# State-Space Models – Introduction

- through two seemingly simple equations, *state-space models* define a rich class of processes that have served well as models for time series
  - special cases: ARMA, ARIMA and SARIMA models
- so-called *Kalman recursions* for state-space models offer an elegant solution not only for forecasting time series, but also for filtering and smoothing (see forthcoming definitions)
- state-space models and Kalman recursions can be readily adapted to handle time series with missing values ('gappy' series)
- just as ARMA processes are built upon white noise, state-space models are built upon a certain type of uncorrelated noise

# Multivariate Uncorrelated Noise: I

- let $\boldsymbol{W}_t$ be a random vector (i.e., a vector of random variables (RVs)), where $t$ is a time index
- as a running example, consider the bivariate case:

$$\boldsymbol{W}_t = \begin{bmatrix} W_{1,t} \\ W_{2,t} \end{bmatrix},$$

  where $W_{1,t}$ and $W_{2,t}$ are two real-valued RVs
- as usual, expected value of random vector is denoted as $E\{\boldsymbol{W}_t\}$, which is a vector of expectations of components of $\boldsymbol{W}_t$:

$$E\{\boldsymbol{W}_t\} = \begin{bmatrix} E\{W_{1,t}\} \\ E\{W_{2,t}\} \end{bmatrix}$$

- $E\{\boldsymbol{W}_t\} = \mathbf{0}$ says all RVs in $\boldsymbol{W}_t$ have mean zero ($\mathbf{0}$ denotes a vector of zeros)

# Multivariate Uncorrelated Noise: II

- assuming $E\{\boldsymbol{W}_t\} = \mathbf{0}$, covariance matrix for $\boldsymbol{W}_t$ is given by

$$R_t \stackrel{\text{def}}{=} E\{\boldsymbol{W}_t \boldsymbol{W}_t'\} = E\left\{ \begin{bmatrix} W_{1,t} \\ W_{2,t} \end{bmatrix} \begin{bmatrix} W_{1,t} & W_{2,t} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \text{var}\,\{W_{1,t}\} & \text{cov}\,\{W_{1,t}, W_{2,t}\} \\ \text{cov}\,\{W_{2,t}, W_{1,t}\} & \text{var}\,\{W_{2,t}\} \end{bmatrix}$$

- $\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots$ said to be *uncorrelated noise* if $E\{\boldsymbol{W}_s \boldsymbol{W}_t'\} = 0$ for all $1 \leq s < t$ (note: here 0 is a matrix of zeros)

- note carefully:
  - $R_t$ need *not* be a diagonal matrix; i.e., RVs within a given $\boldsymbol{W}_t$ can be correlated (in contrast to RVs from distinct $\boldsymbol{W}_t$'s)
  - RVs in $\boldsymbol{W}_t$ need *not* have the same variance
  - $R_s$ and $R_t$ need *not* be the same

# Multivariate Uncorrelated Noise: III

- $\boldsymbol{W}_t \sim \mathrm{UN}(\boldsymbol{\mu}_t, R_t)$ denotes uncorrelated noise with mean vectors $\boldsymbol{\mu}_t$ and covariance matrices $R_t$, $t = 1, 2, \ldots$
- $\boldsymbol{W}_t \sim \mathrm{WN}(\boldsymbol{\mu}, R)$ denotes white noise, i.e., uncorrelated noise with time-independent mean vectors and covariance matrices
- note: Brockwell & Davis use $\boldsymbol{W}_t \sim \mathrm{WN}(\mathbf{0}, R_t)$ to denote uncorrelated noise with zero mean, but with (possibly) time-varying covariance matrices (precisely defined, but, when $\boldsymbol{W}_t$ is one dimensional, does *not* reduce to what we've been calling white noise)

# State-Space Models: I

- let $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, … be a time series of dimension $w$
- state-space model for this series consists of two equations

1. *observation equation* takes form

$$\boldsymbol{Y}_t = G_t \boldsymbol{X}_t + \boldsymbol{W}_t, \;\; t = 1, 2, \ldots,$$

   where

   - $\boldsymbol{X}_t$ is $v$-dimensional *state vector* (stochastic in general)
   - $G_t$ is $w \times v$ observation matrix (deterministic)
   - $\boldsymbol{W}_t \sim \mathrm{UN}(\boldsymbol{0}, R_t)$ is observation noise, with $\boldsymbol{W}_t$ & $R_t$ having dimensions $w$ & $w \times w$ (can be degenerate, i.e., $R_t = 0$)

- observation equation essentially says that we can observe linear combinations of variables in state vector, but typically only in presence of noise (noise present if $R_t \neq 0$)

# State-Space Models: II

2. *state-transition equation* takes form

$$\boldsymbol{X}_{t+1} = F_t \boldsymbol{X}_t + \boldsymbol{V}_t, \ \ t = 1, 2, \ldots,$$

where

– $F_t$ is $v \times v$ state transition matrix (deterministic)

– $\boldsymbol{V}_t \sim \mathrm{UN}(\mathbf{0}, Q_t)$ is state-transition noise, with $\boldsymbol{V}_t$ & $Q_t$ having dimensions $v$ & $v \times v$

- two additional assumptions

a. $E\{\boldsymbol{W}_s \boldsymbol{V}_t'\} = 0$ for all $s$ and $t$ (here 0 is a $w \times v$ matrix of zeros; in words, every observation noise RV is uncorrelated with every state-transition noise RV)

b. assuming $E\{\boldsymbol{X}_1\} = \mathbf{0}$ for convenience, $E\{\boldsymbol{X}_1 \boldsymbol{W}_t'\} = 0$ & $E\{\boldsymbol{X}_1 \boldsymbol{V}_t'\} = 0$ for all $t$ (in words, initial state vector RVs are uncorrelated with observation & state-transition noise)

# AR(1) Process as a State-Space Model: I

- state-transition equation

$$\boldsymbol{X}_{t+1} = F_t \boldsymbol{X}_t + \boldsymbol{V}_t$$

is reminiscent of a causal AR(1) model:

$$\mathcal{X}_{t+1} = \phi \mathcal{X}_t + \mathcal{Z}_{t+1} \text{ with } \{\mathcal{Z}_t\} \sim \text{WN}(0, \sigma^2) \text{ and } |\phi| < 1$$

- can express AR(1) in state-space formulation by setting
  - $\boldsymbol{X}_{t+1} = \mathcal{X}_{t+1}$
  - $F_t = \phi$
  - $\boldsymbol{V}_t = \mathcal{Z}_{t+1}$ along with $Q_t \stackrel{\text{def}}{=} E\{\boldsymbol{V}_t \boldsymbol{V}_t'\} = E\{\mathcal{Z}_{t+1}^2\} = \sigma^2$

  and by using a degenerate form of the observation equation $\boldsymbol{Y}_t = G_t \boldsymbol{X}_t + \boldsymbol{W}_t$ in which

$$G_t = 1 \text{ and } \boldsymbol{W}_t = 0 \text{ (implying } R_t = 0) \text{ so that } \boldsymbol{Y}_t = \mathcal{X}_t$$

## AR(1) Process as a State-Space Model: II

- to complete model, need to define initial state $\boldsymbol{X}_1$
- natural choice is
$$\mathcal{X}_1 = \sum_{j=0}^{\infty} \phi^j \mathcal{Z}_{1-j}, \text{ for which } \operatorname{var}\{\mathcal{X}_1\} = \frac{\sigma^2}{1-\phi^2}$$
- with this choice, required conditions, namely, $E\{\boldsymbol{X}_1\boldsymbol{W}_t'\} = 0$ & $E\{\boldsymbol{X}_1\boldsymbol{V}_t'\} = 0$, hold: $E\{\boldsymbol{X}_1\boldsymbol{W}_t'\} = E\{\mathcal{X}_1 \times 0\} = 0$ and
$$E\{\boldsymbol{X}_1\boldsymbol{V}_t'\} = E\{\mathcal{X}_1\mathcal{Z}_{t+1}\} = \sum_{j=0}^{\infty} \phi^j E\{\mathcal{Z}_{1-j}\mathcal{Z}_{t+1}\} = 0 \text{ for } t = 1, 2, \ldots$$
- could also set $\boldsymbol{X}_1 = \mathcal{Z}_1 \frac{\sigma}{\sqrt{(1-\phi^2)}}$ to get AR(1) process, but using, e.g., $\boldsymbol{X}_1 = \mathcal{Z}_1$ would lead to a valid state-space model that is *not* a true AR(1) model (is such in an asymptotic sense)

# AR(1) Process as a State-Space Model: III

- AR(1) process with $0 < \phi < 1$ is known as 'red noise'
- red noise is related to a 1st-order stochastic differential equation, rendering it a model for various geophysical processes
- can usually only observe red noise process of interest in presence of observational noise (often taken to be white noise)
  – Problem 20: AR(1) + white noise yields ARMA(1,1)
- by modifying state-space formulation for an AR(1) process, can model this setup by changing observational noise from $\boldsymbol{W}_t = 0$ to $\boldsymbol{W}_t = \mathcal{W}_t \sim \text{WN}(0, \sigma_W^2)$ (with $R_t$ changing from 0 to $\sigma_W^2$), where $\mathcal{W}_t$ is uncorrelated with $\mathcal{Z}_t$'s (and hence $\mathcal{X}_1$ also)
- observation and state-transition equations become

$$\mathcal{Y}_t = \mathcal{X}_t + \mathcal{W}_t \text{ and } \mathcal{X}_{t+1} = \phi\mathcal{X}_t + \mathcal{Z}_{t+1}$$

# ARMA(1,1) Process as a State-Space Model: I

- formulating ARMA(1,1) process $\mathcal{Y}_t - \phi\mathcal{Y}_t = \mathcal{Z}_t + \theta\mathcal{Z}_{t-1}$ as a state-space model is trickier
- expressing model as $\phi(B)\mathcal{Y}_t = \theta(B)\mathcal{Z}_t$, note that we can create $\mathcal{Y}_t$ by taking causal AR(1) process $\phi(B)\mathcal{X}_t = \mathcal{Z}_t$, i.e., $\mathcal{X}_t = \phi^{-1}(B)\mathcal{Z}_t$, and subjecting it to a $\theta(B)$ filter to obtain output $\mathcal{Y}_t = \theta(B)\mathcal{X}_t = \theta(B)\phi^{-1}(B)\mathcal{Z}_t$
- can express filtering of AR(1) process by

$$\mathcal{Y}_t = \begin{bmatrix} 1 & \theta \end{bmatrix} \begin{bmatrix} \mathcal{X}_t \\ \mathcal{X}_{t-1} \end{bmatrix},$$

which matches up with observation equation

$$\boldsymbol{Y}_t = G_t \boldsymbol{X}_t + \boldsymbol{W}_t$$

if $\boldsymbol{Y}_t = \mathcal{Y}_t$, $G_t = \begin{bmatrix} 1 & \theta \end{bmatrix}$, $\boldsymbol{X}_t = \begin{bmatrix} \mathcal{X}_t \\ \mathcal{X}_{t-1} \end{bmatrix}$ & $\boldsymbol{W}_t = 0$ (thus $R_t = 0$)

## ARMA(1,1) Process as a State-Space Model: II

- given $\boldsymbol{X}_t = \begin{bmatrix} \mathcal{X}_t \\ \mathcal{X}_{t-1} \end{bmatrix}$, can express $\mathcal{X}_{t+1} = \phi\mathcal{X}_t + \mathcal{Z}_{t+1}$ in 1st row of matrix equation

$$\begin{bmatrix} \mathcal{X}_{t+1} \\ \mathcal{X}_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}_t \\ \mathcal{X}_{t-1} \end{bmatrix} + \begin{bmatrix} \mathcal{Z}_{t+1} \\ 0 \end{bmatrix},$$

which matches up with state-transition equation

$$\boldsymbol{X}_{t+1} = F_t \boldsymbol{X}_t + \boldsymbol{V}_t$$

if $F_t = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix}$ & $\boldsymbol{V}_t = \begin{bmatrix} \mathcal{Z}_{t+1} \\ 0 \end{bmatrix}$ with $Q_t \stackrel{\text{def}}{=} E\{\boldsymbol{V}_t \boldsymbol{V}_t'\} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$

- to complete model, let

$$\boldsymbol{X}_1 = \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_0 \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{\infty} \phi^j \mathcal{Z}_{1-j} \\ \sum_{j=0}^{\infty} \phi^j \mathcal{Z}_{-j} \end{bmatrix},$$

noting that $\boldsymbol{X}_1$ & $\boldsymbol{V}_t$ for $t \geq 1$ are uncorrelated, as required

# ARMA(1,1) Process as a State-Space Model: III

- since

$$E\{\boldsymbol{X}_1\boldsymbol{X}_1'] = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} = \frac{\sigma^2}{1-\phi^2}\begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix},$$

can alternatively stipulate

$$\boldsymbol{X}_1 = \begin{bmatrix} 1 & \frac{\phi}{\sqrt{(1-\phi^2)}} \\ 0 & \frac{1}{\sqrt{(1-\phi^2)}} \end{bmatrix} \begin{bmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_0 \end{bmatrix},$$

yielding

$$E\{\boldsymbol{X}_1\boldsymbol{X}_1'] = \begin{bmatrix} 1 & \frac{\phi}{\sqrt{(1-\phi^2)}} \\ 0 & \frac{1}{\sqrt{(1-\phi^2)}} \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\phi}{\sqrt{(1-\phi^2)}} & \frac{1}{\sqrt{(1-\phi^2)}} \end{bmatrix} = \frac{\sigma^2}{1-\phi^2}\begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix},$$

as required (recall that, if $\boldsymbol{U}$ has covariance $\Sigma$, then $A\boldsymbol{U}$ has covariance $A\Sigma A'$)

# ARMA(1,1) Process as a State-Space Model: IV

- stipulation
$$\boldsymbol{X}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_0 \end{bmatrix}$$
with arbitrary $a$, $b$, $c$ and $d$ still leads to a valid state-space model, but one that need not correspond to elements of $\boldsymbol{X}_t$ (and hence $\boldsymbol{Y}_t$) being part of a stationary process (but asymptotically they will be so)

- as in case of AR(1) process, can formulate state-space model for ARMA(1,1) process + observational noise by replacing $\boldsymbol{W}_t = 0$ and $R_t = 0$ with $W_t \sim \text{WN}(0, \sigma_W^2)$ and $R_t = \sigma_W^2$, where $W_t$'s are uncorrelated with $Z_t$'s

## Classical Decomposition Model Revisited (Again!)

- recall classical decomposition model for time series $Y_t$, namely,
$$Y_t = m_t + s_t + W_t,$$
where $m_t$ is trend; $s_t$ is periodic; and $W_t$ is a stationary process
- $m_t$ & $s_t$ can be treated as deterministic or stochastic
- to introduce basic ideas behind state-space methodology, will consider simple version of above known as a *local level* model in which $m_t$ is stochastic and $s_t = 0$:
$$\begin{aligned} Y_t &= X_t + W_t, \quad \{W_t\} \sim \mathrm{WN}(0, \sigma_W^2) \\ X_{t+1} &= X_t + V_t, \quad \{V_t\} \sim \mathrm{WN}(0, \sigma_V^2), \end{aligned}$$
where $E\{W_s V_t\} = 0$ for all $s$ & $t$
- déjà vu all over again: same model as used to motivate seasonal component in SARIMA models (overheads XV–40 & 41)

## Local Level Model: I

- comparing observation equation

$$\boldsymbol{Y}_t = G_t \boldsymbol{X}_t + \boldsymbol{W}_t, \;\; \boldsymbol{W}_t \sim \mathrm{UN}(\boldsymbol{0}, R_t)$$

with

$$Y_t = X_t + W_t, \;\; \{W_t\} \sim \mathrm{WN}(0, \sigma_W^2)$$

yields correspondences

$$\boldsymbol{Y}_t = Y_t, \;\; G_t = 1, \;\; \boldsymbol{X}_t = X_t, \;\; \boldsymbol{W}_t = W_t \text{ and } R_t = \sigma_W^2$$

- comparing state-transition equation

$$\boldsymbol{X}_{t+1} = F_t \boldsymbol{X}_t + \boldsymbol{V}_t, \;\; \boldsymbol{V}_t \sim \mathrm{UN}(\boldsymbol{0}, Q_t)$$

with

$$X_{t+1} = X_t + V_t, \;\; \{V_t\} \sim \mathrm{WN}(0, \sigma_V^2)$$

yields correspondences

$$F_t = 1, \;\; \boldsymbol{V}_t = V_t \text{ and } Q_t = \sigma_V^2$$

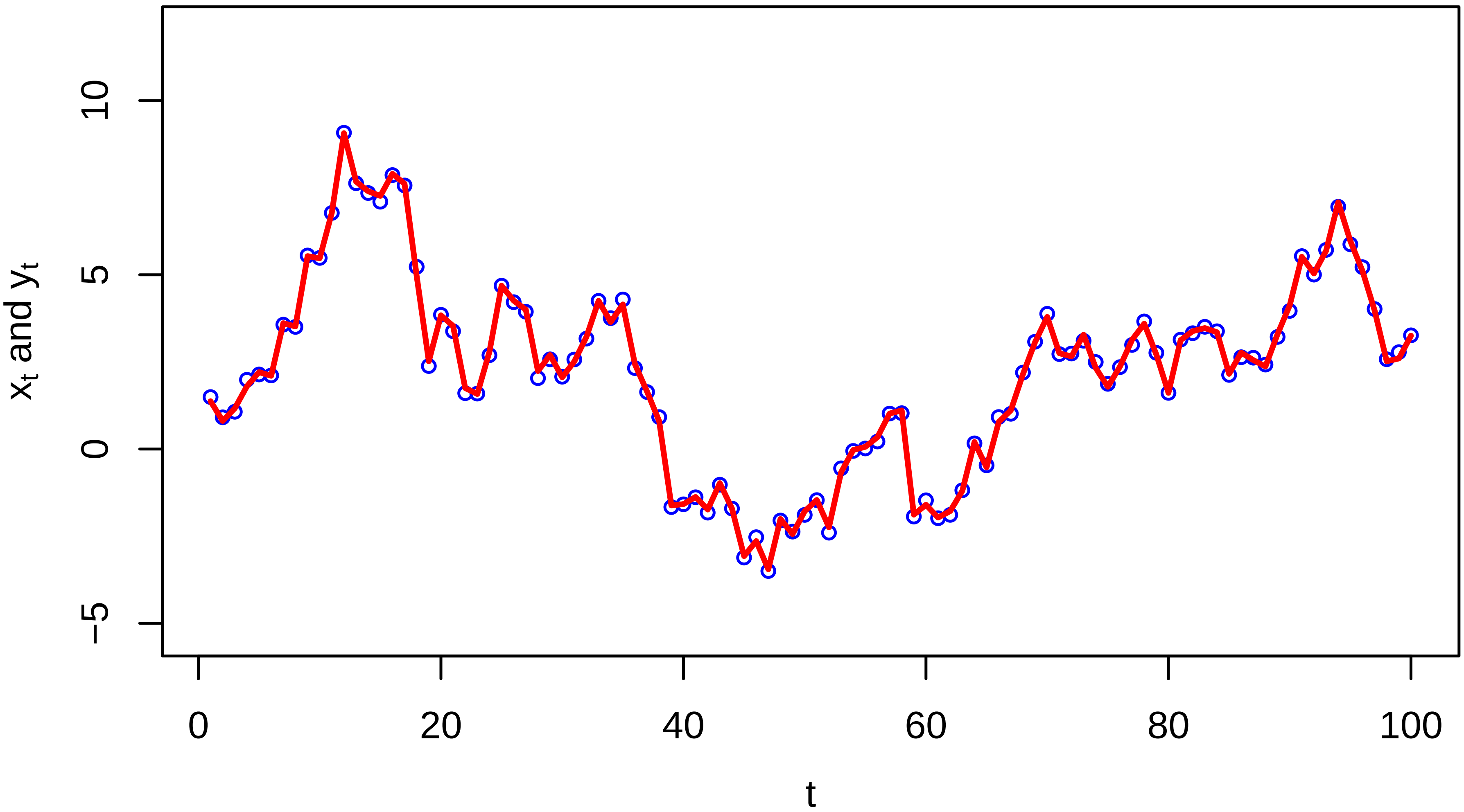
# Local Level Model: II

- to fully specify state-space model, need to define initial state $\boldsymbol{X}_1 = X_1$ to be an RV that is uncorrected with $W_t$'s and $V_t$'s
- in addition, will assume $E\{X_1\} = m_1$ and $\operatorname{var}\{X_1\} = P_1$
- model thus has 4 parameters: $\sigma_W^2 = R_t$, $\sigma_V^2 = Q_t$, $m_1$ & $P_1$
- since $X_{t+1} = X_t + V_t$, state variable $X_t$ is a random walk starting from $m_1$ (intended to model a slowly varying trend)
- since $V_t$ and $X_t$ are uncorrelated (why?),

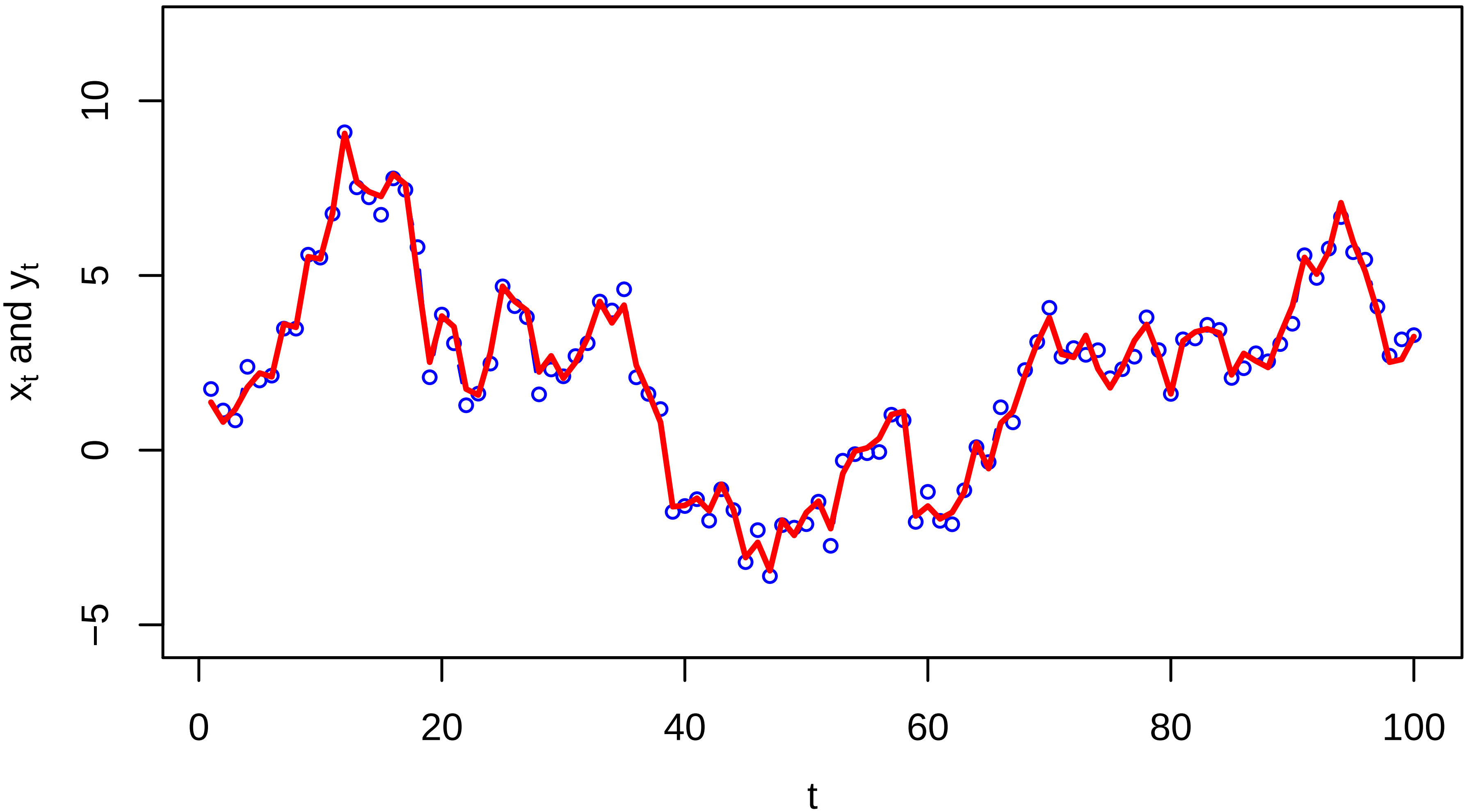
$$E\{X_{t+1} \mid X_t\} = E\{X_t + V_t \mid X_t\} = X_t + E\{V_t\} = X_t;$$

  i.e., if state variable is at a certain 'level' at time $t$, we can expect no change in its level at time $t + 1$
- when $\sigma_W^2 > 0$, trend is corrupted by noise, so ability to pick out trend depends upon 'signal to noise' (SNR) ratio $\sigma_V^2/\sigma_W^2$

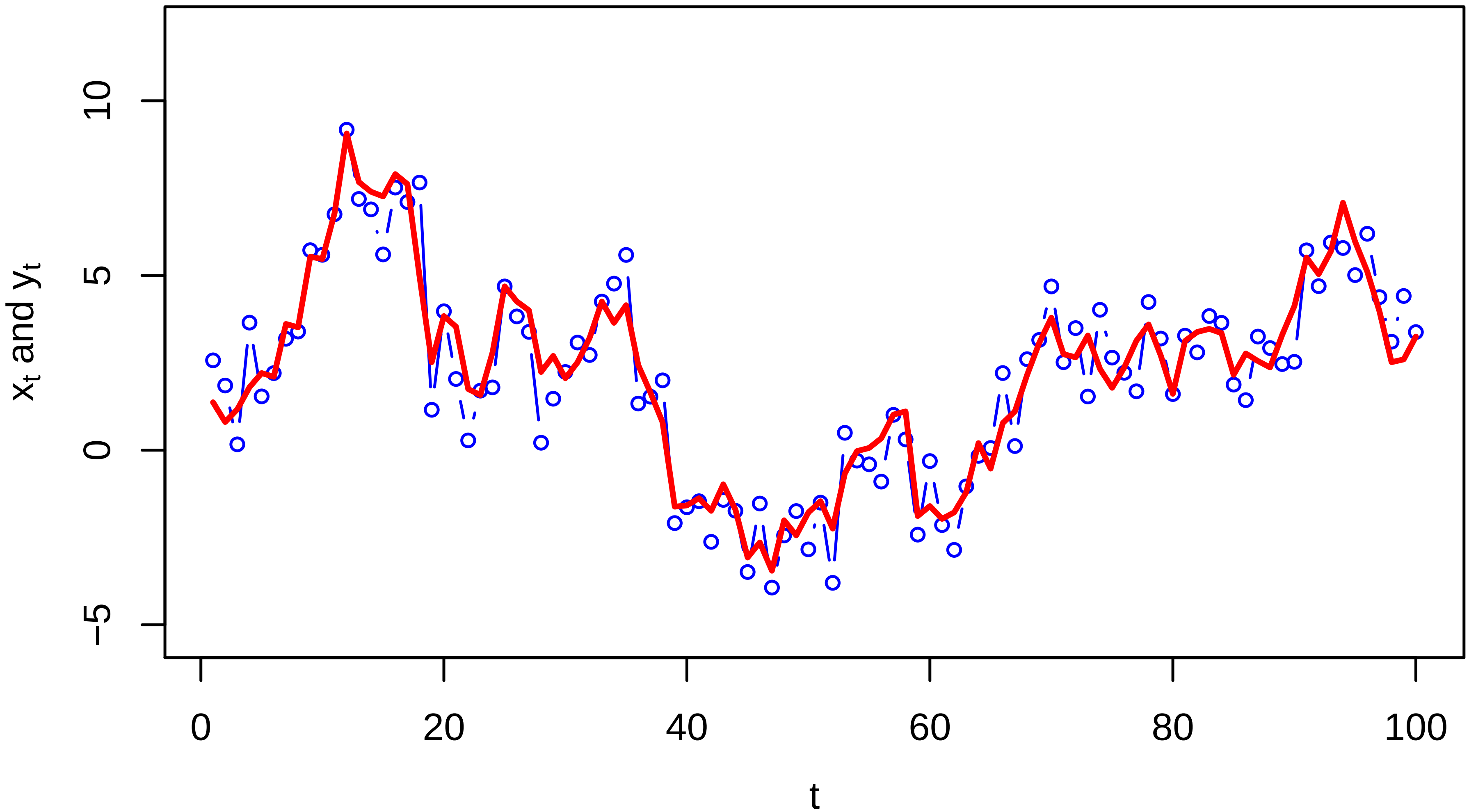
## Time Series $Y_t$ and Underlying State $X_t$, SNR = 100



**Time Series $Y_t$ and Underlying State $X_t$, SNR = 10**



## Time Series $Y_t$ and Underlying State $X_t$, SNR = 1



# Four Classical Problems in State-Space Models

- given observations $Y_1, \ldots, Y_t$ of a local level process,
  1. what is best predictor of state $X_t$? (filtering)
  2. what is best predictor of state $X_{t+1}$? (forecasting)
  3. what is best predictor of state $X_s$ for $s < t$? (smoothing)
  4. what are best estimates of model parameters? (estimation)
- note: if $\sigma_W^2 = 0$ (no observation noise), filtering and smoothing are not of interest (unless some observations are missing)
- will concentrate first on filtering & forecasting problems with 'best' taken to be minimum mean square error (MSE)
- to facilitate discussion, will assume that $X_1$, $V_t$'s and $W_t$ are multivariate normal (Gaussian)
  - implies that $Y_t$ and remaining $X_t$'s are also such

# Regression Lemma: I

- suppose random vectors $\boldsymbol{X}$ & $\boldsymbol{Y}$ are jointly normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma$, to be denoted by

$$\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$$

- can partition both $\boldsymbol{\mu}$ and $\Sigma$:

$$\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} \boldsymbol{\mu}_{\boldsymbol{X}} \\ \boldsymbol{\mu}_{\boldsymbol{Y}} \end{bmatrix}, \begin{bmatrix} \Sigma_{\boldsymbol{X}\boldsymbol{X}} & \Sigma_{\boldsymbol{X}\boldsymbol{Y}} \\ \Sigma'_{\boldsymbol{X}\boldsymbol{Y}} & \Sigma_{\boldsymbol{Y}\boldsymbol{Y}} \end{bmatrix} \right);$$

here $\boldsymbol{\mu}_{\boldsymbol{X}}$ & $\Sigma_{\boldsymbol{X}\boldsymbol{X}}$ are mean vector & covariance matrix for $\boldsymbol{X}$ and $\boldsymbol{\mu}_{\boldsymbol{Y}}$ & $\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}$ are those for $\boldsymbol{Y}$; i.e.,

$$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}}, \Sigma_{\boldsymbol{X}\boldsymbol{X}}) \text{ and } \boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}\boldsymbol{Y}});$$

$\Sigma_{\boldsymbol{X}\boldsymbol{Y}}$ is cross-covariance matrix between $\boldsymbol{X}$ & $\boldsymbol{Y}$; i.e., $(i, j)$th element is covariance between elements $i$ & $j$ of $\boldsymbol{X}$ & $\boldsymbol{Y}$

# Regression Lemma: II

- lemma: conditional distribution of $\boldsymbol{X}$ given $\boldsymbol{Y} = \boldsymbol{y}$ is multivariate normal with mean vector
$$\boldsymbol{\mu}_{\boldsymbol{X}|\boldsymbol{y}} = \boldsymbol{\mu}_{\boldsymbol{X}} + \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}(\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{Y}}) \qquad (*)$$
and covariance matrix
$$\Sigma_{\boldsymbol{X}|\boldsymbol{y}} = \Sigma_{\boldsymbol{X}\boldsymbol{X}} - \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\Sigma'_{\boldsymbol{X}\boldsymbol{Y}}$$
- $\boldsymbol{\mu}_{\boldsymbol{X}|\boldsymbol{y}}$ is mean of $\boldsymbol{X}$ given $\boldsymbol{Y} = \boldsymbol{y}$
- replacing realization $\boldsymbol{y}$ in right-hand side of $(*)$ with RV $\boldsymbol{Y}$ yields $E\{\boldsymbol{X} \mid \boldsymbol{Y}\}$, the conditional mean of $\boldsymbol{X}$ given $\boldsymbol{Y}$:
$$E\{\boldsymbol{X} \mid \boldsymbol{Y}\} = \boldsymbol{\mu}_{\boldsymbol{X}} + \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}(\boldsymbol{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}}) \qquad (\dagger)$$
- best predictor (here linear) of $\boldsymbol{X}$ given $\boldsymbol{Y}$ is $E\{\boldsymbol{X} \mid \boldsymbol{Y}\}$, which $(\dagger)$ makes clear is an RV constructed from $\boldsymbol{Y}$
- can see from $(\dagger)$ that $E\{E\{\boldsymbol{X} \mid \boldsymbol{Y}\}\} = \boldsymbol{\mu}_{\boldsymbol{X}}$

# Regression Lemma: III

- recall that, if random vector $\boldsymbol{\mathcal{U}}$ has covariance matrix $\Sigma_{\boldsymbol{\mathcal{U}}}$, then covariance matrix for $A\boldsymbol{\mathcal{U}}$ is $A\Sigma_{\boldsymbol{\mathcal{U}}}A'$
  - implies covariance matrix for $\boldsymbol{c} + A(\boldsymbol{\mathcal{U}} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{U}}})$ is also $A\Sigma_{\boldsymbol{\mathcal{U}}}A'$ ($\boldsymbol{c}$ and $\boldsymbol{\mu}_{\boldsymbol{\mathcal{U}}}$ are vectors with constants)
- covariance matrix for

$$E\{\boldsymbol{X} \mid \boldsymbol{Y}\} = \boldsymbol{\mu}_{\boldsymbol{X}} + \Sigma_{\boldsymbol{XY}}\Sigma_{\boldsymbol{YY}}^{-1}(\boldsymbol{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})$$

is thus

$$\Sigma_{\boldsymbol{XY}}\Sigma_{\boldsymbol{YY}}^{-1}\Sigma_{\boldsymbol{YY}}\Sigma_{\boldsymbol{YY}}^{-1}\Sigma'_{\boldsymbol{XY}} = \Sigma_{\boldsymbol{XY}}\Sigma_{\boldsymbol{YY}}^{-1}\Sigma'_{\boldsymbol{XY}}$$

(note: *not* the same as $\Sigma_{\boldsymbol{X}|\boldsymbol{y}} = \Sigma_{\boldsymbol{XX}} - \Sigma_{\boldsymbol{XY}}\Sigma_{\boldsymbol{YY}}^{-1}\Sigma'_{\boldsymbol{XY}}$, the covariance matrix for $\boldsymbol{X}$ given $\boldsymbol{Y} = \boldsymbol{y}$)

# Regression Lemma: IV

- consider prediction error $\boldsymbol{U}$ associated with best linear predictor of $\boldsymbol{X}$:

$$\boldsymbol{U} = \boldsymbol{X} - E\{\boldsymbol{X} \mid \boldsymbol{Y}\}$$

- since $E\{E\{\boldsymbol{X} \mid \boldsymbol{Y}\}\} = \boldsymbol{\mu}_{\boldsymbol{X}}$, follows that $E\{\boldsymbol{U}\} = \boldsymbol{0}$
- covariance matrix for $\boldsymbol{U}$ is given by

$$\begin{aligned} E\{\boldsymbol{U}\boldsymbol{U}'\} &= E\{(\boldsymbol{X} - E\{\boldsymbol{X} \mid \boldsymbol{Y}\})(\boldsymbol{X} - E\{\boldsymbol{X} \mid \boldsymbol{Y}\})'\} \\ &= E\{\boldsymbol{X}\boldsymbol{X}'\} + E\{E\{\boldsymbol{X} \mid \boldsymbol{Y}\}E\{\boldsymbol{X} \mid \boldsymbol{Y}\}'\} \\ &\quad - E\{\boldsymbol{X}E\{\boldsymbol{X} \mid \boldsymbol{Y}\}'\} - E\{E\{\boldsymbol{X} \mid \boldsymbol{Y}\}\boldsymbol{X}'\} \\ &= \Sigma_{\boldsymbol{X}\boldsymbol{X}} - \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\Sigma_{\boldsymbol{X}\boldsymbol{Y}}', \end{aligned}$$

which *is* equal to $\Sigma_{\boldsymbol{X}|\boldsymbol{y}}$, the conditional covariance matrix (see next overhead for details)

## Some Details

- without loss of generality, can assume $\boldsymbol{\mu}_{\boldsymbol{X}} = \mathbf{0}$ & $\boldsymbol{\mu}_{\boldsymbol{Y}} = \mathbf{0}$
- by definition, $E\{\boldsymbol{X}\boldsymbol{X}'\} = \Sigma_{\boldsymbol{X}\boldsymbol{X}}$
- since $E\{\boldsymbol{X} \mid \boldsymbol{Y}\} = \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\boldsymbol{Y}$,

$$\begin{aligned}
E\{E\{\boldsymbol{X} \mid \boldsymbol{Y}\}E\{\boldsymbol{X} \mid \boldsymbol{Y}\}'\} &= E\{\Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\boldsymbol{Y}\boldsymbol{Y}'\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\Sigma_{\boldsymbol{X}\boldsymbol{Y}}'\} \\
&= \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\Sigma_{\boldsymbol{X}\boldsymbol{Y}}' \\
E\{\boldsymbol{X}E\{\boldsymbol{X} \mid \boldsymbol{Y}\}'\} &= E\{\boldsymbol{X}\boldsymbol{Y}'\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\Sigma_{\boldsymbol{X}\boldsymbol{Y}}'\} \\
&= \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\Sigma_{\boldsymbol{X}\boldsymbol{Y}}' \\
E\{E\{\boldsymbol{X} \mid \boldsymbol{Y}\}\boldsymbol{X}'\} &= E\{\Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\boldsymbol{Y}\boldsymbol{X}'\} \\
&= \Sigma_{\boldsymbol{X}\boldsymbol{Y}}\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\Sigma_{\boldsymbol{X}\boldsymbol{Y}}',
\end{aligned}$$

  the last line being because $E\{\boldsymbol{Y}\boldsymbol{X}'\} = \Sigma_{\boldsymbol{Y}\boldsymbol{X}} = \Sigma_{\boldsymbol{X}\boldsymbol{Y}}'$
- putting all the pieces together yields desired result

# Regression Corollary

- specialize now to case where $\boldsymbol{X}$ has just one element, say, $X$
- corollary: conditional distribution of $X$ given $\boldsymbol{Y} = \boldsymbol{y}$ is normal with mean
$$\mu_X + \boldsymbol{\Sigma}'_{X\boldsymbol{Y}} \Sigma^{-1}_{\boldsymbol{Y}\boldsymbol{Y}} (\boldsymbol{y} - \boldsymbol{\mu}_{\boldsymbol{Y}})$$
and variance
$$\Sigma_{X|\boldsymbol{y}} = \sigma^2_X - \boldsymbol{\Sigma}'_{X\boldsymbol{Y}} \Sigma^{-1}_{\boldsymbol{Y}\boldsymbol{Y}} \boldsymbol{\Sigma}_{X\boldsymbol{Y}},$$
where $\sigma^2_X = \text{var}\{X\}$ and $\boldsymbol{\Sigma}_{X\boldsymbol{Y}}$ is a column vector containing covariance between $X$ and RVs in $\boldsymbol{Y}$
- since conditional variance is same as MSE for $X$, will refer to $\Sigma_{X|\boldsymbol{y}}$ as MSE

## Aside – Revisiting an Old Friend: I

- suppose $\{X_t\}$ is zero mean stationary process with ACVF $\gamma(h)$
- set $X$ to $X_{n+1}$ and put $X_1, \ldots, X_n$ into $\boldsymbol{Y}$
- corollary says best linear predictor $\widehat{X}_{n+1}$ of $X_{n+1}$ given $X_n$, …, $X_1$ is

$$\widehat{X}_{n+1} = \boldsymbol{\Sigma}'_{X\boldsymbol{Y}} \Sigma^{-1}_{\boldsymbol{Y}\boldsymbol{Y}} \boldsymbol{Y} = \boldsymbol{\gamma}'_n \Gamma_n^{-1} \boldsymbol{Y} \stackrel{\text{def}}{=} \boldsymbol{\phi}'_n \boldsymbol{Y},$$

  where

  – $\boldsymbol{\gamma}_n = [\gamma(1), \gamma(2), \ldots, \gamma(n)]' = \boldsymbol{\Sigma}_{X\boldsymbol{Y}}$

  – $(i,j)$th entry of matrix $\Gamma_n = \Sigma_{\boldsymbol{Y}\boldsymbol{Y}}$ is $\gamma(i-j)$

  – $\boldsymbol{\phi}'_n \stackrel{\text{def}}{=} \boldsymbol{\gamma}'_n \Gamma_n^{-1}$ and hence $\boldsymbol{\phi}_n = \Gamma_n^{-1} \boldsymbol{\gamma}_n$

- $\boldsymbol{\phi}_n$ solves $\Gamma_n \boldsymbol{\phi}_n = \boldsymbol{\gamma}_n$, a result previously seen (overhead XI–2)

## Aside – Revisiting an Old Friend: II

- overhead XI–3 states that MSE for $\widehat{X}_{n+1}$ is

$$
\begin{aligned}
v_n &= \operatorname{var}\{X_{n+1}\} - \boldsymbol{\phi}_n'\boldsymbol{\gamma}_n \\
&= \sigma_X^2 - \boldsymbol{\gamma}_n'\Gamma_n^{-1}\boldsymbol{\gamma}_n \text{ (making use of } \boldsymbol{\phi}_n' \stackrel{\text{def}}{=} \boldsymbol{\gamma}_n'\Gamma_n^{-1}) \\
&= \sigma_X^2 - \boldsymbol{\Sigma}_{X\boldsymbol{Y}}'\Sigma_{\boldsymbol{Y}\boldsymbol{Y}}^{-1}\boldsymbol{\Sigma}_{X\boldsymbol{Y}} \\
&= \Sigma_{X|\boldsymbol{y}}
\end{aligned}
$$

- thus 'old friend' is a special case of regression corollary

# Filtering for Local Level Model: I

- return now to local level model:

$$\begin{aligned} Y_t &= X_t + W_t, \ \ \{W_t\} \sim \mathrm{WN}(0, \sigma_W^2) \\ X_{t+1} &= X_t + V_t, \quad \ \{V_t\} \sim \mathrm{WN}(0, \sigma_V^2), \end{aligned}$$

  and $\boldsymbol{X}_1 = X_1$ is an RV that
  - is uncorrected with $W_t$'s and $V_t$'s
  - has $E\{X_1\} = m_1$ and $\mathrm{var}\,\{X_1\} = P_1$
- filtering problem is to predict unknown state $X_t$ based on data up to time $t$, i.e., $Y_1, \ldots, Y_t$
- in what follows, let $\boldsymbol{Y}_t \stackrel{\mathrm{def}}{=} [Y_1, \ldots, Y_t]'$

# Filtering for Local Level Model: II

- best linear predictor of $X_t$ given $\boldsymbol{Y}_t$ is

$$\widehat{X}_{t|t} \stackrel{\text{def}}{=} E\{X_t \mid \boldsymbol{Y}_t\} = m_t + \boldsymbol{\Sigma}'_{t,t}\Sigma_t^{-1}\left(\boldsymbol{Y}_t - \boldsymbol{m}_t\right), \quad \text{where}$$

  – $m_t = E\{X_t\}$
  – vector $\boldsymbol{\Sigma}_{t,t}$ contains covariances between $X_t$ & $Y_1, \ldots, Y_t$
  – $(j, k)$th element of matrix $\Sigma_t$ is covariance between $Y_j$ & $Y_k$
  – $\boldsymbol{m}_t$ is a vector containing, for $j = 1, \ldots, t$,

$$m_j \stackrel{\text{def}}{=} E\{X_j\} = E\{X_j + W_j\} = E\{Y_j\}$$

- note: $E\{\widehat{X}_{t|t}\} = E\{E\{X_t \mid \boldsymbol{Y}_t\}\} = E\{X_t\} = m_t$
- with $P_t \stackrel{\text{def}}{=} \text{var}\,\{X_t\}$, MSE for predictor is

$$E\{(X_t - \widehat{X}_{t|t})^2\} = P_t - \boldsymbol{\Sigma}'_{t,t}\Sigma_t^{-1}\boldsymbol{\Sigma}_{t,t} \stackrel{\text{def}}{=} P_{t|t} \text{ (see overhead XVIII–26)}$$

## Forecasting for Local Level Model: I

- best linear predictor of $X_{t+1}$ given $\boldsymbol{Y}_t$ is

$$\widehat{X}_{t+1|t} \stackrel{\text{def}}{=} E\{X_{t+1} \mid \boldsymbol{Y}_t\} = m_{t+1} + \boldsymbol{\Sigma}'_{t+1,t}\Sigma_t^{-1}(\boldsymbol{Y}_t - \boldsymbol{m}_t),$$

where vector $\boldsymbol{\Sigma}_{t+1,t}$ has covariances between $X_{t+1}$ & $\boldsymbol{Y}_t$

- note: $E\{\widehat{X}_{t+1|t}\} = E\{E\{X_{t+1} \mid \boldsymbol{Y}_t\}\} = E\{X_{t+1}\} = m_{t+1}$
- MSE for predictor is

$$E\{(X_{t+1} - \widehat{X}_{t+1|t})^2\} = P_{t+1} - \boldsymbol{\Sigma}'_{t+1,t}\Sigma_t^{-1}\boldsymbol{\Sigma}_{t+1,t} \stackrel{\text{def}}{=} P_{t+1|t}$$

# Forecasting for Local Level Model: II

- let's also consider best linear predictor of $Y_{t+1}$ given $\boldsymbol{Y}_t$:

$$\widehat{Y}_{t+1|t} \stackrel{\text{def}}{=} E\{Y_{t+1} \mid \boldsymbol{Y}_t\} = m_{t+1} + \widetilde{\boldsymbol{\Sigma}}'_{t+1,t}\Sigma_t^{-1}\left(\boldsymbol{Y}_t - \boldsymbol{m}_t\right),$$

  where vector $\widetilde{\boldsymbol{\Sigma}}_{t+1,t}$ has covariances between $Y_{t+1}$ & $\boldsymbol{Y}_t$

- however, note that, for $j = 1, \ldots, t$

$$\text{cov}\left\{Y_{t+1}, Y_j\right\} = \text{cov}\left\{X_{t+1} + W_{t+1}, Y_j\right\} = \text{cov}\left\{X_{t+1}, Y_j\right\}$$

- thus $\widetilde{\boldsymbol{\Sigma}}_{t+1,t} = \boldsymbol{\Sigma}_{t+1,t}$ (since latter has covariances between $X_{t+1}$ & $\boldsymbol{Y}_t$), yielding

$$\widehat{Y}_{t+1|t} = m_{t+1} + \boldsymbol{\Sigma}'_{t+1,t}\Sigma_t^{-1}\left(\boldsymbol{Y}_t - \boldsymbol{m}_t\right) = \widehat{X}_{t+1|t},$$

  an 'obvious' result: difference between $Y_{t+1}$ and $X_{t+1}$ is $W_{t+1}$, for which best predictor at time $t$ is zero!

## Forecasting for Local Level Model: III

- although $\widehat{Y}_{t+1|t}$ & $\widehat{X}_{t+1|t}$ are identical, their MSEs differ:

$$\begin{aligned} E\{(Y_{t+1} - \widehat{Y}_{t+1|t})^2\} &= E\{(X_{t+1} + W_{t+1} - \widehat{X}_{t+1|t})^2\} \\ &= E\{(X_{t+1} - \widehat{X}_{t+1|t})^2\} + \sigma_W^2 \\ &= P_{t+1|t} + \sigma_W^2 \end{aligned}$$

## Filtering for Local Level Model: III

- to implement filtering (i.e., compute $\widehat{X}_{t|t}$), need to determine
  1. $m_j = E\{X_j\}$, $j = 1, \ldots, t$
  2. elements of $\boldsymbol{\Sigma}_{t,t}$, i.e., covariances between $X_t$ & $Y_1, \ldots, Y_t$
  3. elements of $\Sigma_t$, i.e., covariances between $Y_j$ & $Y_k$, $1 \leq j \leq k \leq t$
- to compute $P_{t|t}$, i.e., MSE for $\widehat{X}_{t|t}$, need $P_t = \text{var}\{X_t\}$ in addition to 2 and 3
- since $X_t = X_{t-1} + V_{t-1}$ and $Y_t = X_t + W_t$, telescoping yields

$$X_j = X_1 + \sum_{l=1}^{j-1} V_l \text{ and } Y_j = X_1 + \sum_{l=1}^{j-1} V_l + W_j, \ \ j = 1, \ldots, t$$

(above holds for $j = 1$ if we interpret $\sum_{l=1}^{0} V_l$ as zero)

## Filtering for Local Level Model: IV

- using

$$X_j = X_1 + \sum_{l=1}^{j-1} V_l \text{ and } Y_j = X_1 + \sum_{l=1}^{j-1} V_l + W_j, \quad j = 1, \ldots, t,$$

get $m_j = E\{X_j\} = E\{X_1\} = m_1$ and (assuming $j \le k \le t$)

$$\begin{aligned}
\operatorname{cov}\{X_t, Y_j\} &= \operatorname{cov}\{X_1 + \sum_{l=1}^{t-1} V_l, X_1 + \sum_{l=1}^{j-1} V_l + W_j\} \\
&= P_1 + (j-1)\sigma_V^2 \\
\operatorname{cov}\{Y_j, Y_k\} &= \operatorname{cov}\{X_1 + \sum_{l=1}^{j-1} V_l + W_j, X_1 + \sum_{l=1}^{k-1} V_l + W_k\} \\
&= P_1 + (j-1)\sigma_V^2 + \delta_{j,k}\sigma_W^2,
\end{aligned}$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$

## Filtering for Local Level Model: V

- using
$$X_t = X_1 + \sum_{l=1}^{t-1} V_l,$$
get
$$P_t = \text{var}\{X_t\} = P_1 + (t-1)\sigma_V^2$$
- now have all the pieces needed to form $\widehat{X}_{t|t}$ and its MSE $P_{t|t}$
- note: similar argument leads to pieces needed to form forecast $\widehat{X}_{t+1|t}$ and its MSE $P_{t+1|t}$

# Kalman Recursions for Filtering/Forecasting: I

- critique: while straightforward conceptually, forming
$$\widehat{X}_{t|t} = m_t + \boldsymbol{\Sigma}'_{t,t}\Sigma_t^{-1}(\boldsymbol{Y}_t - \boldsymbol{m}_t)$$
and
$$\widehat{X}_{t+1|t} = m_{t+1} + \boldsymbol{\Sigma}'_{t+1,t}\Sigma_t^{-1}(\boldsymbol{Y}_t - \boldsymbol{m}_t)$$
via these equations requires inversion of matrix $\Sigma_t$ whose dimension $t \times t$ can become problematic as $t$ gets large
- celebrated *Kalman recursions* give a recipe that avoids explicit matrix inversion
- mindset is that, at time $t-1$, we have 4 quantities of interest:
  - filtered value $\widehat{X}_{t-1|t-1}$ and forecast $\widehat{X}_{t|t-1}$
  - associated MSEs $P_{t-1|t-1}$ and $P_{t|t-1}$
- note: $\widehat{X}_{t-1|t-1} = \widehat{X}_{t|t-1}$ for local level model (but not others)

## Kalman Recursions for Filtering/Forecasting: II

- at time $t$, new observation $Y_t$ becomes available
- Kalman recursion takes $\widehat{X}_{t|t-1}$, $P_{t|t-1}$ & $Y_t$ and yields
  - filtered value $\widehat{X}_{t|t}$ and forecast $\widehat{X}_{t+1|t}$
  - associated MSEs $P_{t|t}$ and $P_{t+1|t}$
- there are six steps in the Kalman recursion:
  - steps 1 & 2 are preparatory
  - steps 3 & 4 yield $\widehat{X}_{t|t}$ and $P_{t|t}$ (filtering)
  - steps 5 & 6 yield $\widehat{X}_{t+1|t}$ and $P_{t+1|t}$ (forecasting)

BD: 270, CC: 224, SS: 295–296

# Kalman Recursions for Filtering/Forecasting: III

1. compute innovation:

$$U_t = Y_t - \widehat{Y}_{t|t-1} = Y_t - \widehat{X}_{t|t-1}$$

2. compute MSE for $\widehat{Y}_{t|t-1}$ (i.e., var $\{U_t\}$ since $E\{U_t\} = 0$):

$$P_{t|t-1} + \sigma_W^2 \stackrel{\text{def}}{=} F_t$$

3. compute new filtered value:

$$\widehat{X}_{t|t} = \widehat{X}_{t|t-1} + K_t U_t,$$

   where $K_t \stackrel{\text{def}}{=} P_{t|t-1}/F_t$ is the so-called *Kalman gain*

4. compute MSE for new filtered value:

$$P_{t|t} = P_{t|t-1}(1 - K_t)$$

## Kalman Recursions for Filtering/Forecasting: IV

5. compute new forecast:

$$\widehat{X}_{t+1|t} = \widehat{X}_{t|t-1} + K_t U_t = \widehat{X}_{t|t}$$

6. compute MSE for new forecast:

$$P_{t+1|t} = P_{t|t-1}(1 - K_t) + \sigma_V^2 = P_{t|t} + \sigma_V^2$$

(makes sense in view of $X_{t+1} = X_t + V_t$: since predictor $\widehat{X}_{t|t}$ for $X_t$ has MSE $P_{t|t}$, MSE for predictor $\widehat{X}_{t+1|t}$ for $X_{t+1}$ must be $P_{t|t} + \sigma_V^2$)

- recursions are carried out for $t = 1, \ldots, n$, with inputs at $t = 1$ being $\widehat{X}_{1|0} \stackrel{\text{def}}{=} m_1 = E\{X_1\}$, $P_{1|0} \stackrel{\text{def}}{=} P_1 = \text{var}\,\{X_1\}$ and $Y_1$

## Another Regression Corollary

- can prove that Kalman recursion works as advertised using regression lemma specialized to case where both $\boldsymbol{X}$ and $\boldsymbol{Y}$ have a single element
- corollary: conditional distribution of $X$ given $Y$ is normal with mean

$$E\{X \mid Y\} = \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(Y - \mu_Y)$$

and conditional variance

$$\sigma_{X|Y}^2 = \sigma_X^2 - \frac{\sigma_{XY}^2}{\sigma_Y^2},$$

where $\mu_X = E\{X\}$, $\mu_Y = E\{Y\}$, $\sigma_X^2 = \text{var}\,\{X\}$, $\sigma_Y^2 = \text{var}\,\{Y\}$ and $\sigma_{XY} = \text{cov}\,\{X, Y\}$

## Kalman Recursions for Filtering/Forecasting: V

- to prove validity of steps 3 and 4, need to show that
  - $\widehat{X}_{t|t-1} + K_t U_t$ is equal to $\widehat{X}_{t|t}$
  - $P_{t|t-1}(1 - K_t)$ is equal to $P_{t|t}$
- in corollary, let $X$ be $X_t$ with $\mathbf{Y}_{t-1}$ held fixed so that

$$\mu_X = E\{X_t \mid \mathbf{Y}_{t-1}\} = \widehat{X}_{t|t-1} \text{ and } \sigma_X^2 = \text{var}\,\{X_t \mid \mathbf{Y}_{t-1}\} = P_{t|t-1}$$

- let $Y$ be $U_t$ so $\mu_Y = E\{U_t\} = 0$, $\sigma_Y^2 = \text{var}\,\{U_t\} = F_t$ and

$$\begin{aligned}
\sigma_{XY} &= \text{cov}\,\{X_t, U_t \mid \mathbf{Y}_{t-1}\} \\
&= \text{cov}\,\{X_t, Y_t - \widehat{Y}_{t|t-1} \mid \mathbf{Y}_{t-1}\} \\
&= \text{cov}\,\{X_t, X_t + W_t \mid \mathbf{Y}_{t-1}\} \\
&= \text{var}\,\{X_t \mid \mathbf{Y}_{t-1}\} = P_{t|t-1}
\end{aligned}$$

# Kalman Recursions for Filtering/Forecasting: VI

- key fact: $X_t$ conditioned on both $U_t = Y_t - \widehat{Y}_{t|t-1}$ and $\boldsymbol{Y}_{t-1}$ is *same* as $X_t$ conditioned on $\boldsymbol{Y}_t$ ($\widehat{Y}_{t|t-1}$ is set if $\boldsymbol{Y}_{t-1}$ is known)
- plugging all the pieces into

  $$E\{X \mid Y\} = \mu_X + \frac{\sigma_{XY}}{\sigma_Y^2}(Y - \mu_Y) \;\text{ and }\; \sigma^2_{X|Y} = \sigma^2_X - \frac{\sigma^2_{XY}}{\sigma^2_Y}$$

  yields

  $$\widehat{X}_{t|t} = \widehat{X}_{t|t-1} + \frac{P_{t|t-1}}{F_t}U_t \;\text{ and }\; P_{t|t} = P_{t|t-1} - \frac{P^2_{t|t-1}}{F_t};$$

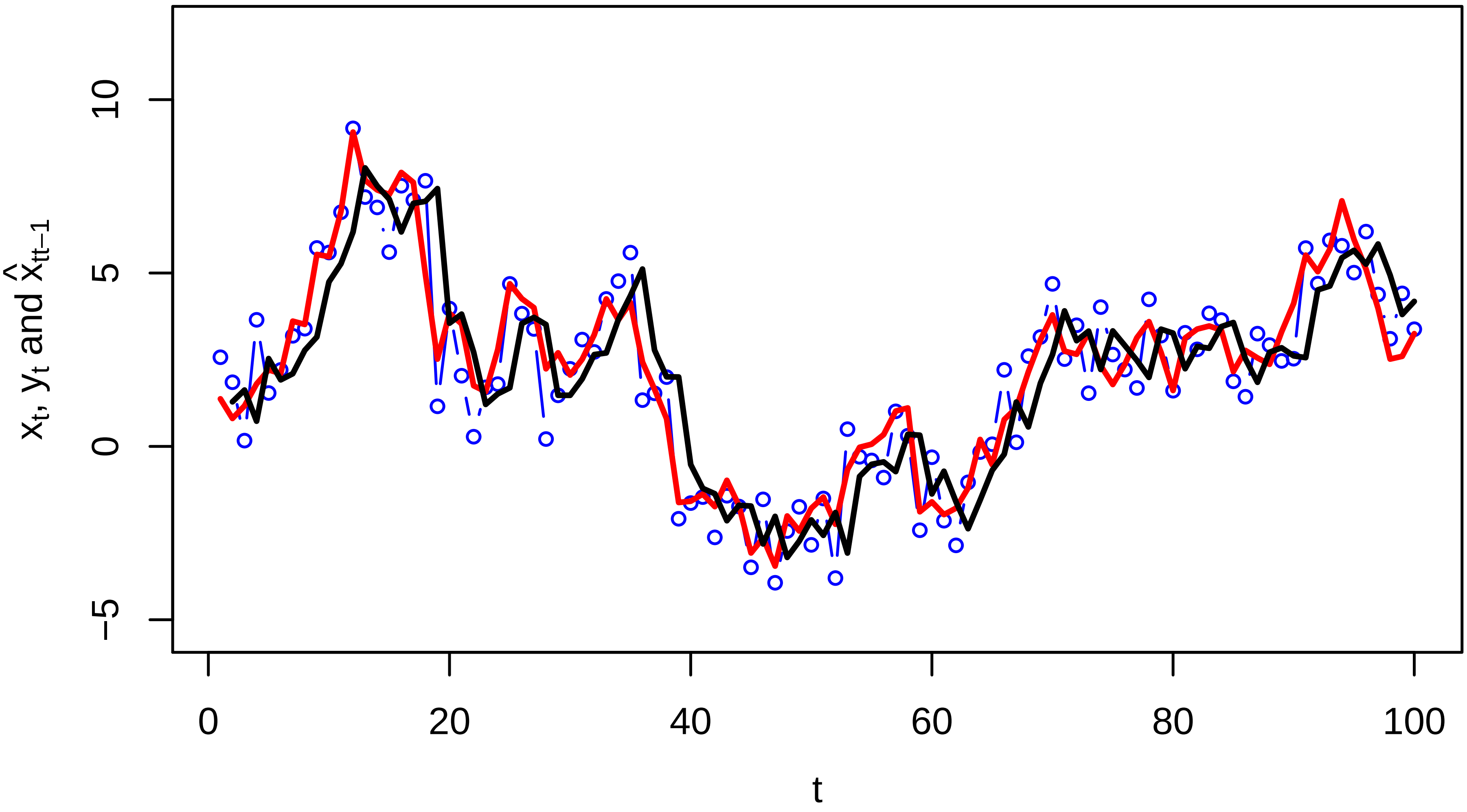
  i.e., since $K_t \stackrel{\text{def}}{=} P_{t|t-1}/F_t$, we get required

  $$\widehat{X}_{t|t} = \widehat{X}_{t|t-1} + K_t U_t \;\text{ and }\; P_{t|t} = P_{t|t-1}(1 - K_t)$$
- note: proof of steps 5 and 6 of Kalman recursion is analogous

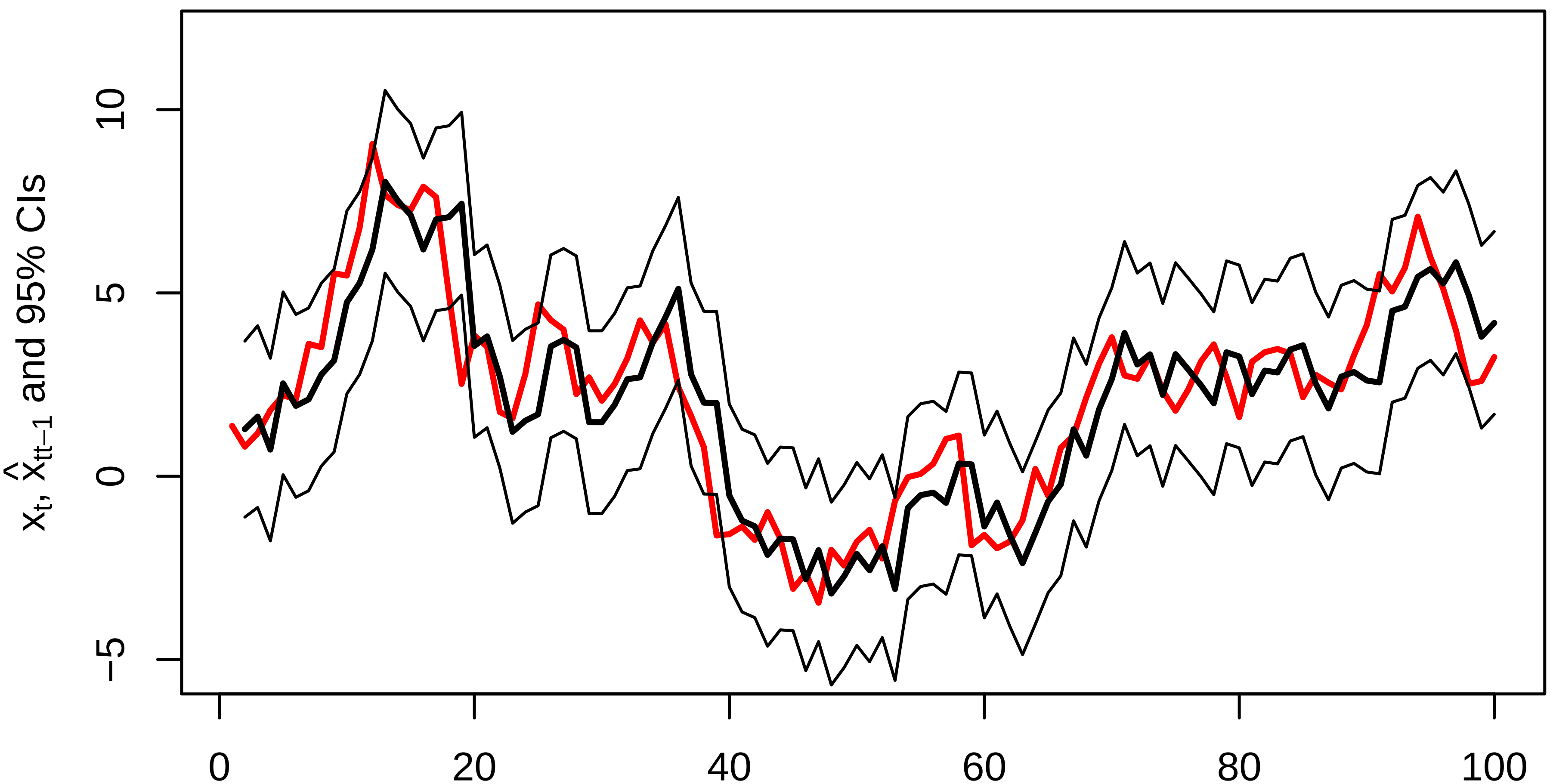
## Kalman Recursions for Filtering/Forecasting: VII

- proof of Kalman recursion for local level model adapted from Durbin & Koopman (2012)
- proofs for general state-space models are in Brockwell & Davis and Shumway & Stoffer textbooks and also in
  - Diderrich (1985)
  - Eubank (2006)
  - Meinhold & Singpurwalla (1983)
  - Petris, Petrone & Campagnoli (2009)
- as an example, let's apply Kalman filter to time series from simulated local level model with SNR = 1

## Time Series $Y_t$, States $X_t$ and Forecasts $\widehat{X}_{t|t-1}$



# States $X_t$, Forecasts $\widehat{X}_{t|t-1}$ & 95% CIs Based on $P_{t|t-1}$



## Kalman Recursions for Filtering/Forecasting: VIII

- for a time series of length $n$, last use of Kalman recusions gives forecast $\widehat{X}_{n+1|n}$ for future state $X_{n+1}$, along with error variance $P_{n+1|n}$ useful for constucting 95% confidence interval:

$$\left[\widehat{X}_{n+1|n} - 1.96\sqrt{P_{n+1|n}},\ \widehat{X}_{n+1|n} + 1.96\sqrt{P_{n+1|n}}\right]$$

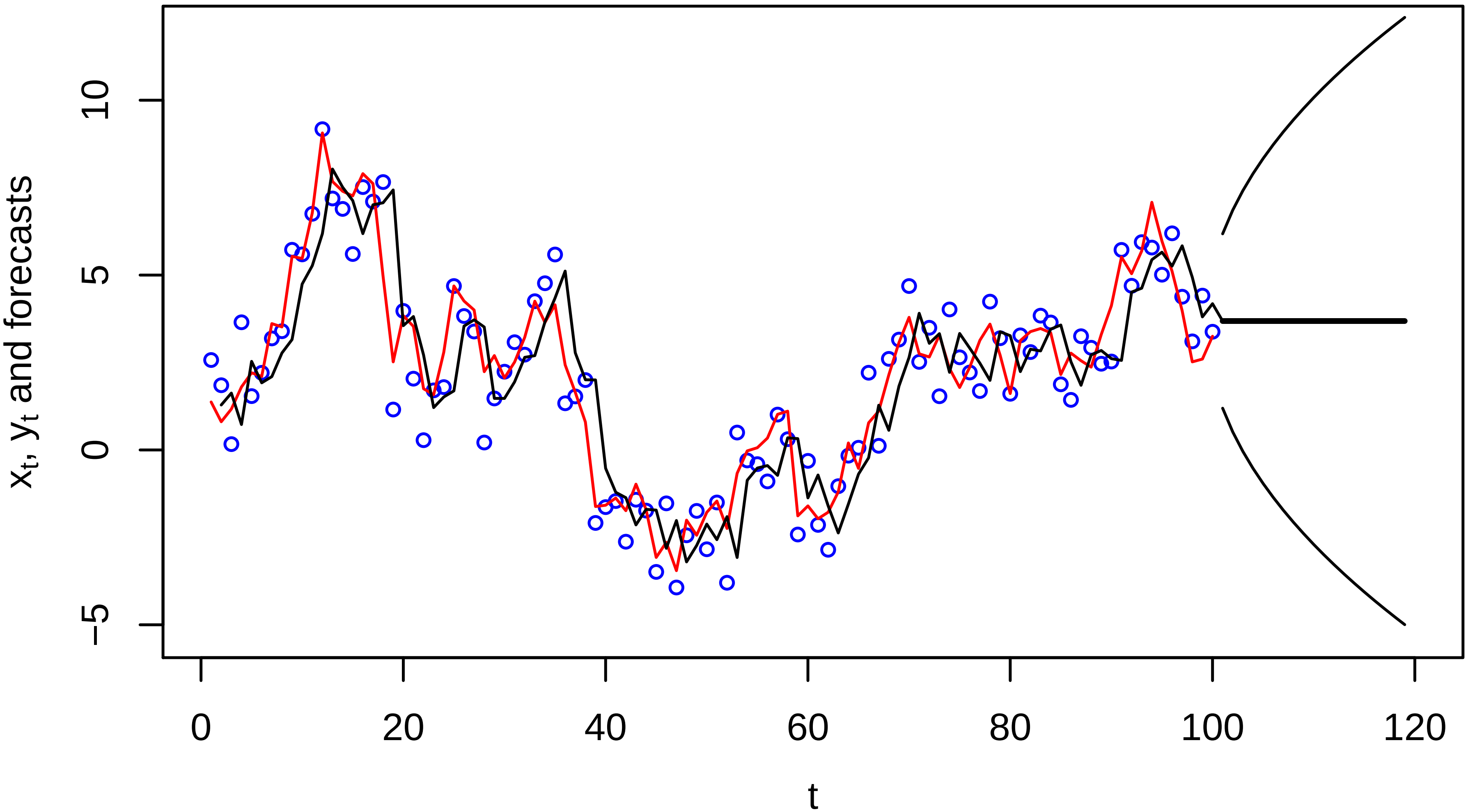
- to forecast $X_{n+2}$, enter $\widehat{X}_{n+1|n}$ and $P_{n+1|n}$ into *modified recursion* that differs from usual six-step recursion as follows:

  1. skip computation of innovation ($Y_{n+1}$ is unavailable)
  2. do as usual
  3. do as usual, but with $K_t U_t$ set to 0
  4. do as usual, but with Kalman gain $K_t$ set to 0
  5. do as usual, but with $K_t U_t$ set to 0
  6. do as usual, but with $K_t$ set to 0

# Kalman Recursions for Filtering/Forecasting: IX

- in short: proceed as usual, but set Kalman gain $K_t$ to 0
- recursion for $t = n + 1$ yields $\widehat{X}_{n+2|n}$ and associated error variance $P_{n+2|n}$
- to forecast $X_{n+3}$, enter $\widehat{X}_{n+2|n}$ and $P_{n+2|n}$ into modified recursion to obtain $\widehat{X}_{n+3|n}$ and $P_{n+3|n}$
- repeat as needed to get forecasts as far into future as desired
- as an example, let's forecast $X_{101}, \ldots, X_{120}$ for simulated local level model with SNR $= 1$

## Time Series $Y_t$, States $X_t$, Forecasts and 95% CIs



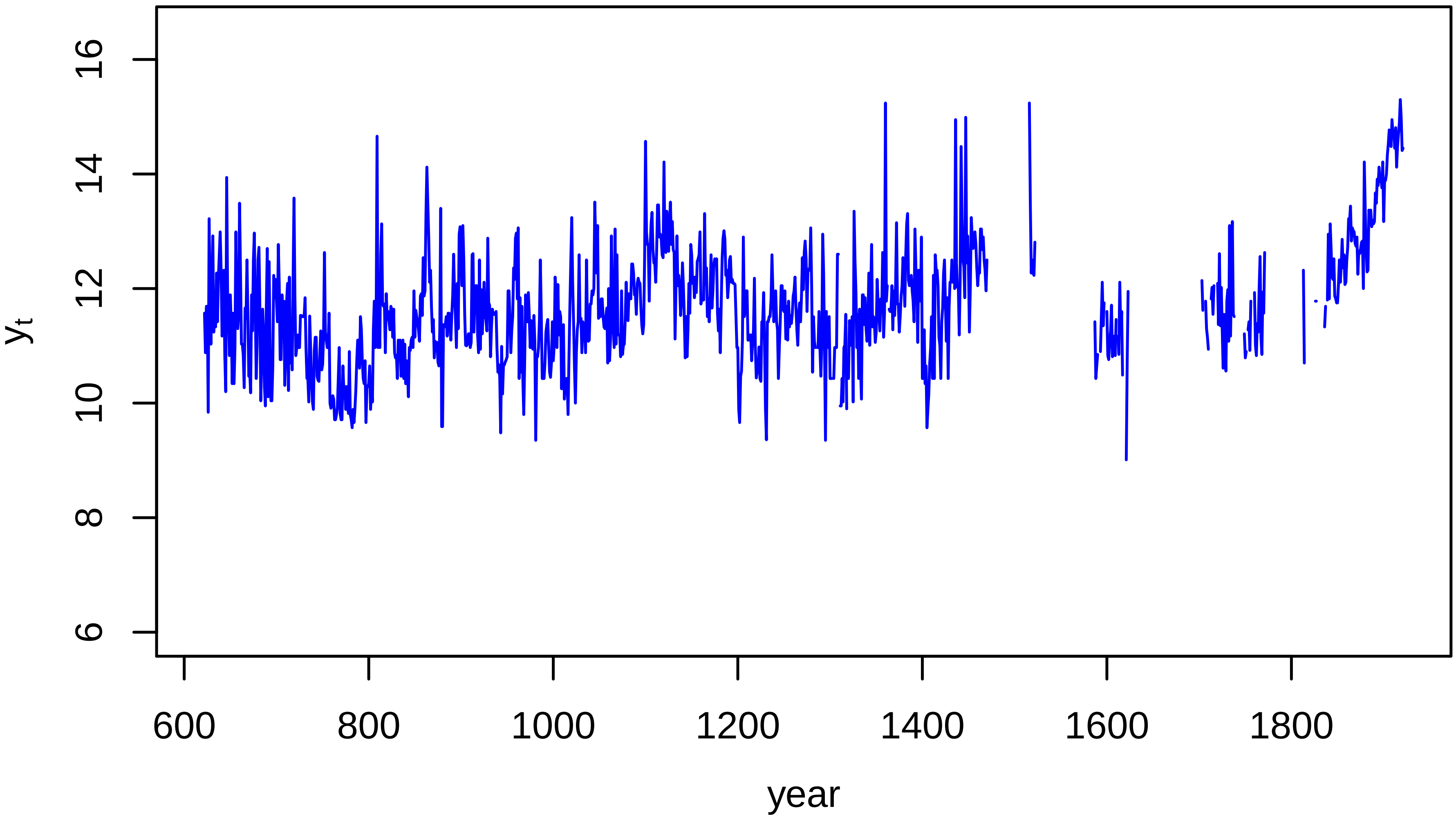
## Kalman Recursions for Gappy Time Series: I

- one of the strengths of state-space formulation is ease with which it can handle time series with missing values (gaps)
- suppose $Y_1, \ldots, Y_t$ & $Y_{t+3}$ are observed, but not $Y_{t+1}$ & $Y_{t+2}$
- use modified recursion
  - with $\widehat{X}_{t+1|t}$ & $P_{t+1|t}$ to get $\widehat{X}_{t+2|t}$ & $P_{t+2|t}$
  - with $\widehat{X}_{t+2|t}$ & $P_{t+2|t}$ to get $\widehat{X}_{t+3|t}$ & $P_{t+3|t}$
- enter $\widehat{X}_{t+3|t}$, $P_{t+3|t}$ and $Y_{t+3}$ into usual recursion to obtain $\widehat{X}_{t+3|t+3}$ & $\widehat{X}_{t+4|t+3}$ along with $P_{t+3|t+3}$ & $P_{t+4|t+3}$
- need to interpret 'given $t + 3$' as conditioning on everything available at time $t + 3$, i.e., $Y_1, \ldots, Y_t$ and $Y_{t+3}$

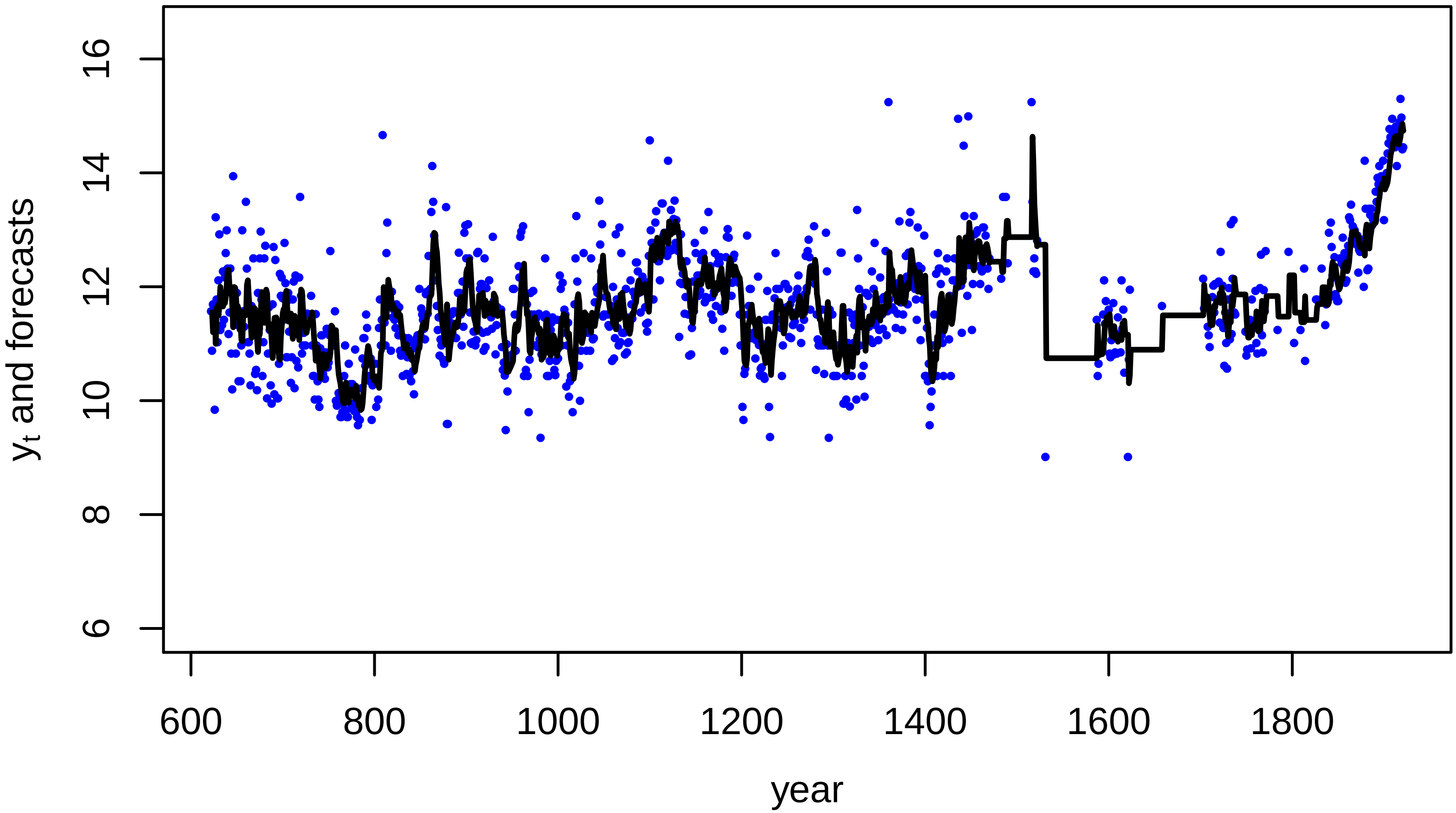
# Kalman Recursions for Gappy Time Series: II

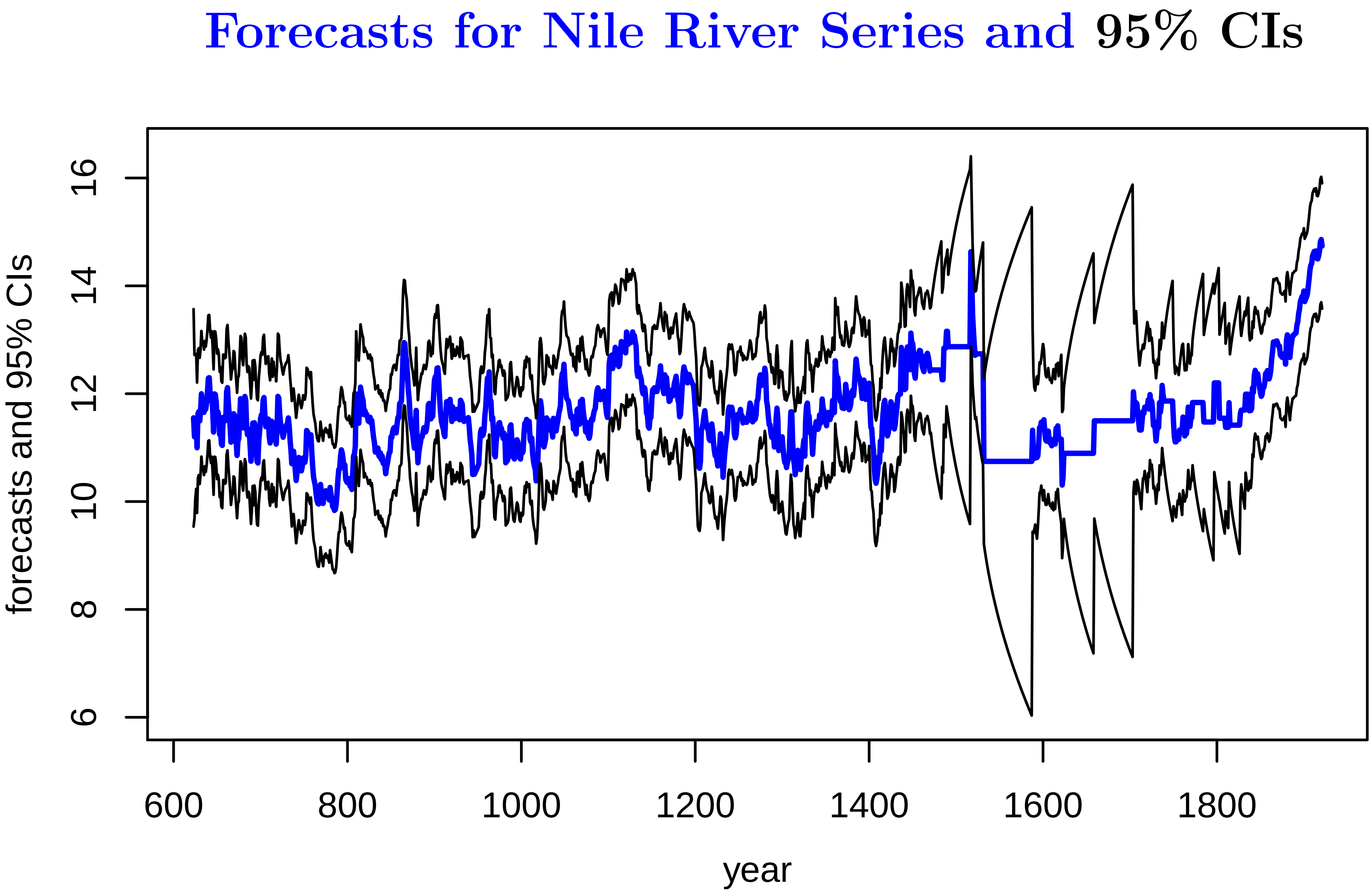
- repeat as needed to handle other gaps in time series
- as an example, let's reconsider Nile River minima

# Nile River Minima Series, 622 to 1921



# Nile River Series and Forecasts





Forecasts for Nile River Series and 95% CIs
forecasts and 95% CIs
6
8
10
12
14
16
600
800
1000
1200
1400
1600
1800
year

# Kalman Recursions for Smoothing: I

- given time series $Y_1, \ldots, Y_n$, Kalman filter recursions give us $\widehat{X}_{t|t}$ for $t = 1, \ldots, n$
- predictor $\widehat{X}_{t|t}$ of unknown state variable $X_t$ only makes use of $Y_1, \ldots, Y_t$ (appropriate mindset for 'real-time' applications)
- makes sense that best linear predictor of $X_t$ given $Y_1, \ldots, Y_n$ might significantly outperform $\widehat{X}_{t|t}$ (particularly for small $t$)
- regression lemma says solution to *smoothing problem* is
$$\widehat{X}_{t|n} \stackrel{\text{def}}{=} E\{X_t \mid \boldsymbol{Y}_n\} = m_t + \boldsymbol{\Sigma}'_{t,n}\Sigma_n^{-1}\left(\boldsymbol{Y}_n - \boldsymbol{m}_n\right),$$
where vector $\boldsymbol{\Sigma}_{t,n}$ has covariances between $X_t$ & $\boldsymbol{Y}_n$
- MSE for predictor, i.e., $E\{(X_t - \widehat{X}_{t|n})^2\}$, is
$$P_t - \boldsymbol{\Sigma}'_{t,n}\Sigma_n^{-1}\boldsymbol{\Sigma}_{t,n} \stackrel{\text{def}}{=} P_{t|n}, \quad \text{where } P_t \stackrel{\text{def}}{=} \text{var}\{X_t\}$$

BD: 275, SS: 299

# Kalman Recursions for Smoothing: II

- using innovations $U_t$, innovation variances $F_t$, Kalman gains $K_t$, forecasts $\widehat{X}_{t|t-1}$ and associated MSEs $P_{t|t-1}$, $t = 1, \ldots, n$ (all computed by Kalman filter recursions), *Kalman smoother recursions* allow efficient computation of $\widehat{X}_{t|n}$, $t = 1, \ldots, n$
- there are four steps in the Kalman recursion
- first two steps yield desired predictor $\widehat{X}_{t|n}$

1. manipulate innovations: starting with $r_n = 0$, recursively form

$$r_{t-1} = \frac{U_t}{F_t} + (1 - K_t)r_t, \;\; t = n, \ldots, 1$$

2. combine manipulated innovations and forecasts:

$$\widehat{X}_{t|n} = \widehat{X}_{t|t-1} + P_{t|t-1} r_{t-1}, \;\; t = 1, \ldots, n$$

# Kalman Recursions for Smoothing: III

- next two steps yield MSE for predictor $\widehat{X}_{t|n}$

3. manipulate innovation variances: starting with $N_n = 0$, recursively form

$$N_{t-1} = \frac{1}{F_t} + (1 - K_t)^2 N_t, \ \ t = n, \ldots, 1$$

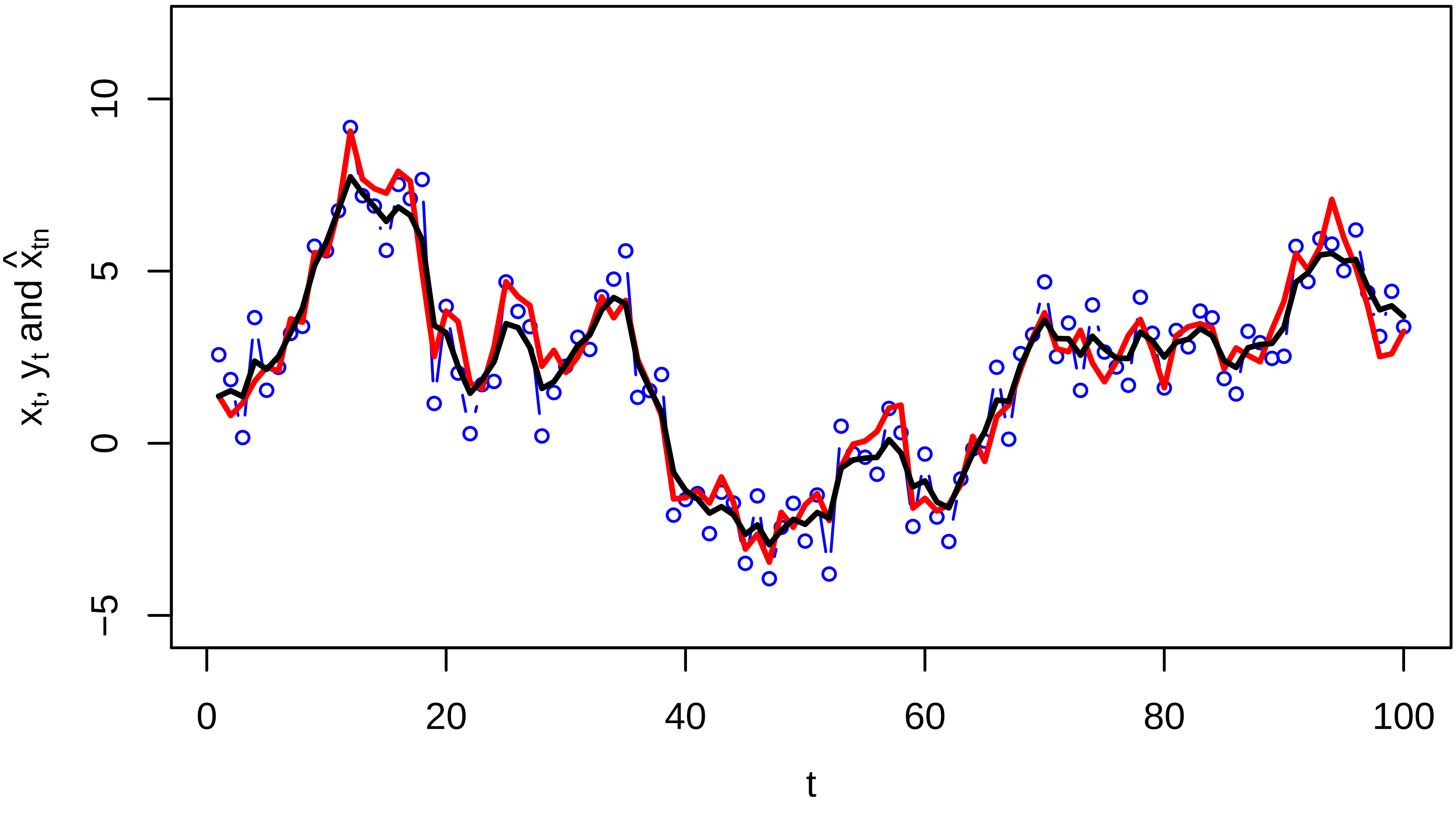
4. combine manipulated innovation variances and forecast MSEs:

$$V_t = P_{t|t-1} - P^2_{t|t-1} N_{t-1}, \ \ t = 1, \ldots, n,$$

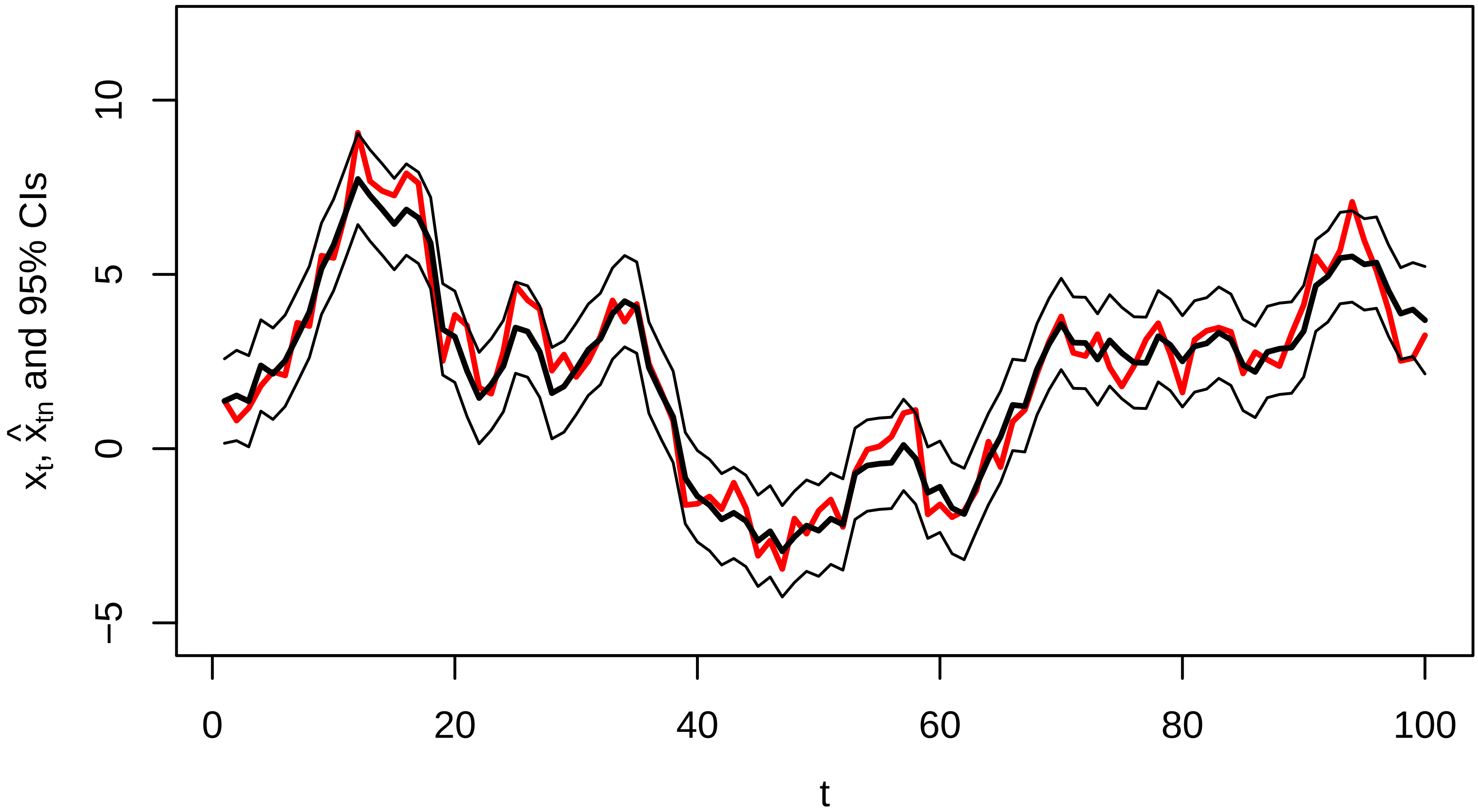
where $V_t$ is desired MSE

- as an example, let's apply Kalman smoother to time series from simulated local level model with SNR $= 1$

## Time Series $Y_t$, States $X_t$ and Smooths $\widehat{X}_{t|n}$

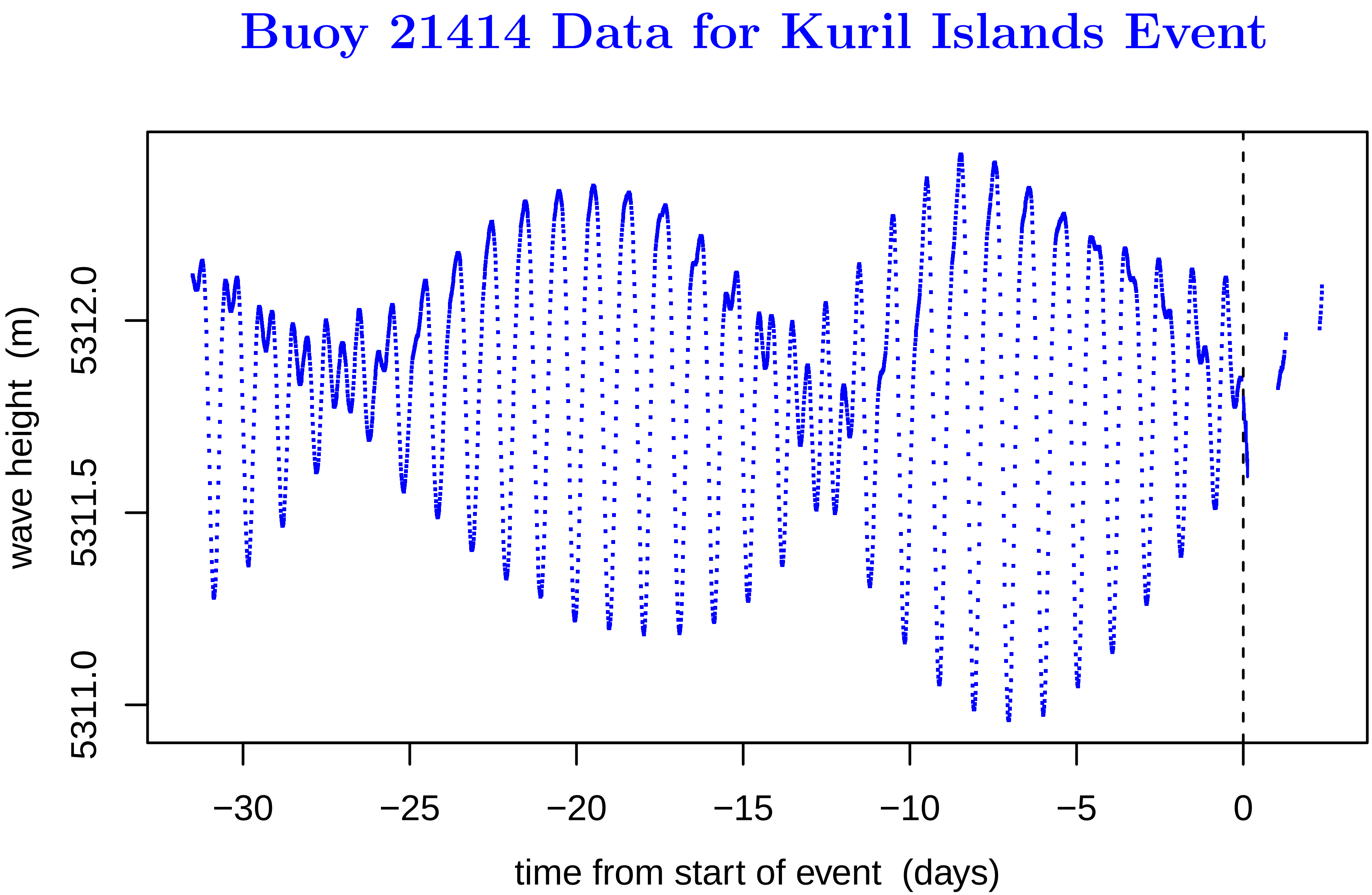


## States $X_t$, Smooths $\widehat{X}_{t|n}$ and 95% CIs

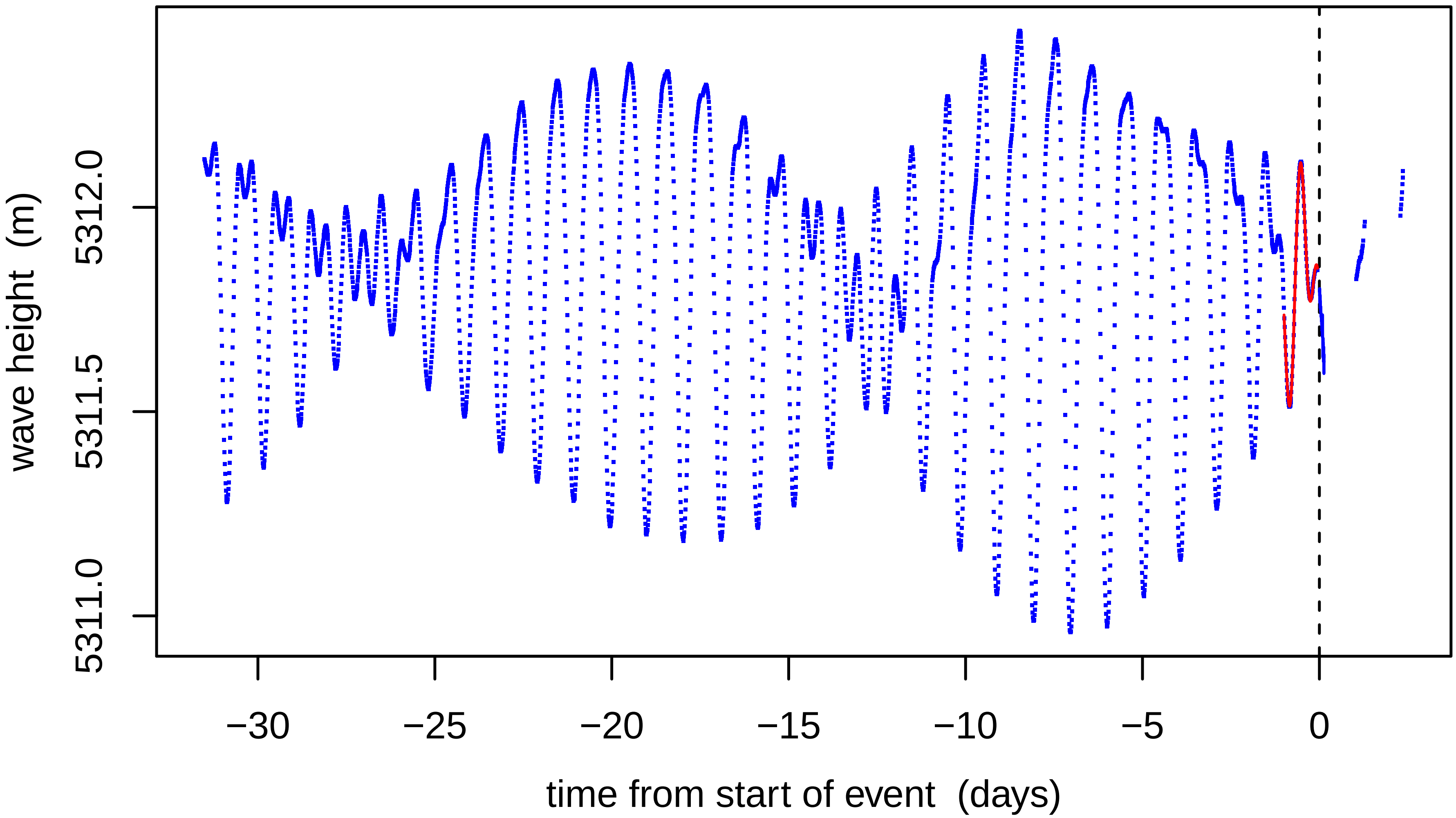


# Detiding DART™ Buoy Data

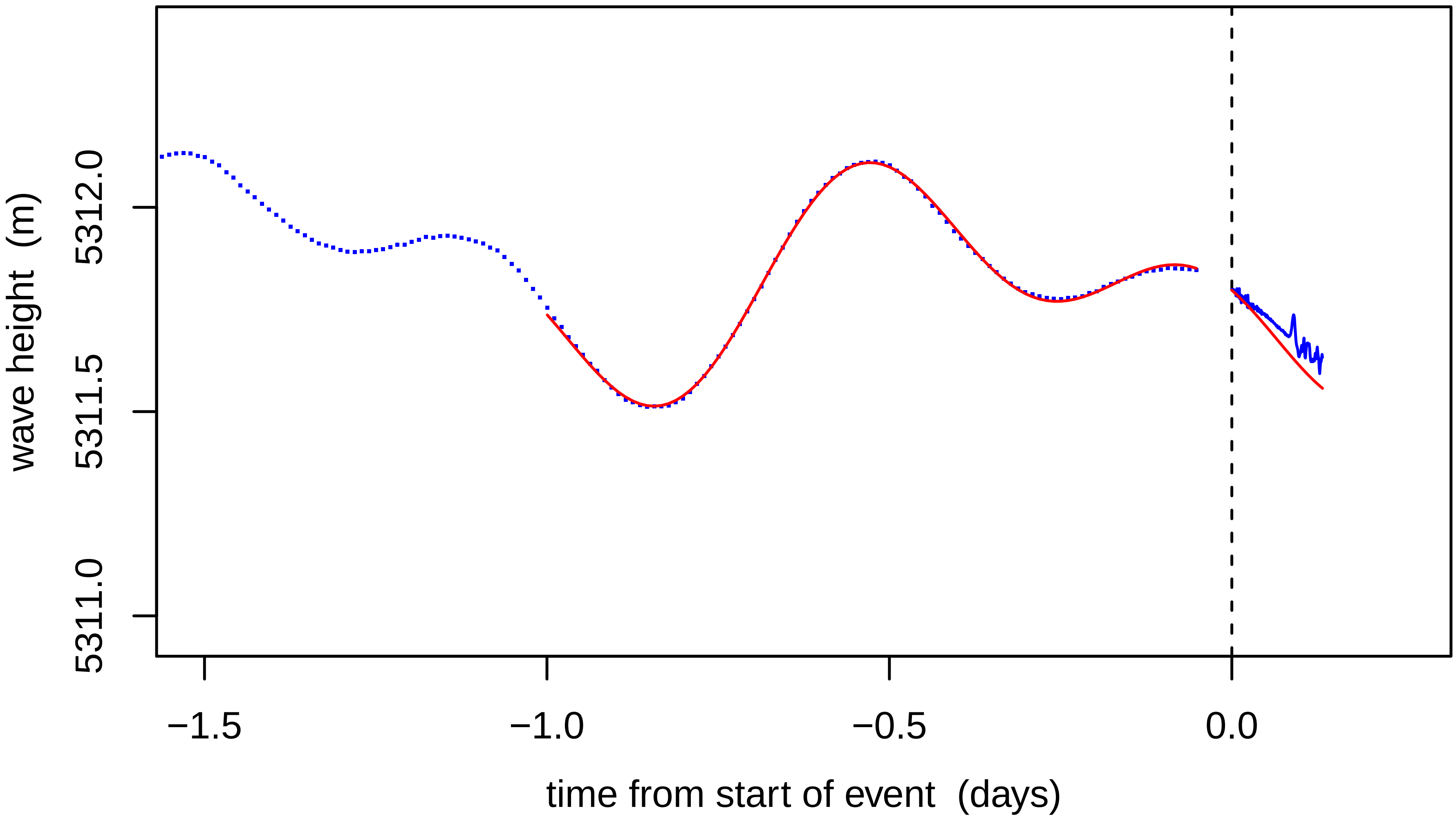
- though simple, local level model useful in practical applications
- as example, consider use of model to remove tidal component from data recorded by DART™ buoy 21414 prior to and after passage of tsunami wave in November, 2006 (generated by an earthquake near Kuril Islands)


Buoy 21414 Data for Kuril Islands Event
wave height (m)
5311.0
5311.5
5312.0
−30
−25
−20
−15
−10
−5
0
time from start of event (days)

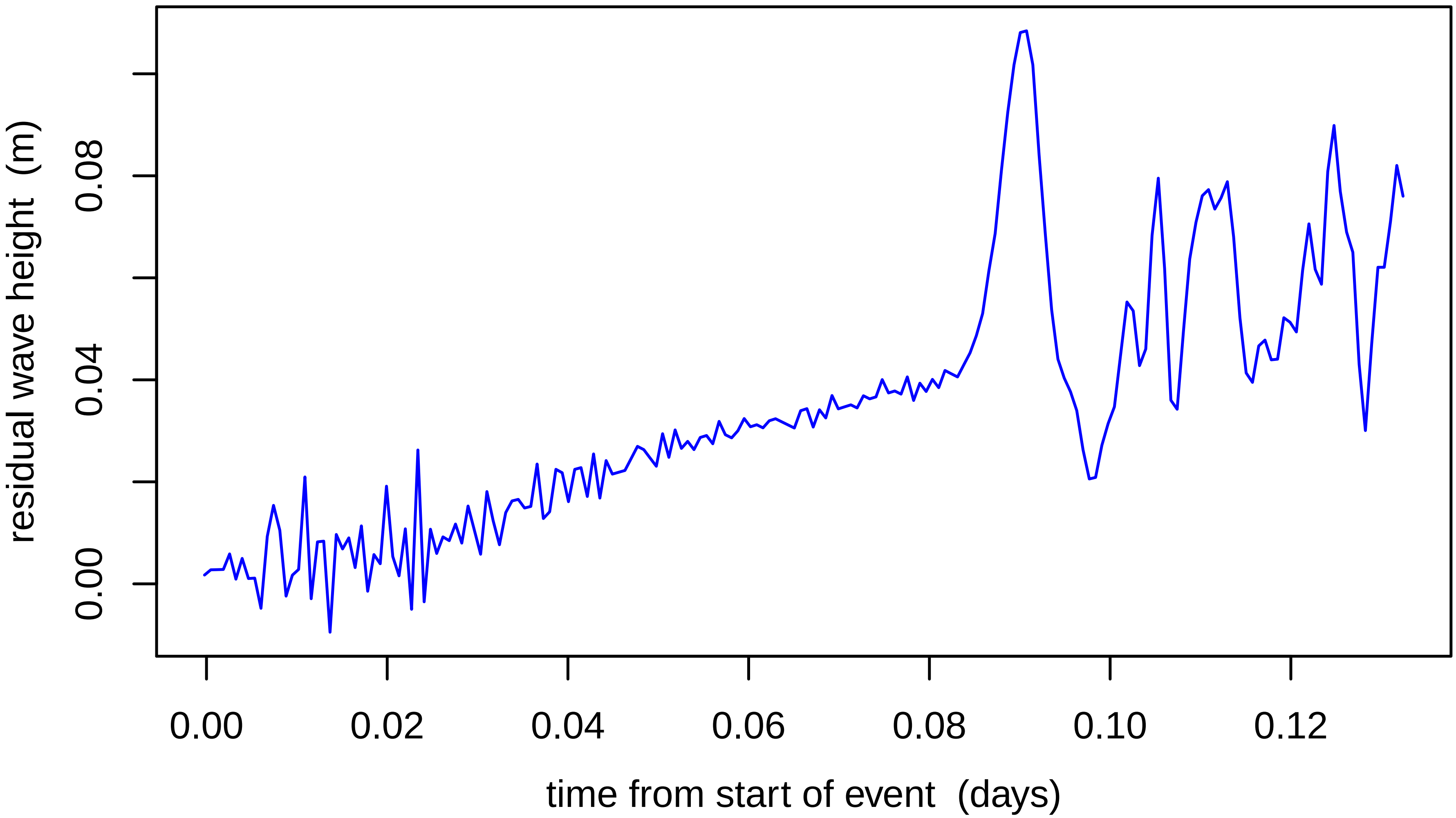
# Model Based on One Day Harmonic Regression



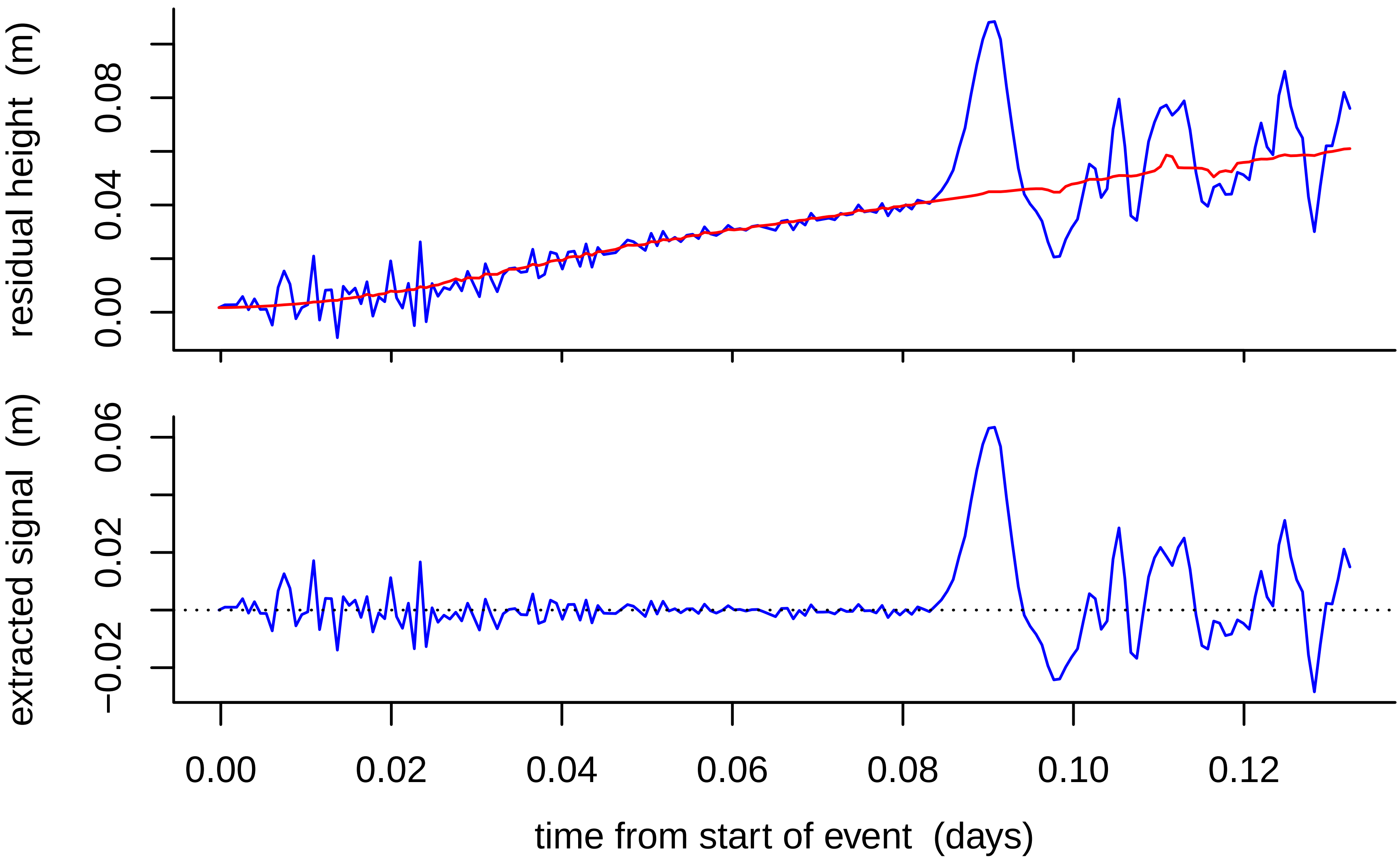
## Forecasts from Harmonic Regression Model



# Prediction Errors from Harmonic Regression Model



# Kalman Filtering/Smoothing (Local Level Model)



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