Statistics 519, Winter Quarter 2020

Problem Set 1

Problem 1 (2 points for each part of the 4 parts). Here we consider some basic properties of covariances and variances (see course overhead II–2). In what follows, all random variables (RVs) are real-valued and are denoted by Z (or subscripted versions thereof), while c with or without a subscript denotes a real-valued constant (an ordinary variable).

a. Show that $\operatorname{cov}\{Z, c\} = 0$.

b. Show that $\operatorname{cov}\{Z_1 + c_1, Z_2 + c_2\} = \operatorname{cov}\{Z_1, Z_2\}$.

c. Show that

$$\operatorname{cov}\Big\{\sum_j c_{1,j} Z_{1,j}, \sum_k c_{2,k} Z_{2,k}\Big\} = \sum_j \sum_k c_{1,j} c_{2,k} \operatorname{cov}\{Z_{1,j}, Z_{2,k}\},$$

where the indices j and k range over finite sets of integers.

d. Show that

$$\operatorname{var}\Big\{\sum_{j=1}^{n} c_j Z_j\Big\} = \sum_{j=1}^{n} c_j^2 \operatorname{var}\{Z_j\} + 2 \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} c_j c_k \operatorname{cov}\{Z_j, Z_k\},$$

where $n \geq 2$.

Problem 2 (2 points for each part of the 6 parts). Let $\{Z_t : t \in \mathbb{Z}\}$ be IID$(0,\sigma^2)$ normal (Gaussian) noise, where $0 < \sigma^2 < \infty$ (see course overhead II–17). Let a, b and c be constants. Which (if any) of the following processes are stationary? For each stationary process, determine its mean and autocovariance function (ACVF). For each process (stationary or not) and with the settings $\sigma^2 = 1$, $a = -2$, $b = 1$ and $c = \pi/4$, use a random number generator to create and then plot two independent realizations of $\{X_1, X_2, \ldots, X_8\}$ versus $\{1, 2, \ldots, 8\}$ (please turn in the code you used to create the realizations).

a. $X_t = a + bZ_t + cZ_{t-2}$

b. $X_t = Z_2 \cos(ct) + Z_1 \sin(ct)$

c. $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

d. $X_t = a + bZ_1$

e. $X_t = Z_1 \sin(ct)$

f. $X_t = Z_t Z_{t-1}$

For one of the parts, you might find the *Isserlis theorem* to be useful: if Z_1, Z_2, Z_3 and Z_4 are *any* four Gaussian RVs with zero mean, then

$$\text{cov}\{Z_1 Z_2, Z_3 Z_4\} = \text{cov}\{Z_1, Z_3\}\text{cov}\{Z_2, Z_4\} + \text{cov}\{Z_1, Z_4\}\text{cov}\{Z_2, Z_3\}$$

(L. Isserlis, 'On a Formula for the Product-Moment Coefficient of Any Order of a Normal Frequency Distribution in Any Number of Variables,' *Biometrika*, vol. 12 (1918), pp. 134–9). (This exercise is adapted from Exercise [1.4] of Brockwell & Davis.)

Problem 3 (5 points). The autocorrelation function (ACF) for an MA(1) process is given by

$$\rho(h) = \begin{cases} 1, & h = 0, \\ \theta/(1+\theta^2), & h = \pm 1, \\ 0, & \text{otherwise,} \end{cases}$$

where θ is an arbitrary constant (see course overhead II–33). Determine (a) the range of values that $\rho(1)$ can assume and (b) the values of θ for which $|\rho(1)|$ is as large as possible.

Solutions are due Friday, January 17, at the beginning of the class.