Problem 1 (2 points for each part of the 4 parts). Here we consider some basic properties of covariances and variances (see course overhead II–2). In what follows, all random variables (RVs) are real-valued and are denoted by $Z$ (or subscripted versions thereof), while $c$ with or without a subscript denotes a real-valued constant (an ordinary variable).

a. Show that $\text{cov}\{Z,c\} = 0$.

b. Show that $\text{cov}\{Z_1 + c_1, Z_2 + c_2\} = \text{cov}\{Z_1, Z_2\}$.

c. Show that

$$\text{cov}\left\{\sum_{j} c_{1,j}Z_{1,j}, \sum_{k} c_{2,k}Z_{2,k}\right\} = \sum_{j} \sum_{k} c_{1,j}c_{2,k}\text{cov}\{Z_{1,j}, Z_{2,k}\},$$

where the indices $j$ and $k$ range over finite sets of integers.

d. Show that

$$\text{var}\left\{\sum_{j=1}^{n} c_{j}Z_{j}\right\} = \sum_{j=1}^{n} c_{j}^{2}\text{var}\{Z_{j}\} + 2\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} c_{j}c_{k}\text{cov}\{Z_{j}, Z_{k}\},$$

where $n \geq 2$.

Problem 2 (2 points for each part of the 6 parts). Let $\{Z_t\}$ be IID$(0,\sigma^2)$ normal (Gaussian) noise, where $0 < \sigma^2 < \infty$ (see course overhead II–16). Let $a$, $b$ and $c$ be constants. Which (if any) of the following processes are stationary? For each stationary process, determine its mean and autocovariance function (ACVF). For each process (stationary or not) and with the settings $\sigma^2 = 1$, $a = 1$, $b = -2$ and $c = \pi/4$, use a random number generator to create and then plot two independent realizations of $X_1, X_2, \ldots, X_{10}$ (please turn in the code you used to create the realizations).

a. $X_t = a + bZ_{t+1} + cZ_{t-2}$

b. $X_t = Z_1 \cos (ct) + Z_2 \sin (ct)$

c. $X_t = Z_t \cos (ct) + Z_{t-1} \sin (ct)$

d. $X_t = a + bZ_0$
e. $X_t = Z_0 \sin (ct)$

f. $X_t = Z_t Z_{t-1}$

For one of the parts, you might find the Isserlis theorem to be useful: if $Z_1$, $Z_2$, $Z_3$ and $Z_4$ are any four Gaussian RVs with zero mean, then

$$\text{cov} \{Z_1 Z_2, Z_3 Z_4\} = \text{cov} \{Z_1, Z_3\} \text{cov} \{Z_2, Z_4\} + \text{cov} \{Z_1, Z_4\} \text{cov} \{Z_2, Z_3\}$$


(This exercise is adapted from Exercise [1.4] of Brockwell & Davis.)

**Problem 3 (5 points).** The autocorrelation function (ACF) for an MA(1) process is given by

$$\rho(h) = \begin{cases} 
1, & h = 0, \\
\theta/(1 + \theta^2), & h = \pm 1, \\
0, & \text{otherwise}, 
\end{cases}$$

where $\theta$ is an arbitrary constant (see course overhead II–32). Determine (a) the range of values that $\rho(1)$ can assume and (b) the values of $\theta$ for which $|\rho(1)|$ is as large as possible.

**Problem 4 (5 points).** Let $\{X_{1,t}\}, \{X_{2,t}\}, \ldots, \{X_{m,t}\}$ each be stationary processes with respective means $\mu_1$, $\mu_2$, $\ldots$, $\mu_m$ and ACVFs given at lag $h$ by $\gamma_1(h), \gamma_2(h), \ldots, \gamma_m(h)$. If $X_{j,t}$ and $X_{k,u}$ are uncorrelated for all $t$, $u$ and $j \neq k$ (i.e., $\text{cov} \{X_{j,t}, X_{k,u}\} = 0$ when $j \neq k$) and if $c_1, c_2, \ldots, c_m$ are arbitrary real-valued variables, show that

$$X_t \equiv \sum_{j=1}^{m} c_j X_{j,t}$$

is a stationary process, and determine its ACVF. (The procedure of forming a new process by making a linear combination of several processes is sometimes called *aggregation*.)

Solutions are due Wednesday, January 13, at the beginning of the class.