Statistics 519, Winter Quarter 2020

Problem Set 6

Problem 18 (2 points for each of the 4 parts). In what follows, assume that $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $0 < \sigma^2 < \infty$. Determine which of the following processes are stationary and, for the ones that are stationary, which are causal. Determine also which are invertible.

a. $X_t + 0.2X_{t-1} + 0.7X_{t-2} = Z_t$

b. $X_t - 0.2X_{t-1} - 0.8X_{t-2} = Z_t$

c. $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$

d. $X_t + 1.25X_{t-1} = Z_t + Z_{t-2}$

Problem 19 (2 points). Consider the stationary process of Problem 2(b), namely,

$$X_t = Z_2 \cos(\omega t) + Z_1 \sin(\omega t), \quad t \in \mathbb{Z},$$

where Z_1 and Z_2 are independent $\mathcal{N}(0, \sigma^2)$ RVs with $0 < \sigma^2 < \infty$, and ω is an arbitrary real-valued constant (we can, however, assume $0 \le \omega < 2\pi$ without loss of generality). Show that $\{X_t\}$ satisfies the difference equation

$$2\cos(\omega)X_{t-1} - X_{t-2} = X_t, \quad t \in \mathbb{Z}$$

(for context, see overhead VIII–35). Hint: rather than evoking trigonometric identities, you might consider complex exponentials, using which you can write

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \text{ and make use of } e^{i(m+n)x} = e^{imx}e^{inx},$$

where m and n are integers.

Problem 20 (1 point for part a and 2 points each for parts b and c). In what follows, assume that $\{Z_t\} \sim \mathrm{WN}(0, \sigma_Z^2)$ with $0 < \sigma_Z^2 < \infty$, that $\{W_t\} \sim \mathrm{WN}(0, \sigma_W^2)$ with $0 < \sigma_W^2 < \infty$ and that $\mathrm{cov}\{W_s, Z_t\} = 0$ for all integers s and t. Let $\{X_t\}$ be a causal AR(1) process constructed from $\{Z_t\}$; i.e., $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j} = \phi X_{t-1} + Z_t$, $t \in \mathbb{Z}$, where $|\phi| < 1$ is a real-valued constant. Define

$$Y_t = X_t + W_t. \qquad (*)$$

a. Argue that $\{Y_t\}$ is a stationary process, and determine its ACVF.

b. Show that $U_t = Y_t - \phi Y_{t-1}$ has the ACVF of an MA(1) process.

c. Assuming that $\{U_t\}$ can be expressed by $U_t = V_t + \theta V_{t-1}$, where $\{V_t\} \sim \text{WN}(0, \sigma_V^2)$ and $|\theta| < 1$, it follows from part b that

$$Y_t - \phi Y_{t-1} = V_t + \theta V_{t-1},$$

i.e., that, in addition to $(*)$, the process $\{Y_t\}$ can also be represented as a causal and invertible ARMA(1,1) process with three parameters ϕ, θ and σ_V^2. Express θ and σ_V^2 in terms of ϕ, σ_Z^2 and σ_W^2, i.e., the three parameters for $\{Y_t\}$ when represented by $(*)$.

(This is an adaptation of Problem 2.9 of Brockwell & Davis.)

Problem 21 (2 points for each of the 5 parts). In Problem 17 of Problem Set 5, you considered the causal and invertible process $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-3}$, $\{Z_t\} \sim \text{WN}(0, \sigma^2)$.

a. Determine the ACVF for this process in terms of the model parameters ϕ, θ and σ^2 using the fourth method for doing so described in overheads IX–47 to IX–50.

b. Determine the ACVF in terms of ϕ, θ and σ^2 using the first method for doing so described in overheads IX–1 and IX–2 and illustrated in overheads IX–3 and IX–4. Verify that you get the same ACVF as in part a. Note: this method requires the ψ_j weights for this process – assume these to be known from the solution to Problem 17:

$$\psi_0 = 1, \ \ \psi_1 = \phi, \ \ \psi_2 = \phi^2 \ \text{ and } \ \psi_j = \phi^{j-3}(\phi^3 + \theta) \ \text{ for } j \geq 3.$$

c. The second method for determining the ACVF for an ARMA process is described starting with overhead IX–19. In the ARMA(1,1) case, this method leads to a matrix formulation displayed in the second bulleted item on overhead IX–23. For the process under consideration here, determine what the equivalent of this matrix equation is. In the ARMA(1,1) case, we solved for the unknowns in the matrix equation using matrix inversion. For the process under consideration, use the results of part a to state what the unknowns must be in terms of ϕ, θ and σ^2.

d. The third method for determining the ACVF is described starting with overhead IX–44 and leads to the matrix equation shown on overhead IX–45. Determine the equivalent of this matrix equation for the process under consideration (in particular, state what the c_k's are). You need not solve this equation explicitly, but its solution would give you the ACVF for a certain number of low lags (and hence can be deduced from part a). Describe as explicitly as possible how you would determine the ACVF for all remaining lags using the third method.

e. With $\sigma^2 = 1$ and $\phi = 0.9$, plot the ACF (note: *not* the ACVF!) at lags $h = 0, 1, \ldots, 20$ for θ set to 0.99, 0, −0.25, −0.5, −0.75 and, finally, −0.99. Comment briefly on how these compare with the ACVFs shown in overheads IX–6, IX–8, IX–9, IX–10, IX–11 and IX–13 for an ARMA(1,1) process with similar settings for σ^2, ϕ and θ.

Solutions are due Friday, February 21, at the beginning of the class.