Statistics 519, Winter Quarter 2020

Problem Set 5

Problem 14 (4 points for each of the 2 parts). Let $\{Z_t\}$ be IID$(0,\sigma^2)$ normal (Gaussian) noise, where $0 < \sigma^2 < \infty$, and let a, b and c be constants. In Problem 2a, you showed that the process

$$X_t = a + bZ_t + cZ_{t-2}, \quad t \in \mathbb{Z},$$

is stationary with an ACVF given by

$$\gamma(h) = \begin{cases} (b^2+c^2)\sigma^2, & h=0; \\ bc\sigma^2, & h=\pm 2; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and hence its ACF is } \rho(h) = \begin{cases} 1, & h=0; \\ bc/(b^2+c^2), & h=\pm 2; \\ 0, & \text{otherwise}. \end{cases}$$

Given a time series $X_1, X_2, \ldots, X_n$ that is a portion of length $n \geq 5$ from this process, consider

$$\hat{\boldsymbol{\rho}}_4 = [\hat{\rho}(1), \hat{\rho}(2), \hat{\rho}(3), \hat{\rho}(4)]',$$

where, as usual,

$$\hat{\rho}(h) = \frac{\sum_{t=1}^{n-|h|}(X_{t+|h|} - \overline{X})(X_t - \overline{X})}{\sum_{t=1}^{n}(X_t - \overline{X})^2} \text{ and } \overline{X} = \frac{1}{n}\sum_{t=1}^{n} X_t$$

(see overheads II–65 and VI–13).

a. Using Bartlett's formula (see overhead VI–13), determine the entries $w_{i,j}$ for the 4×4 matrix W that forms the large-sample covariance matrix W/n for $\hat{\boldsymbol{\rho}}_4$ (please state the entries in terms of the ACF rather than in terms of b and c).

b. Letting $\sigma^2 = 1$, $a = 42$, $b = 1$ and $c = 0.9$, generate a realization of $X_1, X_2, \ldots, X_{100}$, and use these to form a realization of the vector $\hat{\boldsymbol{\rho}}_4$ – denote this realization as $\hat{\boldsymbol{\rho}}_{4,1}$. Repeat this procedure 999 more times to obtain realizations $\hat{\boldsymbol{\rho}}_{4,1}, \hat{\boldsymbol{\rho}}_{4,2}, \ldots, \hat{\boldsymbol{\rho}}_{4,1000}$. Compute the sample mean and the sample covariance and correlation matrices for these 1000 realizations, and compare them to what large-sample theory suggests (see overhead VI–13; hint: you might find the R functions `colMeans`, `cov` and `cor` useful). Finally, comment on what would happen if σ^2 and/or a were set to values other than 1 and 42.

Problem 15 (4 points). Suppose that the process $\{X_t\}$ satisfies the equation

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{Z},$$

where ϕ is a real-valued constant, and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $0 < \sigma^2 < \infty$. Show that this process does *not* have a stationary solution when $|\phi| = 1$. Hint: suppose that a solution exists, and show that this leads to a contradiction by (i) using a 'telescoping' argument à la overhead VII–12 to derive an expression for $X_t - \phi^J X_{t-J}$, where J is a positive integer, and (ii) considering $\mathrm{var}\,\{X_t - \phi^J X_{t-J}\}$. (This is essentially Problem 2.8 of Brockwell & Davis.)

Problem 16 (5 points). Let $\{X_t\}$ denote the stationary solution for the equation

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{Z},$$

where $|\phi| > 1$ is a real-valued constant, and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$; i.e., as per overheads VII–14 and VII–13,

$$X_t \stackrel{\mathrm{def}}{=} -\sum_{j=1}^{\infty} \frac{1}{\phi^j} Z_{t+j}.$$

Define a new process

$$W_t = X_t - \frac{1}{\phi} X_{t-1}.$$

Show that $\{W_t\} \sim \mathrm{WN}(0, \sigma_W^2)$, and find an expression for σ_W^2 in terms of ϕ and σ^2. (This is essentially Problem 3.8 of Brockwell & Davis; to place it in context, see overhead VII–15.)

Problem 17 (4 points for each of the 2 parts). For the casual and invertible ARMA(1,1) process $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$, we can express the ψ_j and π_j weights as $\psi_0 = \pi_0 = 1$ along with $\psi_j = (\phi + \theta)\phi^{j-1}$ and $\pi_j = -(\phi + \theta)(-\theta)^{j-1}$ for $j \geq 1$ (see overheads VII–25, VII–29 and VIII–18). Consider now the process

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-3}$$

which we assume to be causal and invertible and where, as usual, $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

a. Derive expressions for the ψ_j and π_j weights for $\{X_t\}$.

b. Using the ψ_j weights derived in part a, verify that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

is a solution to the equation

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-3}.$$

Solutions are due Friday, February 14, at the beginning of the class.