Name: ______________________________

This exam has 5 questions, each of which is worth 12 points. Good luck!

[1] Consider the process

$$X_t = \phi X_{t-12} + Z_t,$$

where $0 < |\phi| < 1$ and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ (the above might serve as a model for a time series of monthly values for which the current month is related to the same month of the previous year, but not to any of the intervening months).

(a) Use a 'telescoping' argument to determine the weights ψ_j needed to represent the process as an infinite moving average process, i.e., as

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

(b) Using the ψ_j weights derived in part (a), verify that $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ is a solution to the equation $X_t - \phi X_{t-12} = Z_t$.

(c) Determine the ACVF for $\{X_t\}$ (an item for your crib sheet: $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$).

(d) Determine the PACF for $\{X_t\}$. Hint: regard $\{X_t\}$ as an AR(p) process.

[2] The invertibility condition for the MA(1) process $X_t = Z_t + \theta Z_{t-1}$ is $|\theta| < 1$, where, as usual, $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Determine the invertibility condition for $X_t = Z_t + \theta Z_{t-q}$, where $q \geq 2$. Hint: 'telescoping' might be helpful.

[3] Suppose that $\{Y_t\}$ is a stationary process with mean 0 and ACVF $\{\gamma_Y(h)\}$. Define

$$X_t = \sum_{j=-J}^{J} \psi_j Y_{t-j},$$

where the $2J+1$ real-valued constants ψ_j are all finite.

(a) Argue that $\{X_t\}$ is a stationary process, and determine its ACVF $\{\gamma_X(h)\}$ in terms of the ACVF for $\{Y_t\}$.

(b) Suppose $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, where $0 < \sigma^2 < \infty$. Determine the ACVF for the causal and invertible ARMA(1,1) process $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$ in terms of ϕ, θ and σ^2. Do so by combining together

(1) your answer from part (a) and
(2) the fact that $\gamma_Y(h) = \phi^{|h|}\sigma^2/(1-\phi^2)$ is the ACVF for $Y_t = \phi Y_{t-1} + Z_t$

(*no* credit will be given for just stating the ARMA(1,1) ACVF!).

[4] Given a stationary process $\{X_t\}$ with zero mean, let $\widehat{X}_{n+1}$ denote the best linear predictor of X_{n+1} given X_1, X_2, $\ldots$, X_n. Argue that $E\{X_{n+1}\widehat{X}_{n+1}\} = \operatorname{var}\{\widehat{X}_{n+1}\}$.

[5] Suppose we have a time series that is a portion $X_1, X_2, \ldots, X_n$ of an AR(1) process $X_t = \phi X_{t-1} + Z_t$ with mean zero, where $0 < |\phi| < 1$, and $\{Z_t\} \sim \text{WN}(0, \sigma^2)$ (we also assume that $n \geq 2$ and that the series is not boring).

(a) By definition the Burg estimator $\bar{\phi}_X$ of ϕ is the value minimizing the sum of squares of forward (one-step-ahead) and backward (one-step-behind) prediction errors when we regard the sum of squares as a function of ϕ. Derive an explicit expression for $\bar{\phi}_X$ in terms $X_1, X_2, \ldots, X_n$.

(b) Define $Y_1 = Y_{n+2} = 0$ and $Y_t = X_{t-1}$ for $t = 2, 3, \ldots, n+1$; i.e., the time series $\{Y_t\}$ is a series of length $n+2$ formed by placing a zero at the beginning and at the end of the X_t series. Let $\bar{\phi}_Y$ denote the Burg estimator for $Y_1, Y_2, \ldots, Y_{n+2}$. Using part (a), what form does $\bar{\phi}_Y$ take, and how is it related to the Yule–Walker estimator $\hat{\phi}_X = \hat{\gamma}_X(1)/\hat{\gamma}_X(0)$ for the X_t series, where $\hat{\gamma}_X(h)$ is the sample ACVF at lag h?

(c) Assume now that $n = 2$, i.e., the time series consists of just X_1 and X_2. How is the Burg estimator $\bar{\phi}_X$ related to the Yule–Walker estimator $\hat{\phi}_X$ for this special case? Given that $|\bar{\phi}_X| \leq 1$ for any sample size $n \geq 2$ (a fact you need *not* prove), what range of values can $\hat{\phi}_X$ assume when $n = 2$?