1}^{\infty}\psi_j^k = 1 + \frac{(\phi+\vartheta)^k}{\phi^k}\sum_{j=1}^{\infty}\phi^{kj} = 1 + \frac{(\phi+\vartheta)^k\phi^k}{\phi^k(1-\phi^k)} = \frac{1 - \phi^k + (\phi+\vartheta)^k}{1-\phi^k}. \tag{A–27}$$

Use of the above with $k = 2$ in combination with $\operatorname{var}\{X_t\} = \sigma_\epsilon^2\sum_{j=0}^{\infty}\psi_j^2$ proves Equation (632d). Similarly use of the above with $k = 2$ and $k = 3$ in combination with Equation (632a) gives Equation (632e). Finally use of Equation (A–27) with $k = 2$ and $k = 4$ in combination with Equation (632c) yields

$$\delta_2(X_t) - 3 = \frac{[1 - \phi^4 + (\phi+\vartheta)^4](1-\phi^2)^2}{[1 - \phi^2 + (\phi+\vartheta)^2]^2(1-\phi^4)}[\delta_2(\epsilon_t) - 3],$$

which leads to Equation (632f) once we note that $(1-\phi^2)^2/(1-\phi^4) = (1-\phi^2)/(1+\phi^2)$.