Statistics 519, Winter Quarter 2020

Problem Set 4

Problem 10 (2 points for each of the 3 parts). Suppose $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $0 < \sigma^2 < \infty$ as usual. Let $\overline{X}_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{t=1}^{n} X_t$ be the sample mean based upon a portion of one of the following three stationary processes, each with mean and ACVF to be denoted by μ and $\gamma_X(h)$:

a. $X_t = \mu + Z_1$ (see part d of Problem 2);

b. $X_t = \mu + Z_t + Z_{t-1}$; and

c. $X_t = \mu + Z_t - Z_{t-1}$

For all three processes, express $\overline{X}_n$ as a linear combination of RVs from $\{Z_t\}$, and determine $\mathrm{var}\,\{\overline{X}_n\}$ directly with this expression. In each case, verify that your equation for $\mathrm{var}\,\{\overline{X}_n\}$ is in agreement with the equation

$$\mathrm{var}\,\{\overline{X}_n\} = \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\left(1 - \frac{|h|}{n}\right)\gamma_X(h)$$

(see overhead V–5). Comment upon how well the approximation

$$\mathrm{var}\,\{\overline{X}_n\} \approx \frac{1}{n}\sum_{h=-\infty}^{\infty}\gamma_X(h)$$

works in each case (see overhead V–6).

Problem 11 (3 points). Given a time series $x_1, x_2, \ldots, x_n$, consider its sample ACVF, defined as usual as

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-|h|}(x_{t+|h|} - \bar{x})(x_t - \bar{x}), \text{ where } \bar{x} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{t=1}^{n} x_t,$$

with h ranging from $-(n-1)$ up to $n-1$. Show that

$$\hat{v} \stackrel{\text{def}}{=} \sum_{h=-(n-1)}^{n-1}\hat{\gamma}(h) = 0.$$

Hint: consider an $n \times n$ matrix whose (s,t)th element is $(x_s - \bar{x})(x_t - \bar{x})$.

Problem 12 (2 points for each of the 3 parts). Let $X_1, X_2, \ldots, X_n$ be a time series that is a portion of a stationary AR(1) process given by

$$X_t = \mu + \phi(X_{t-1} - \mu) + Z_t, \quad t \in \mathbb{Z},$$

where μ is a constant, $|\phi| < 1$ and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $0 < \sigma^2 < \infty$. Let $\overline{X}_n$ denote the sample mean of $X_1, X_2, \ldots, X_n$.

a. Show that

$$\mathrm{var}\,\{\overline{X}_n\} = \frac{\mathrm{var}\,\{X_t\}}{n} \times \frac{1+\phi}{1-\phi}\left(1 + \frac{f(\phi, n)}{n}\right),$$

where $f(\phi, n)$ is a function to be determined as part of the exercise. While this function depends upon n, it does so only weakly in that sense that, for a fixed ϕ, $|f(\phi, n)|$ is bounded by a constant. Thus, for large enough n, the following should be a decent approximation:

$$\frac{\mathrm{var}\,\{\overline{X}_n\}}{\mathrm{var}\,\{X_t\}} \approx \frac{1+\phi}{n(1-\phi)}$$

(you don't need to justify this statement).

b. For $n = 100$, plot the log of the ratio $\mathrm{var}\,\{\overline{X}_n\}/\mathrm{var}\,\{X_t\}$ versus $\phi = -0.99, -0.98, \ldots, 0.98, 0.99$, and also plot the above approximation in the same manner. Comment briefly on how well the approximation does.

c. For $\phi = -0.9$, $\phi = 0$ and finally $\phi = 0.9$, plot $\log_{10}\left(\mathrm{var}\,\{\overline{X}_n\}/\mathrm{var}\,\{X_t\}\right)$ versus $\log_{10}(n)$ for $n = 1, 2, \ldots, 1000$. Comment briefly on the appearance of the three plots (in particular, how they differ and how they are similar).

Problem 13 (2 points for each of the 5 parts). Suppose that the RVs X_1 and X_2 are part of a real-valued Gaussian stationary process $\{X_t\}$ with unknown mean μ and unknown ACVF $\gamma(h)$, which is arbitrary except for the mild restriction $\gamma(1) < \gamma(0)$. Based upon just these two RVs, consider the so-called unbiased and biased estimators of $\gamma(1)$ (see overhead VI–9):

$$\hat{\gamma}^{(\mathrm{U})}(1) = (X_1 - \overline{X})(X_2 - \overline{X}) \text{ and } \hat{\gamma}^{(\mathrm{B})}(1) = \frac{(X_1 - \overline{X})(X_2 - \overline{X})}{2}, \text{ where } \overline{X} = \frac{X_1 + X_2}{2}.$$

a. For this special case of $n = 2$, determine $E\{\hat{\gamma}^{(\mathrm{U})}(1)\}$ and $E\{\hat{\gamma}^{(\mathrm{B})}(1)\}$, and use these to derive the biases of $\hat{\gamma}^{(\mathrm{U})}(1)$ and $\hat{\gamma}^{(\mathrm{B})}(1)$ as estimators of $\gamma(1)$. How does the squared bias of $\hat{\gamma}^{(\mathrm{U})}(1)$ compare with that of $\hat{\gamma}^{(\mathrm{B})}(1)$? Under what conditions (if any) are $\hat{\gamma}^{(\mathrm{U})}(1)$ and $\hat{\gamma}^{(\mathrm{B})}(1)$ unbiased?

b. Determine the variances of $\hat{\gamma}^{(\mathrm{U})}(1)$ and $\hat{\gamma}^{(\mathrm{B})}(1)$ How does $\mathrm{var}\,\{\hat{\gamma}^{(\mathrm{U})}(1)\}$ compare with $\mathrm{var}\,\{\hat{\gamma}^{(\mathrm{B})}(1)\}$? Hint: recall the Isserlis theorem, which was stated in Problem 2 of Problem Set 1.

c. Using the results of parts a and b, verify that, for this special case of $n = 2$, we have

$$\text{mse}\{\hat{\gamma}^{(\text{B})}(1)\} \stackrel{\text{def}}{=} E\{(\hat{\gamma}^{(\text{B})}(1) - \gamma(1))^2\} < E\{(\hat{\gamma}^{(\text{U})}(1) - \gamma(1))^2\} \stackrel{\text{def}}{=} \text{mse}\{\hat{\gamma}^{(\text{U})}(1)\},$$

thus demonstrating a point mentioned on overhead VI–12 about the relative merits of the unbiased and biased estimators of the ACVF.

d. Calculate $\mathbf{P}\left[\hat{\gamma}^{(\text{U})}(1) < -1\right]$ when $\text{var}\{X_1\} = 4/5$ and $\rho(1) \stackrel{\text{def}}{=} \gamma(1)/\gamma(0) = 1/2$, and calculate $\mathbf{P}\left[\hat{\gamma}^{(\text{U})}(1) > 1\right]$ also. Hint: if Z is a Gaussian RV with zero mean and unit variance, then Z^2 obeys a chi-square distribution with one degree of freedom.

e. Verify that, for the special case of $n = 2$,

$$\hat{\rho}^{(\text{U})}(1) \stackrel{\text{def}}{=} \frac{\hat{\gamma}^{(\text{U})}(1)}{\hat{\gamma}^{(\text{U})}(0)} = -1 \text{ and } \hat{\rho}^{(\text{B})}(1) \stackrel{\text{def}}{=} \frac{\hat{\gamma}^{(\text{B})}(1)}{\hat{\gamma}^{(\text{B})}(0)} = -\frac{1}{2}$$

no matter what values the RVs X_1 and X_2 assume (we can ignore the special case $X_1 = X_2 = 0$ since it has probability zero). Thus, while part d indicates that realizations of $\hat{\gamma}^{(\text{U})}(1)$ can take on different values, those of $\hat{\rho}^{(\text{U})}(1)$ cannot.

Solutions are due Friday, February 7, at the beginning of the class.