Classical Decomposition Model Revisited

- recall classical decomposition model

$$X_t = m_t + s_t + W_t \qquad (*)$$

where m_t is trend; s_t is periodic with known period s (i.e., $s_{t-s} = s_t$ for all $t \in \mathbb{Z}$) satisfying $\sum_{j=1}^{s} s_j = 0$; and W_t is a zero-mean stationary process

- recall definition of lag-s seasonal differencing operator:

$$\nabla_s X_t = X_t - X_{t-s} = (1 - B^s)X_t$$

- application of this operator to $(*)$ yields

$$\begin{aligned}\nabla_s X_t &= m_t - m_{t-s} + s_t - s_{t-s} + W_t - W_{t-s} \\ &= m_t - m_{t-s} + W_t - W_{t-s}\end{aligned}$$

because $\{s_t\}$ has period s

Seasonal ARMA Models: I

- assumption of *exact* repetition of seasonal pattern often dicey
- while differencing eliminates s_t, also complicates W_t component
- motivates period s seasonal ARMA$(p, q) \times (P, Q)_s$ model
- time series X_t is said to obey this so-called SARMA model if X_t is a causal ARMA process defined by

$$\phi(B)\Phi(B^s)X_t = \theta(B)\Theta(B^s)Z_t, \ \ \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

where

$$\begin{aligned} \phi(z) &= 1 - \phi_1 z - \cdots - \phi_p z^p \\ \Phi(z) &= 1 - \Phi_1 z - \cdots - \Phi_P z^P \\ \theta(z) &= 1 + \theta_1 z + \cdots + \theta_q z^q \\ \Theta(z) &= 1 + \Theta_1 z + \cdots + \Theta_Q z^Q \end{aligned}$$

Seasonal ARMA Models: II

- as an example, consider SARMA$(1,0) \times (1,0)_{12}$ model:

$$\begin{aligned} \phi(B)\Phi(B^{12})X_t &= Z_t \\ (1-\phi_1 B)(1-\Phi_1 B^{12})X_t &= Z_t \\ (1-\phi_1 B-\Phi_1 B^{12}+\phi_1\Phi_1 B^{13})X_t &= Z_t \\ X_t-\phi_1 X_{t-1}-\Phi_1 X_{t-12}+\phi_1\Phi_1 X_{t-13} &= Z_t \end{aligned}$$

- above is a special case of AR(13) model with
  - 10 coefficients set to zero
  - 3 remaining coefficients set using 2 parameters
- SARMA$(p,q) \times (P,Q)_s$ models are ARMA$(p+Ps, q+Qs)$ models with certain constraints on AR and MA coefficients

Seasonal ARIMA Models: I

- can extend SARMA model to include ordinary and/or seasonal differencing – leads to seasonal ARIMA$(p, d, q) \times (P, D, Q)_s$ model
- time series X_t is said to obey a SARIMA model if

$$Y_t = (1 - B)^d (1 - B^s)^D X_t$$

is a causal ARMA process defined by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t, \ \ \{Z_t\} \sim \text{WN}(0, \sigma^2)$$

- as an example, consider SARIMA$(0, 1, 1) \times (0, 1, 1)_{12}$ model, which is a popular model for nonstationary monthly economic data with a seasonal pattern:

$$(1 - B)(1 - B^{12})X_t = \theta(B)\Theta(B^{12})Z_t$$

Seasonal ARIMA Models: II

- expanding out

$$(1-B)(1-B^{12})X_t = \theta(B)\Theta(B^{12})Z_t$$

yields

$$\begin{aligned}(1-B-B^{12}+B^{13})X_t &= (1-\theta_1 B)(1-\Theta_1 B^{12})Z_t \\ (1-B-B^{12}+B^{13})X_t &= (1-\theta_1 B-\Theta_1 B^{12}+\theta_1\Theta_1 B^{13})Z_t\end{aligned}$$

and hence

$$X_t = X_{t-1} + X_{t-12} - X_{t-13} + Z_t - \theta_1 Z_{t-1} - \Theta_1 Z_{t-12} + \theta_1\Theta_1 Z_{t-13}$$

Seasonal ARMA Models: III

- to build up an understanding of SARMA models, let's look at time series Y_t consisting of r years of monthly data ($s = 12$)

year/month	$j = 1$	$j = 2$	$\cdots$	$j = 12$
1	Y_1	Y_2	$\cdots$	Y_{12}
2	Y_{13}	Y_{14}	$\cdots$	Y_{24}
3	Y_{25}	Y_{26}	$\cdots$	Y_{36}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	$Y_{1+12(r-1)}$	$Y_{2+12(r-1)}$	$\cdots$	$Y_{12+12(r-1)}$

- consider each column as time series by itself (12 series in all)
- suppose each series obeys ARMA(P, Q) model of *same* form:

$$\begin{aligned} Y_{j+12t} = {} & \Phi_1 Y_{j+12(t-1)} + \cdots + \Phi_P Y_{j+12(t-P)} \\ & + Z_{j,t} + \Theta_1 Z_{j,t-1} + \cdots + \Theta_Q Z_{j,t-Q}, \quad \{Z_{j,t}\} \sim \mathrm{WN}(0, \sigma^2) \end{aligned}$$

Seasonal ARMA Models: IV

- interleave $Z_{j,t}$'s from models for individual months to form

$$Z_{j+12t} = Z_{j,t},$$

  using which we can now write

$$\begin{aligned} Y_{j+12t} = {} & \Phi_1 Y_{j+12(t-1)} + \cdots + \Phi_P Y_{j+12(t-P)} \\ & + Z_{j+12t} + \Theta_1 Z_{j+12(t-1)} + \cdots + \Theta_Q Z_{j+12(t-Q)} \end{aligned}$$

- since same equation holds for all j and t, can write

$$\Phi(B^{12}) Y_t = \Theta(B^{12}) Z_t$$

- while $E\{Z_t\} = 0$ and $\text{var}\,\{Z_t\} = \sigma^2$, $\{Z_t\}$ need *not* be white noise: we have $E\{Z_t Z_{t+h}\} = 0$ if h is an nonzero integer multiple of 12, but this need *not* be zero for other nonzero h

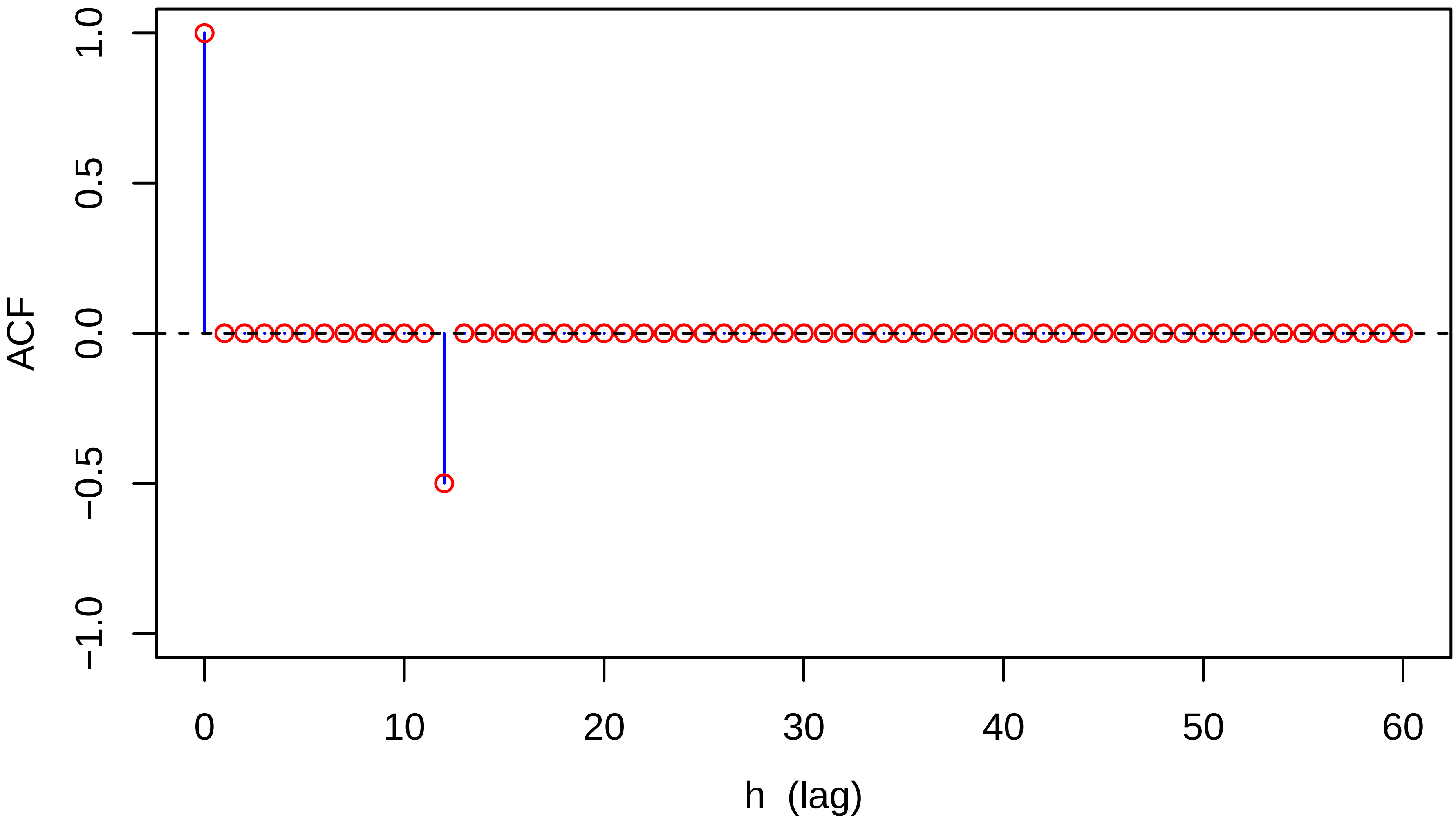
Seasonal ARMA Models: V

- between-year (interannual) model $\Phi(B^{12})Y_t = \Theta(B^{12})Z_t$ *not* a SARMA$(0,0) \times (P,Q)_{12}$ model unless we further assume $E\{Z_t Z_{t+h}\} = 0$ for *all* $h \neq 0$, allowing us to write

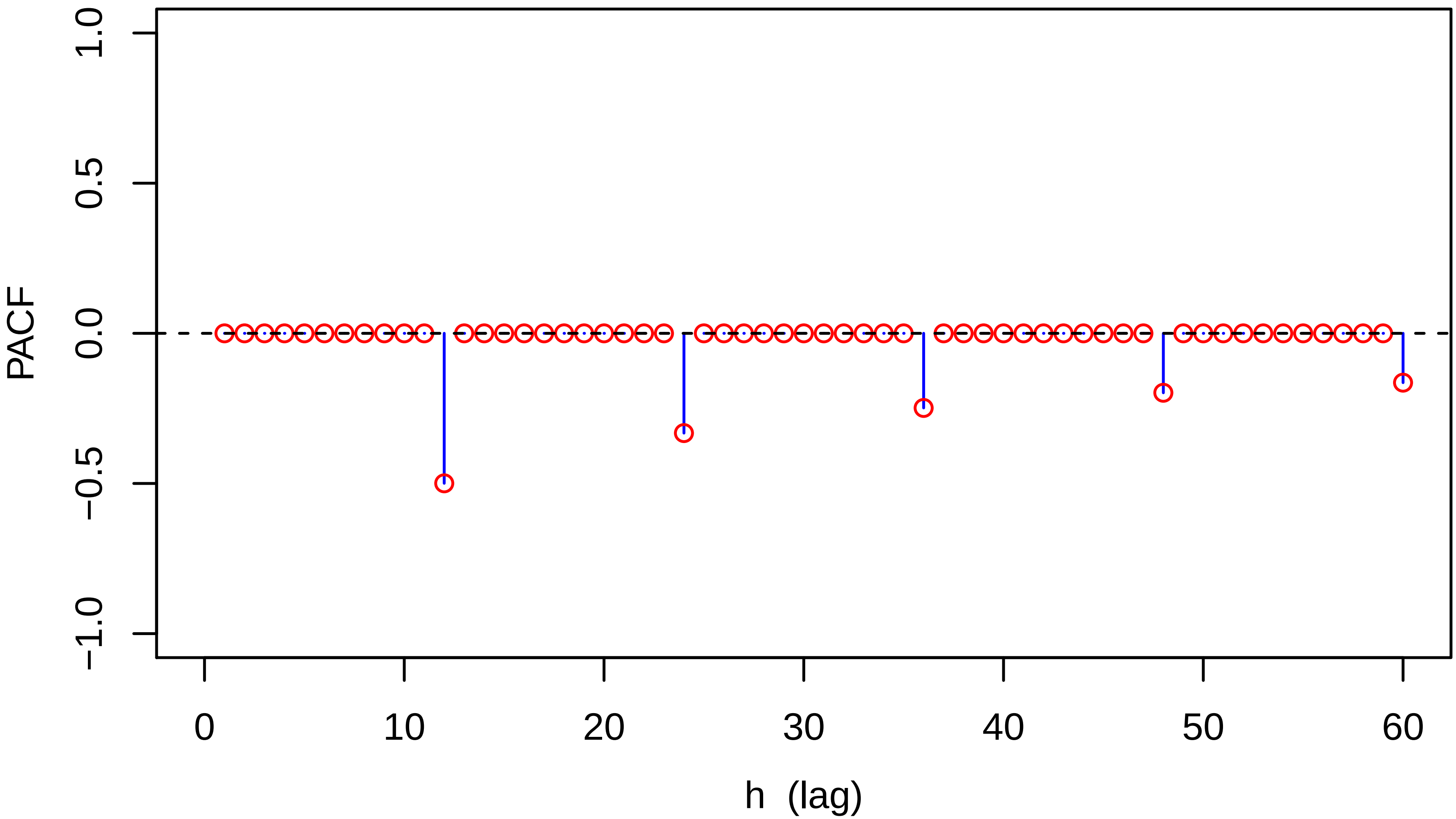
$$\Phi(B^{12})Y_t = \Theta(B^{12})Z_t, \ \ \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

- so-called *pure SARMA model* (i.e., $p = q = 0$) is instructive, but of questionable practical value: says, e.g., every RV that is part of February time series is uncorrelated with every RV that is part of March series (even if months are within same year!)
- keeping still to $s = 12$, consider ACF, PACF and simulated series from following pure SARMA models:
  - $P = 0$, $Q = 1$ and $\Theta_1 = -0.95$
  - $P = 1$, $Q = 0$ and $\Phi_1 = 0.95$

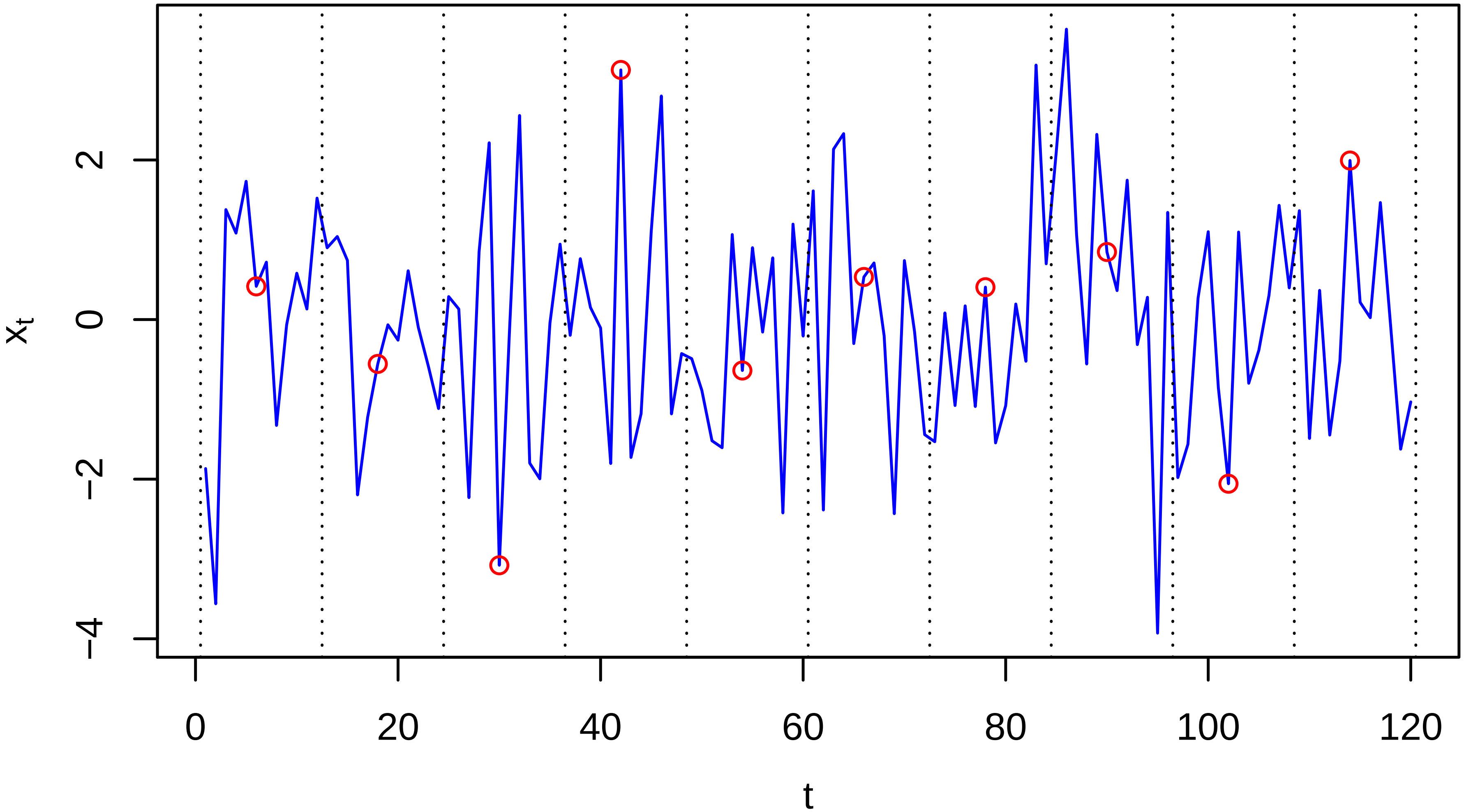
ACF for SARMA$(0,0) \times (0,1)_{12}$ Model

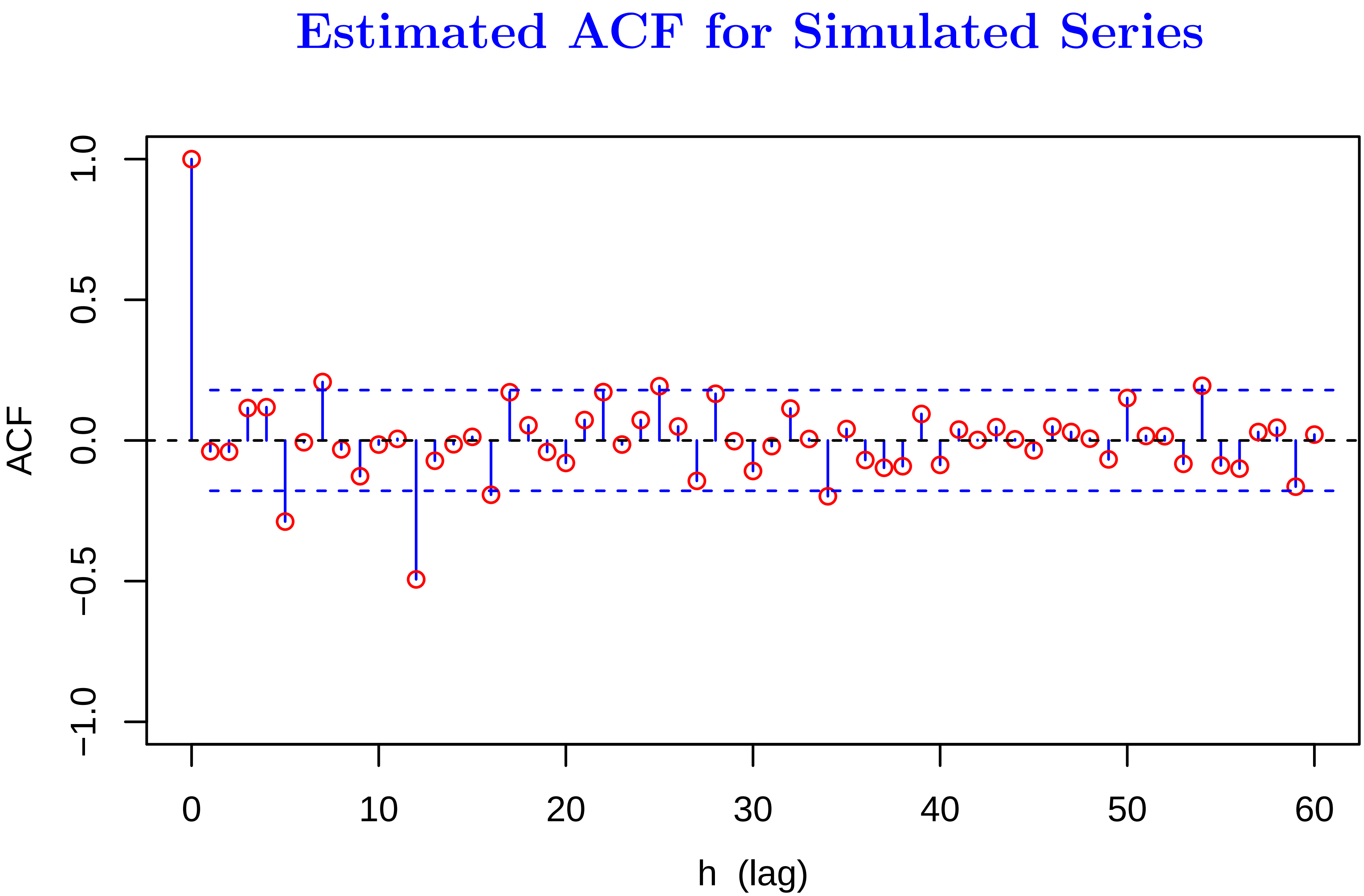


PACF for SARMA$(0,0) \times (0,1)_{12}$ Model

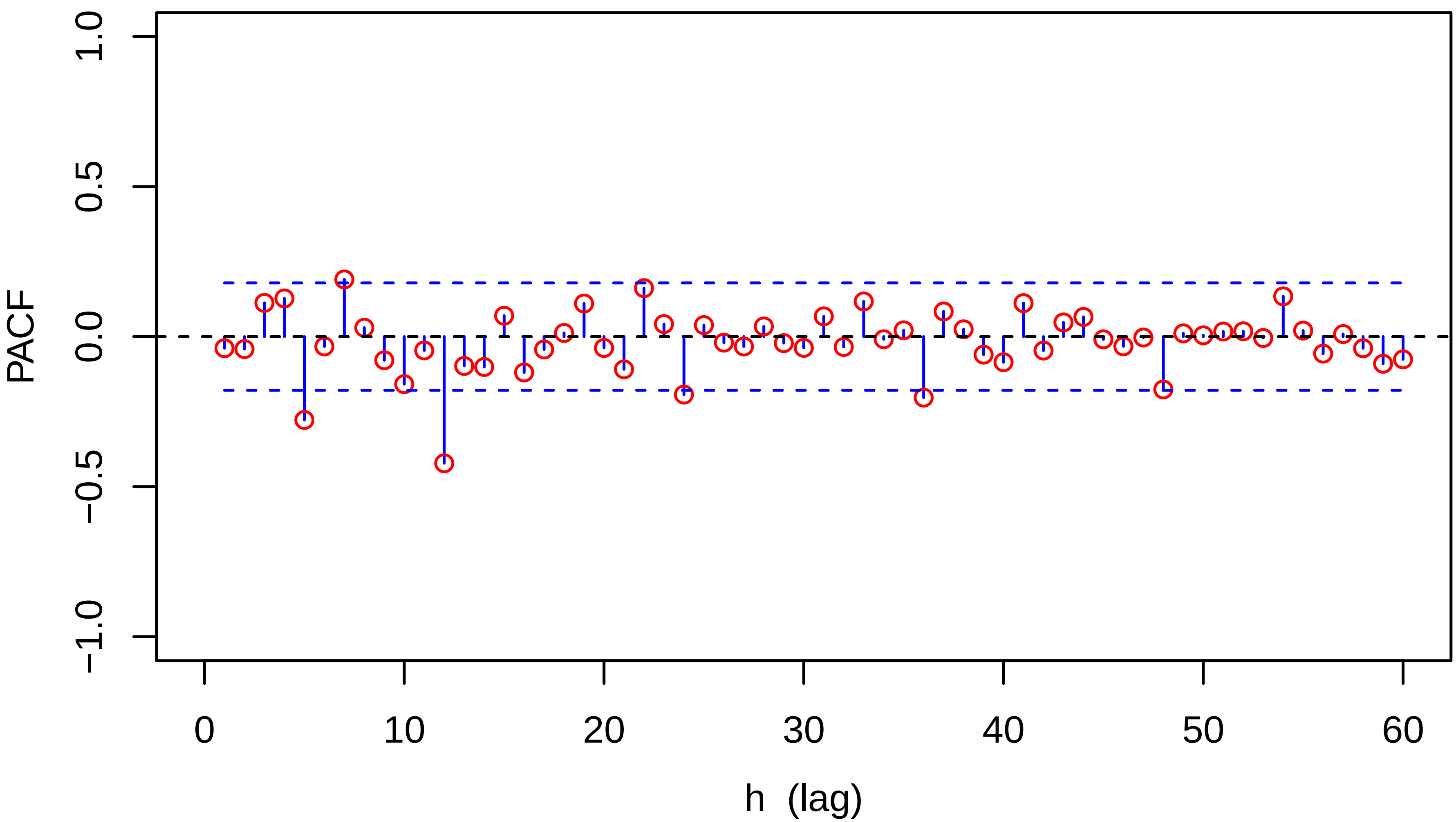


Simulated SARMA$(0,0)\times(0,1)_{12}$ **Series (10 Years)**

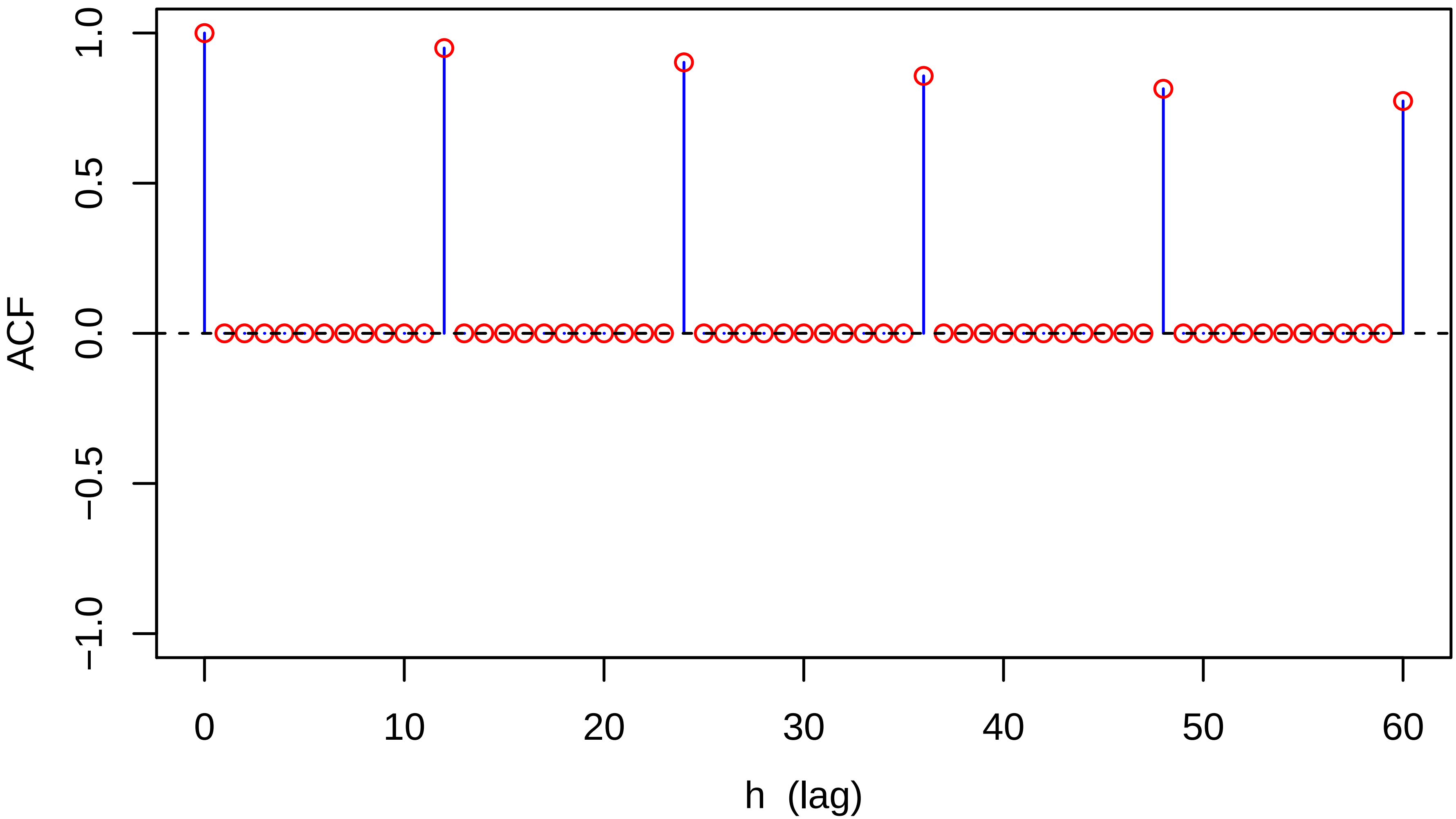



Estimated ACF for Simulated Series
ACF
1.0
0.5
0.0
−0.5
−1.0
0
10
20
30
40
50
60
h (lag)

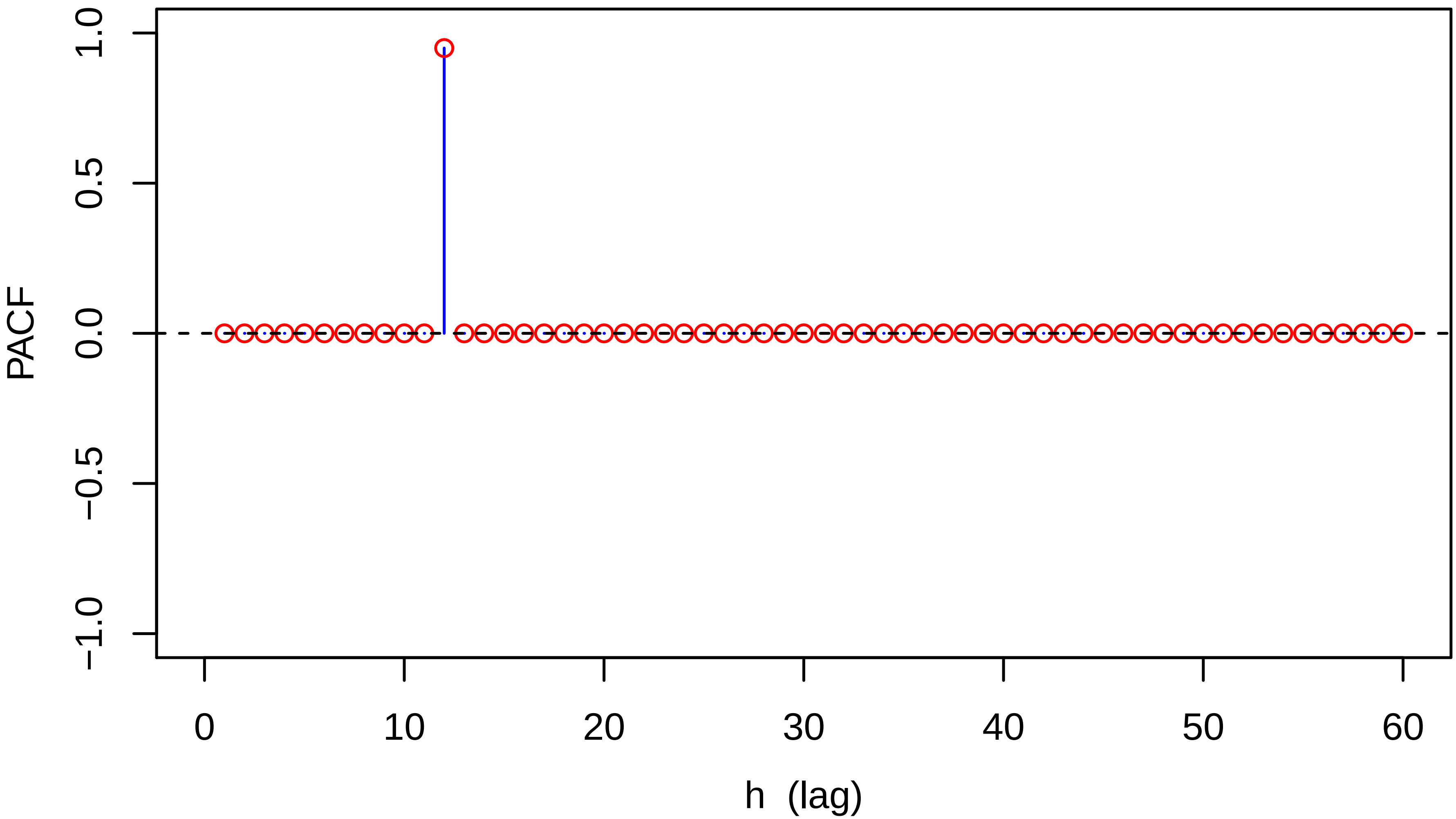
Estimated PACF for Simulated Series



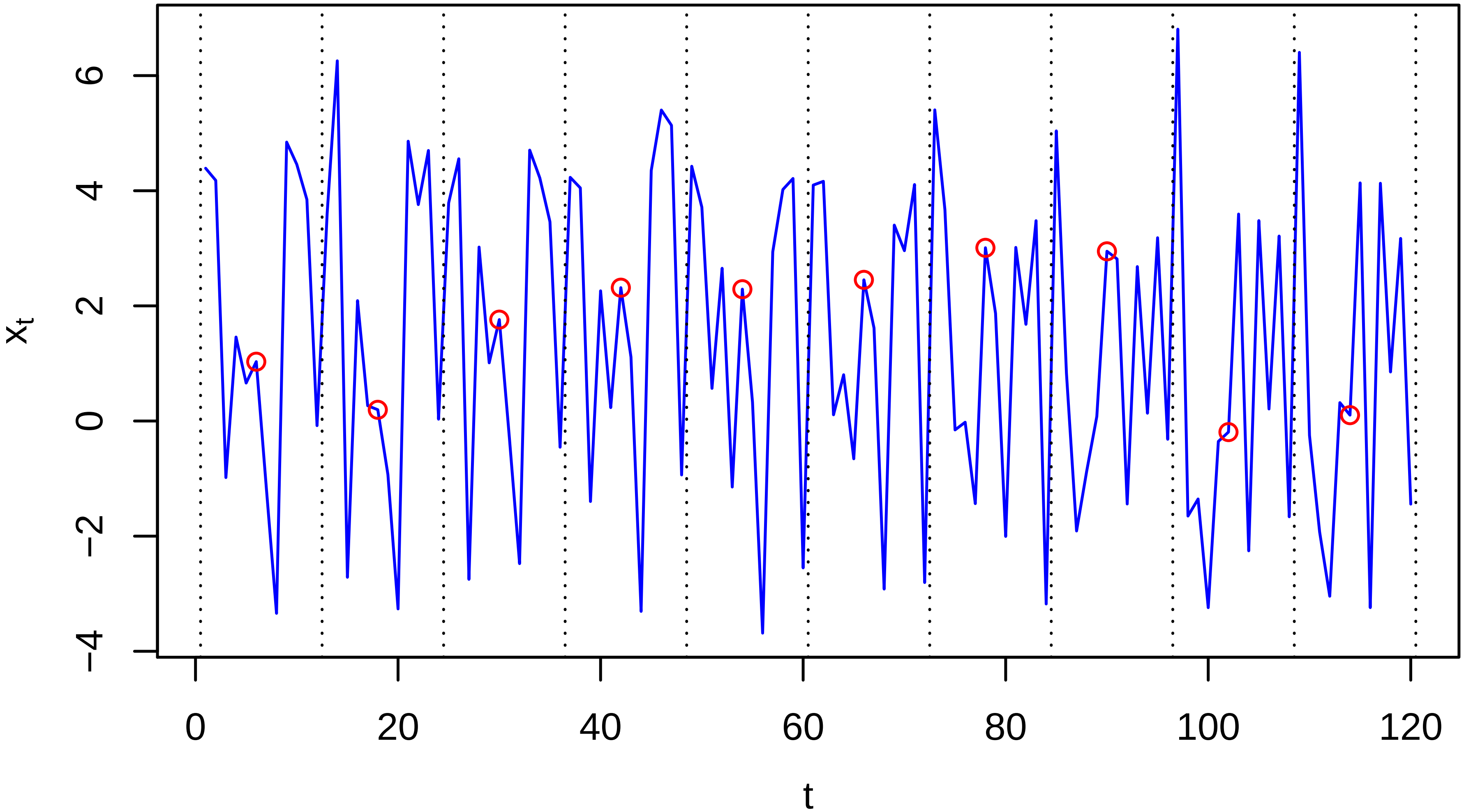
ACF for SARMA$(0,0) \times (1,0)_{12}$ Model



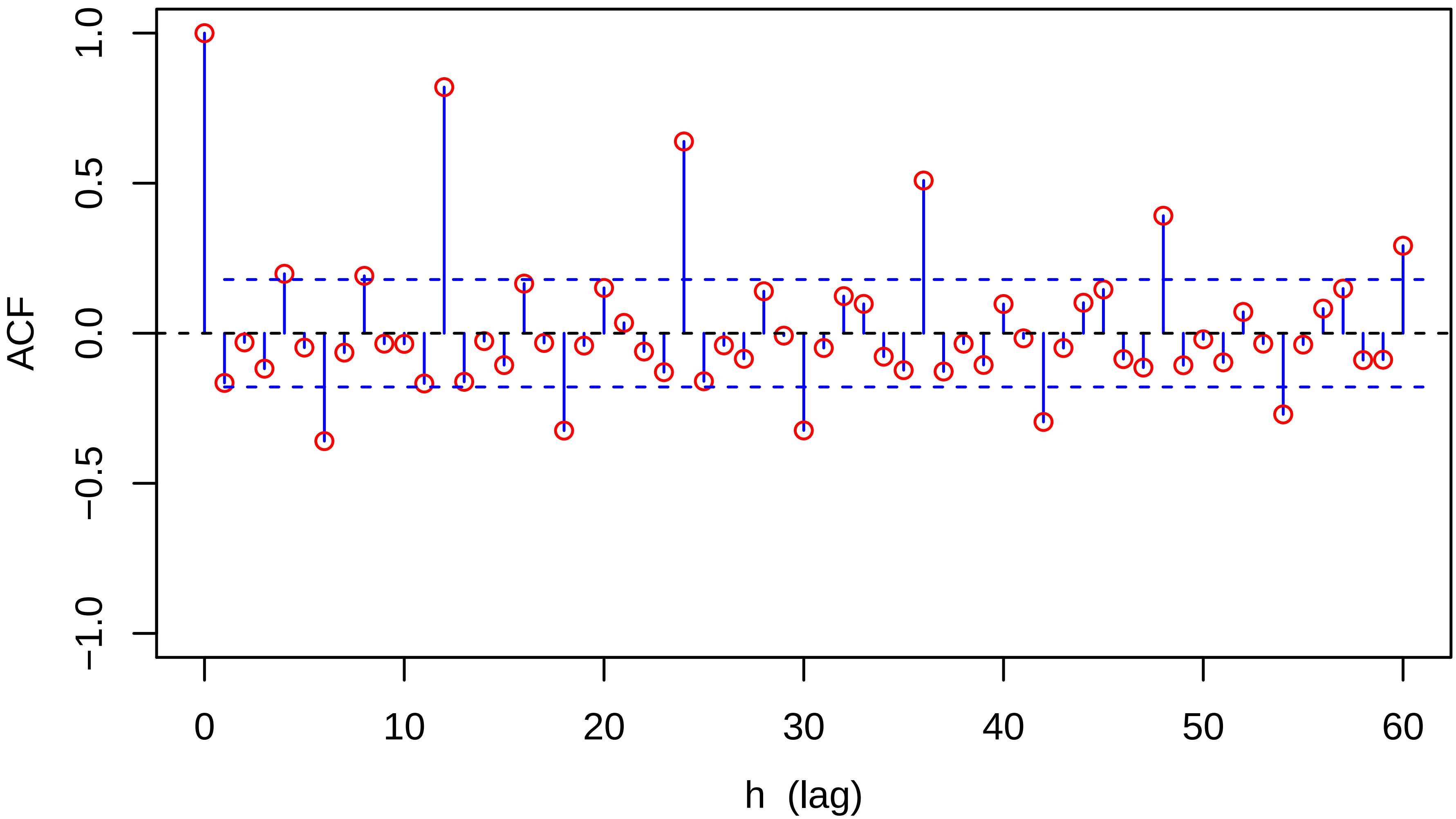
PACF for SARMA$(0,0) \times (1,0)_{12}$ Model



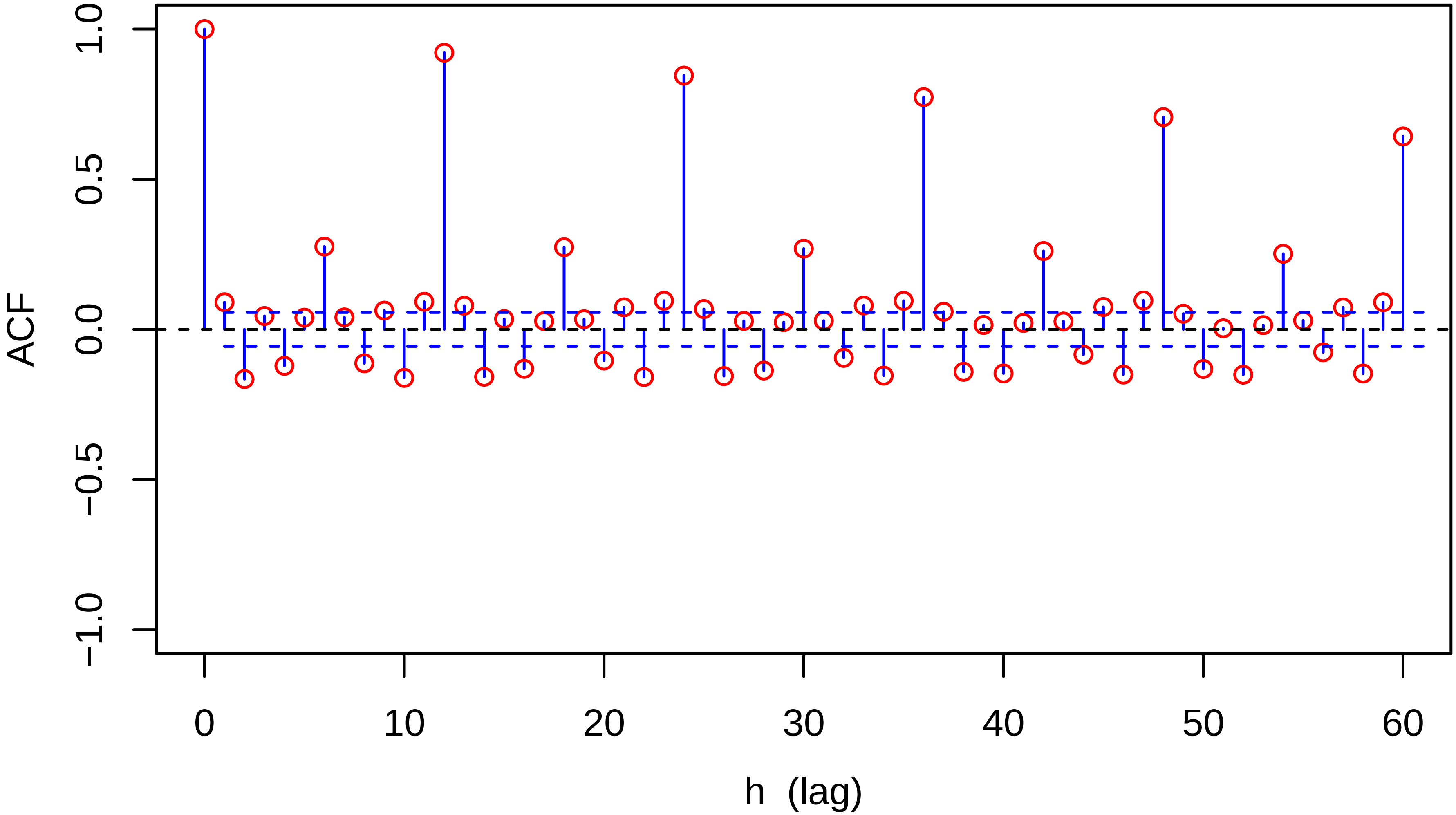
Simulated SARMA$(0,0) \times (1,0)_{12}$ Series (10 Years)



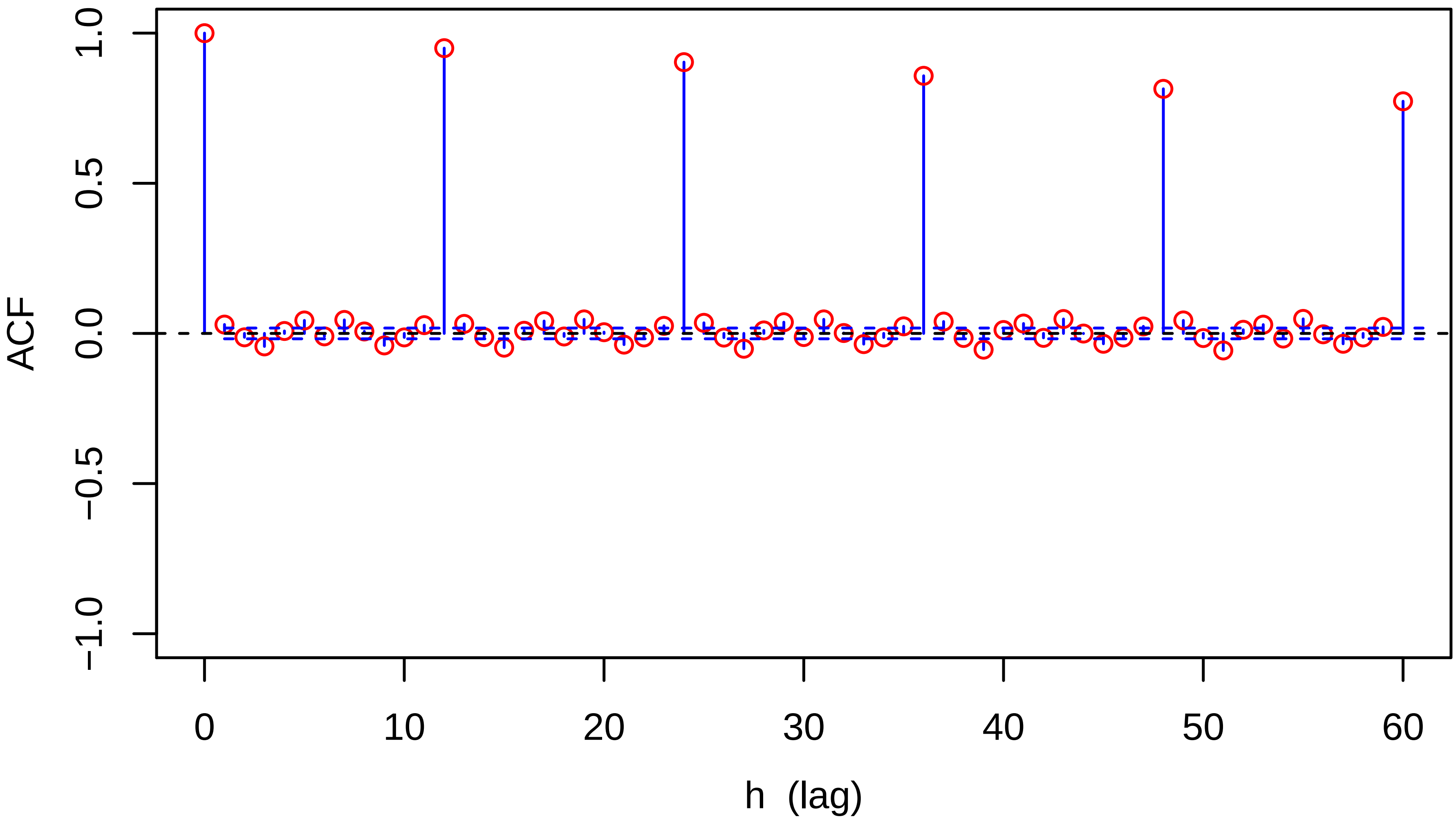
Estimated ACF for Simulated Series (10 Years)



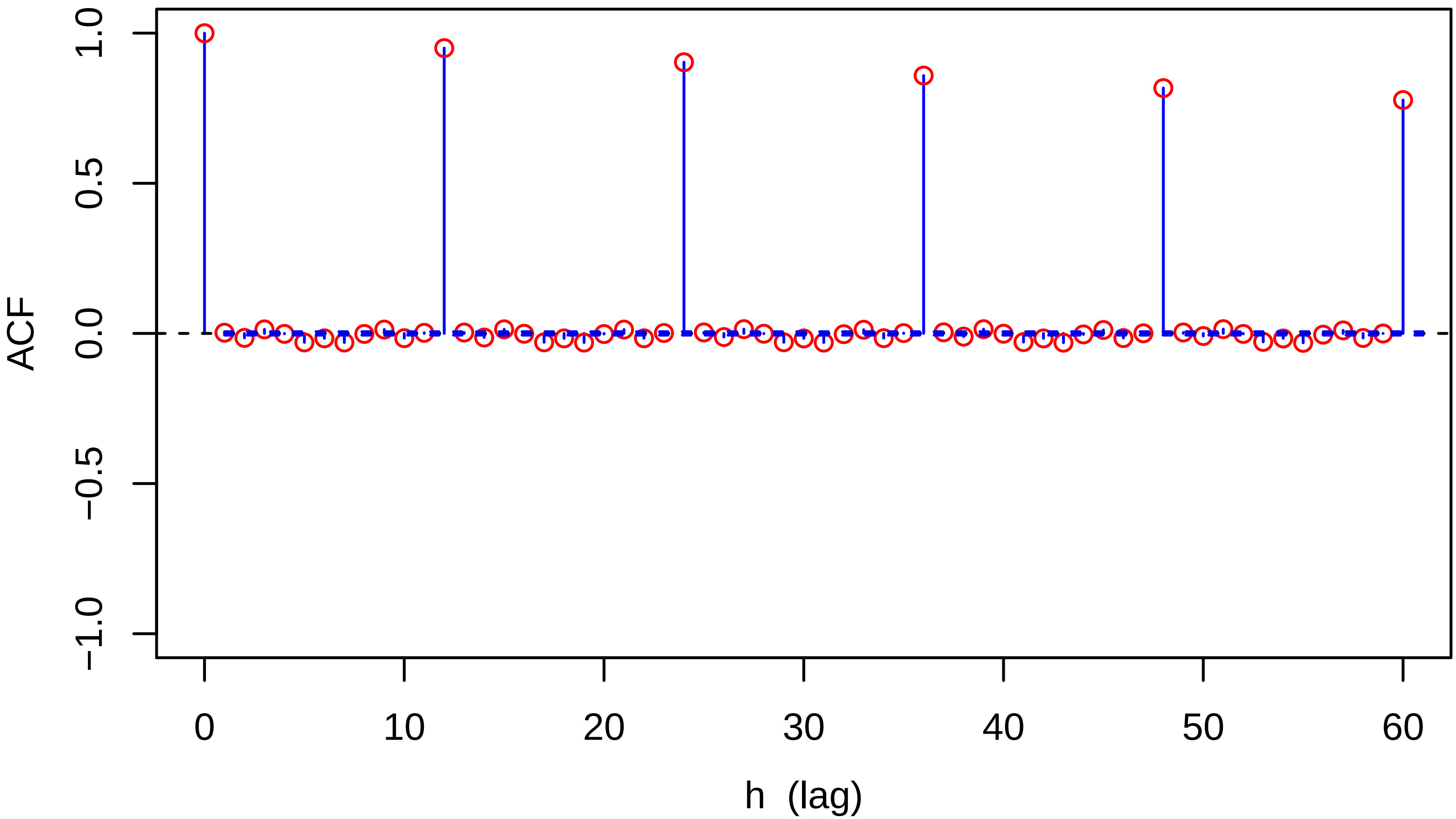
Estimated ACF for Simulated Series (100 Years)



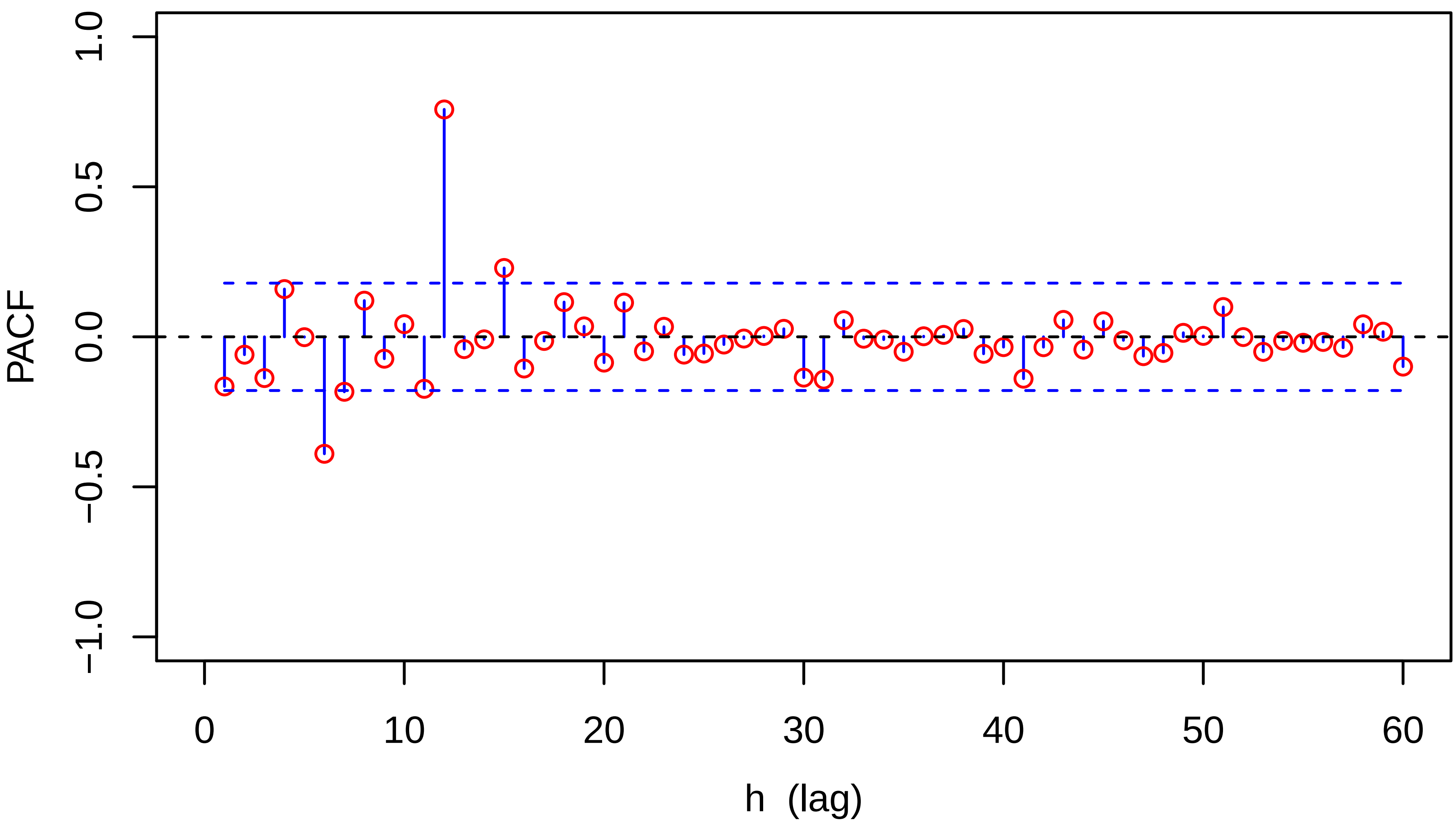
Estimated ACF for Simulated Series (1000 Years)



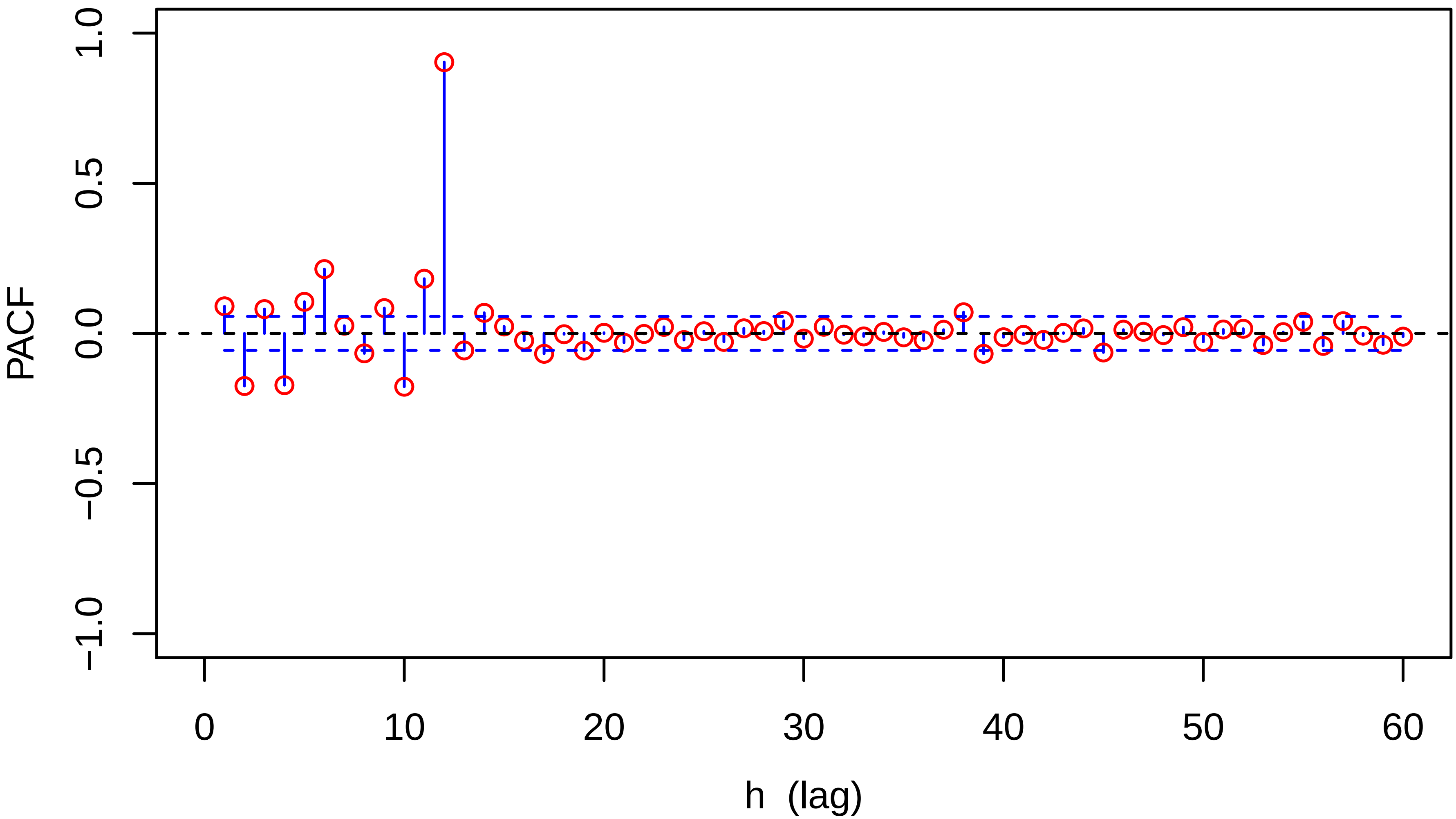
Estimated ACF for Simulated Series (10,000 Years)



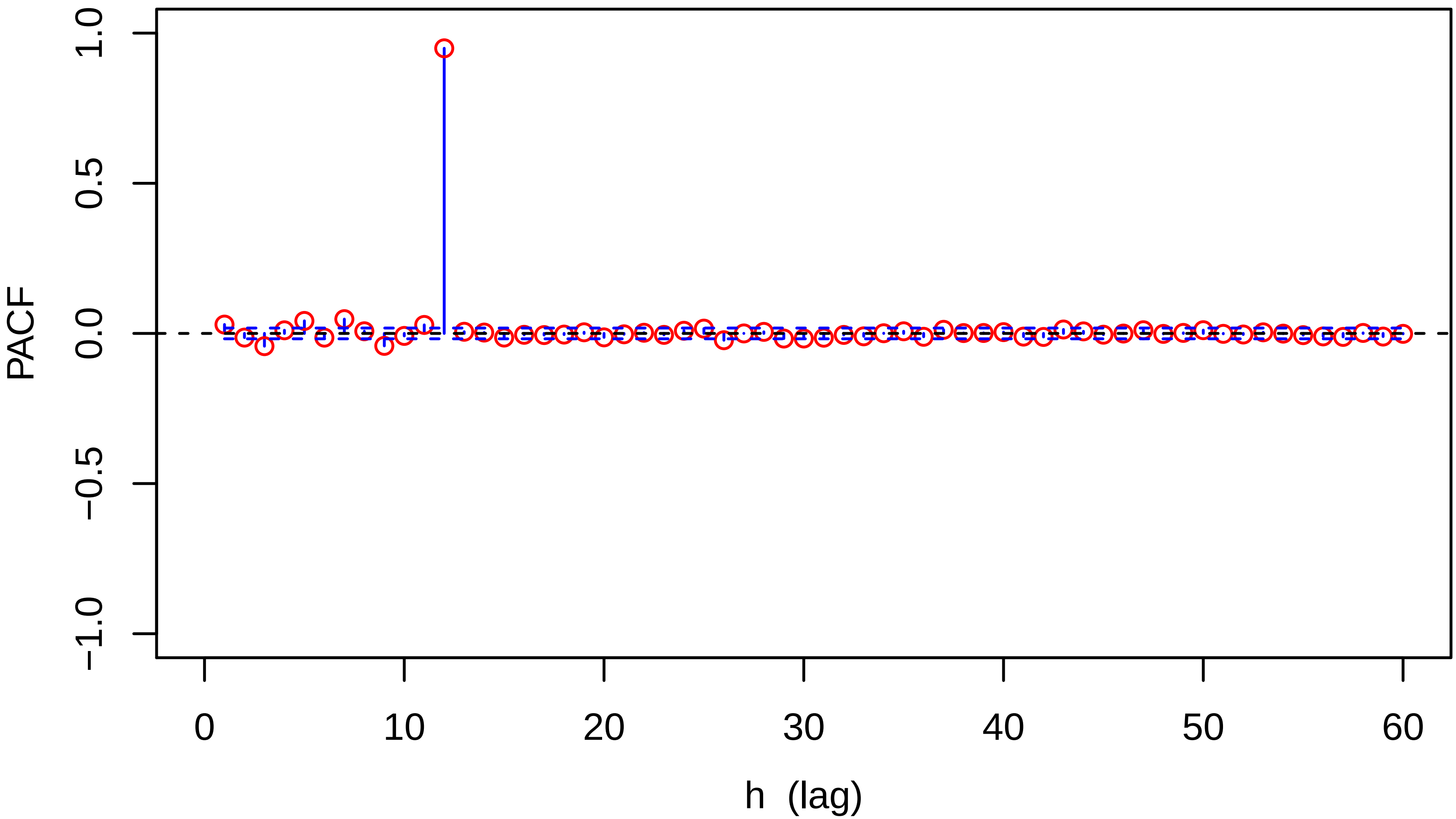
Estimated PACF for Simulated Series (10 Years)



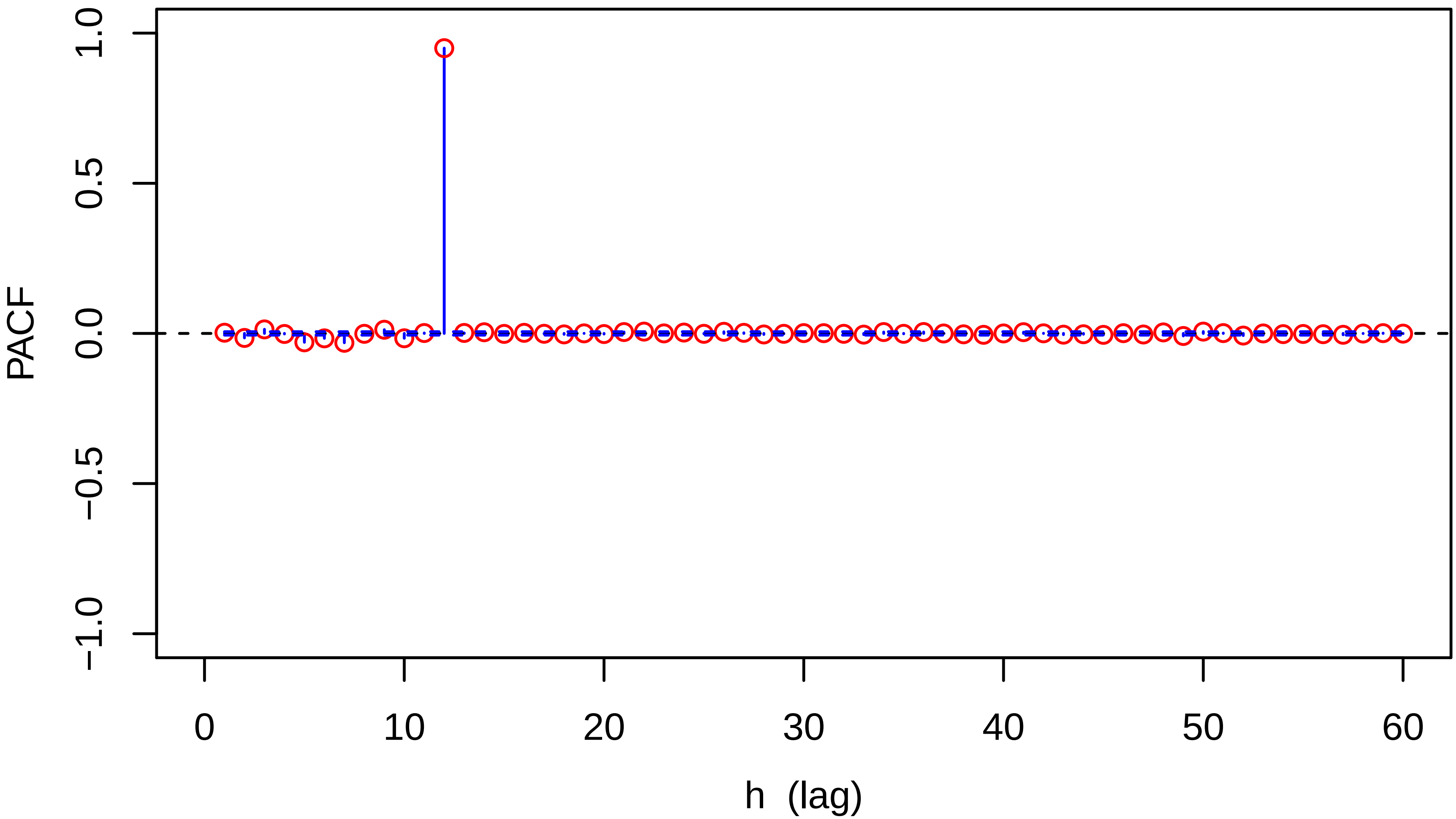
Estimated PACF for Simulated Series (100 Years)



Estimated PACF for Simulated Series (1000 Years)



Estimated PACF for Simulated Series (10,000 Years)



Seasonal ARMA Models: VI

- if Y_t is a pure SARMA model with period $s = 12$, can induce correlations between adjacent months by subjecting it to a filter
- for example, consider three-point running average
$$X_t = \frac{Y_{t-1} + Y_t + Y_{t+1}}{3}$$
- whereas Y_t is uncorrelated with Y_{t-2}, Y_{t-1}, Y_{t+1}, Y_{t+2}, X_t is now correlated with X_{t-2}, X_{t-1}, X_{t+1}, X_{t+2}
- now suppose we subject Y_t to suitable AR and MA filtering operations described by $\phi(B)$ and $\theta(B)$:
$$\phi^{-1}(B)\theta(B)Y_t \stackrel{\text{def}}{=} X_t, \text{ yielding } Y_t = \phi(B)\theta^{-1}(B)X_t$$
- substituting above into $\Phi(B^{12})Y_t = \Theta(B^{12})Z_t$ yields
$$\Phi(B^{12})\phi(B)\theta^{-1}(B)X_t = \Theta(B^{12})Z_t, \text{ i.e., } \phi(B)\Phi(B^{12})X_t = \theta(B)\Theta(B^{12})Z_t$$

Seasonal ARMA Models: VII

- conclusion: in general SARMA model

$$\phi(B)\Phi(B^{12})X_t = \theta(B)\Theta(B^{12})Z_t,$$

  primary role of

  - $\phi(B)$ & $\theta(B)$ is to model correlations between months *within* a single year (intra-annual variations)
  - $\Phi(B)$ & $\Theta(B)$ is to model correlations for a given month *across* several years (inter-annual variations)

Seasonal ARMA Models: VIII

- keeping with $s = 12$, let's consider four SARMA models:
  - $p = 0$, $P = 0$, $q = 1$, $Q = 1$, $\theta_1 = 0.9$, $\Theta_1 = -0.4$
  - $p = 0$, $P = 0$, $q = 1$, $Q = 1$, $\theta_1 = 0.9$, $\Theta_1 = 0.4$
  - $p = 1$, $P = 1$, $q = 0$, $Q = 0$, $\phi_1 = 0.5$, $\Phi_1 = 0.7$
  - $p = 1$, $P = 1$, $q = 0$, $Q = 0$, $\phi_1 = 0.9$, $\Phi_1 = 0.7$
- first two are examples of SARMA$(0,1) \times (0,1)_{12}$ model

$$X_t = Z_t + \theta_1 Z_{t-1} + \Theta_1 Z_{t-12} + \theta_1 \Theta_1 Z_{t-13}$$

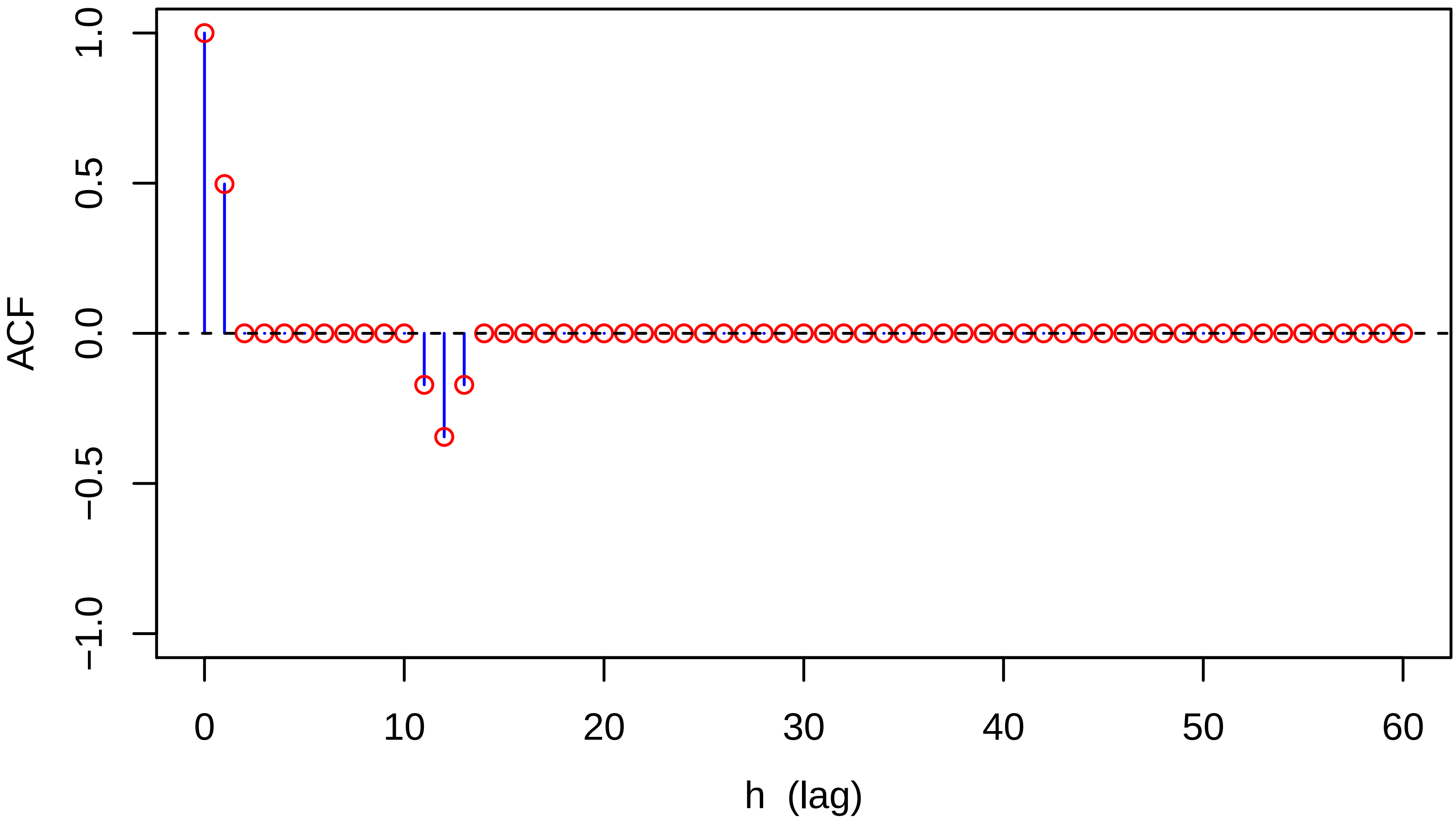
(special case of MA(13) model)

- second two are examples of SARMA$(1,0) \times (1,0)_{12}$ model

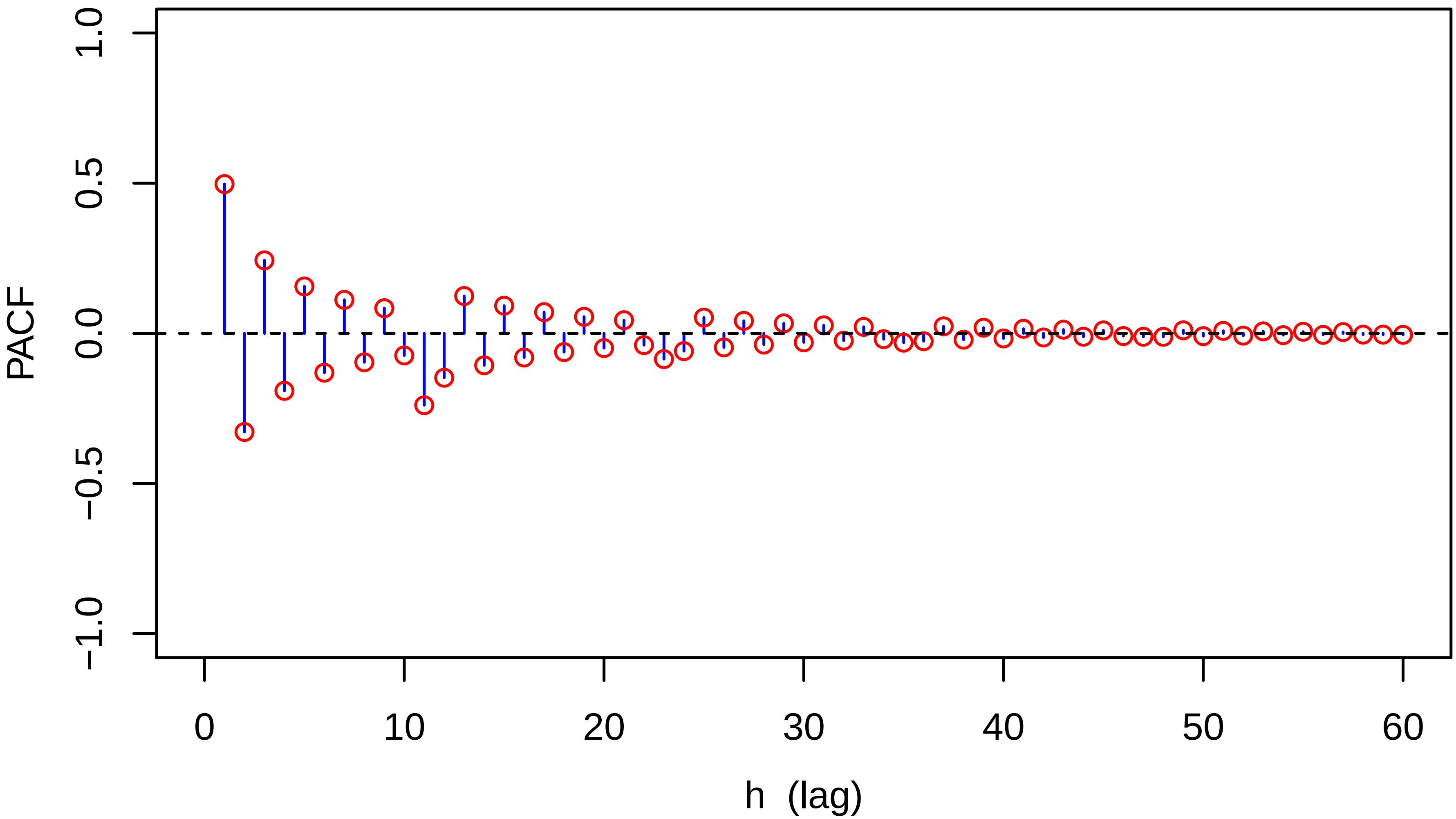
$$X_t - \phi_1 X_{t-1} - \Phi_1 X_{t-12} + \phi_1 \Phi_1 X_{t-13} = Z_t$$

(special case of AR(13) model)

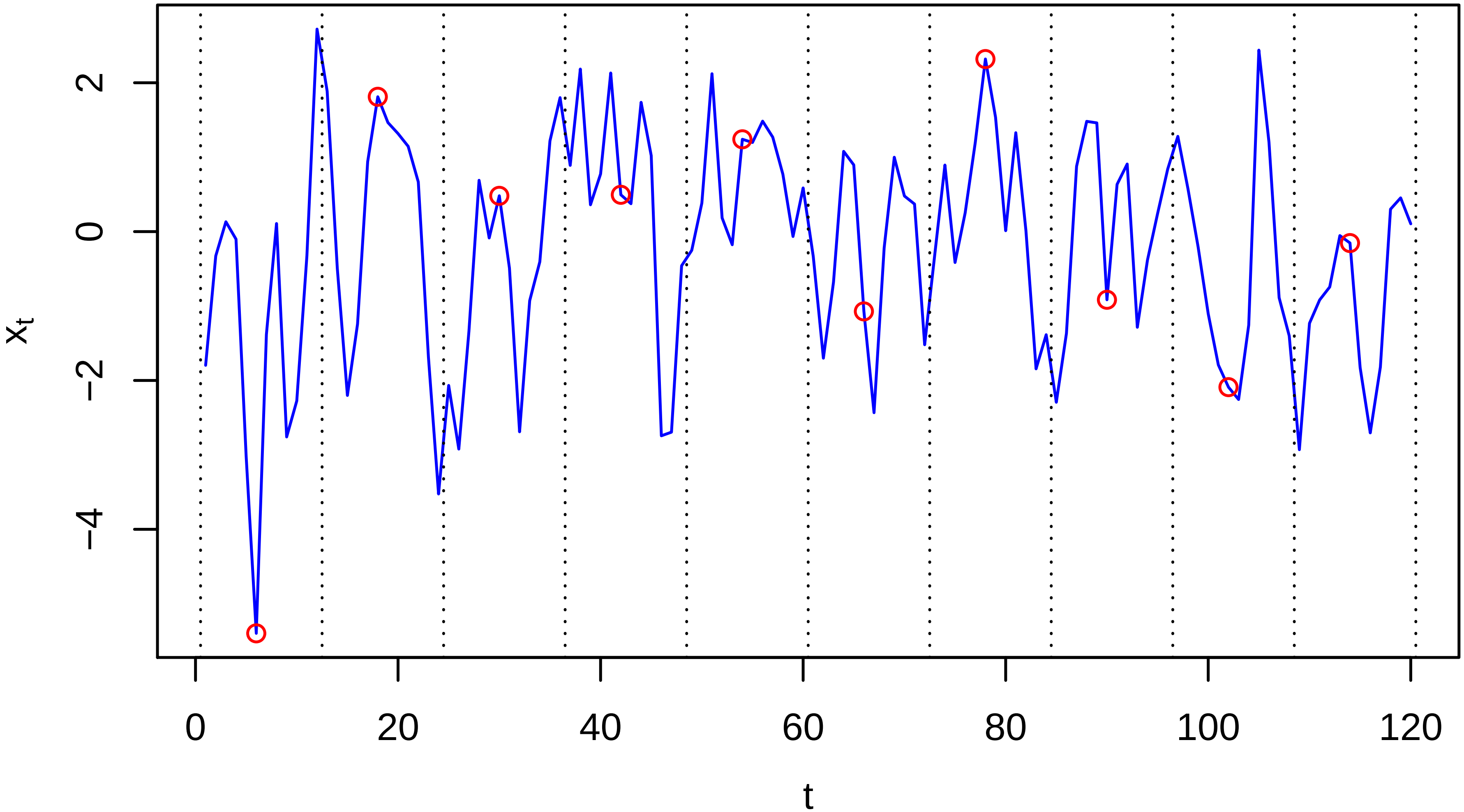
ACF for 1st SARMA$(0,1) \times (0,1)_{12}$ Model



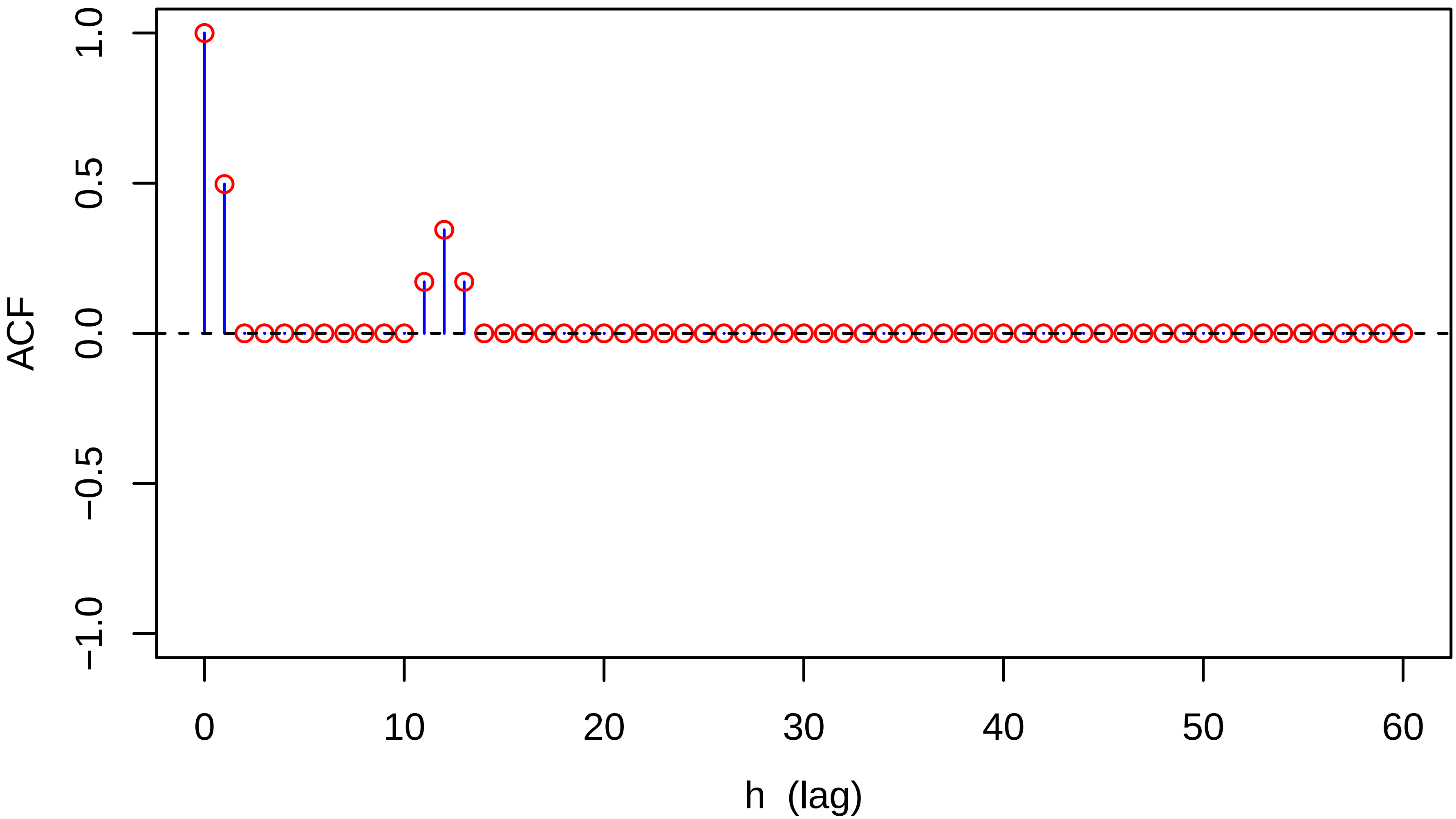
PACF for 1st SARMA$(0,1) \times (0,1)_{12}$ Model



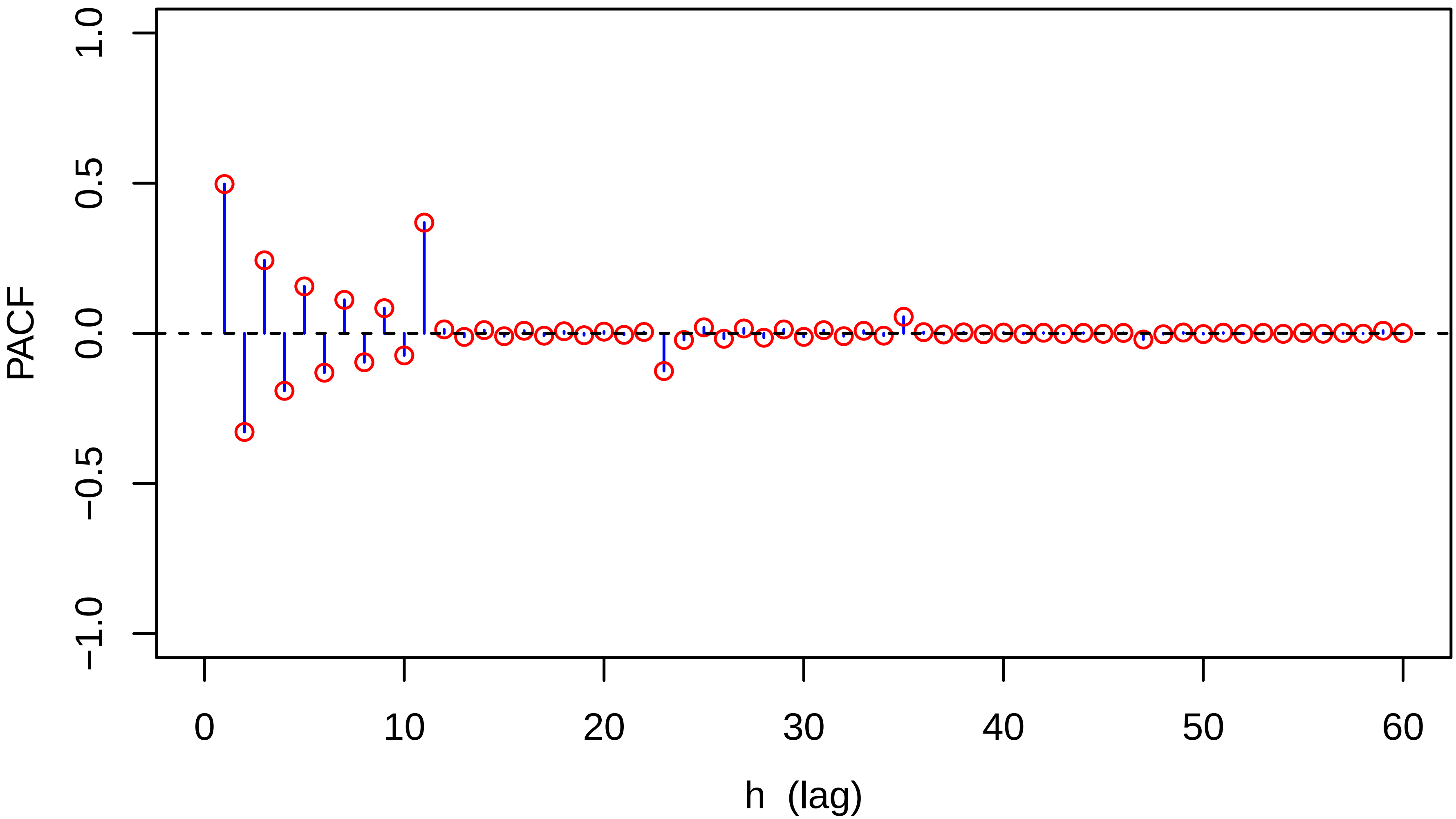
Simulated Series for 1st SARMA$(0,1) \times (0,1)_{12}$ Model



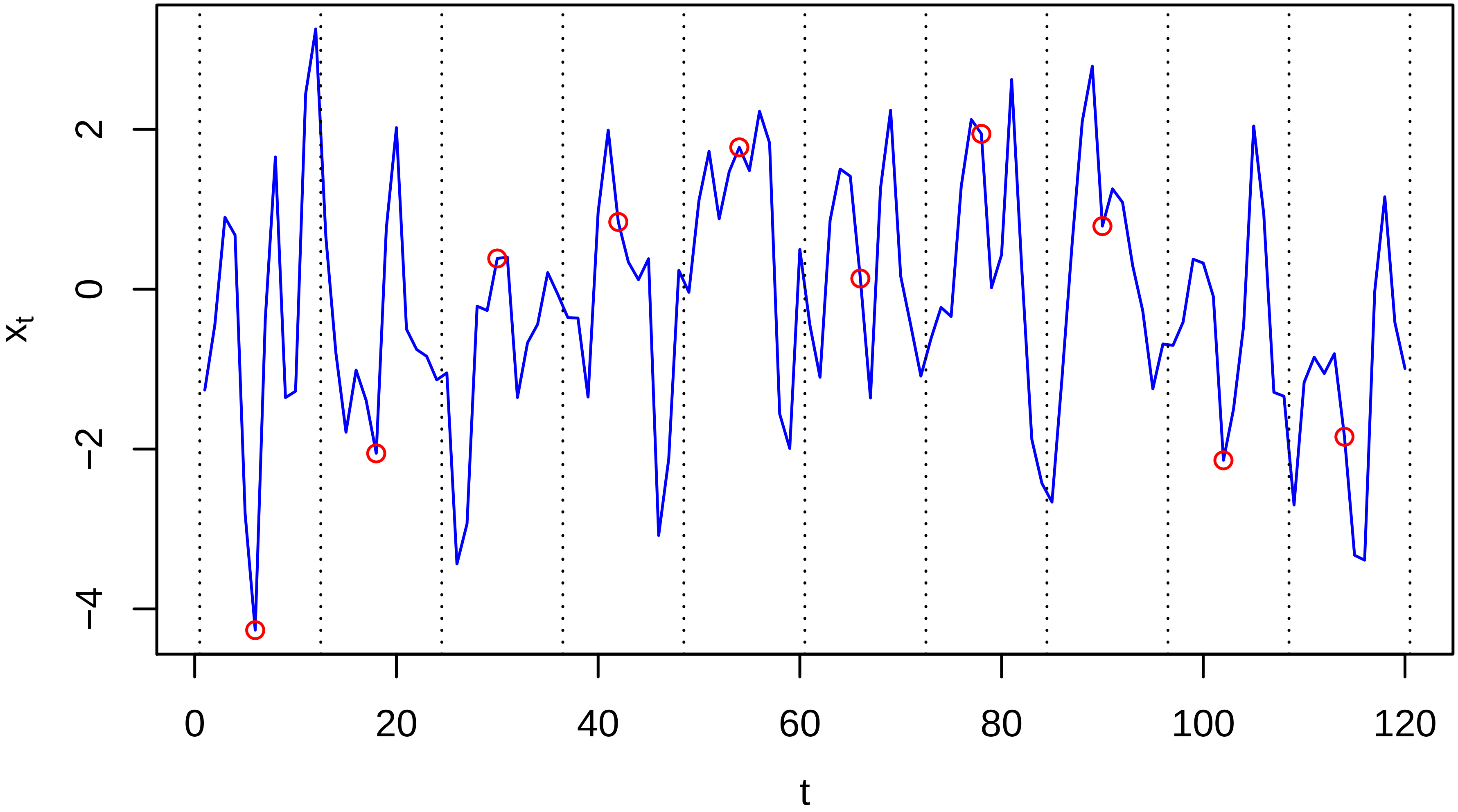
ACF for 2nd SARMA$(0,1) \times (0,1)_{12}$ Model



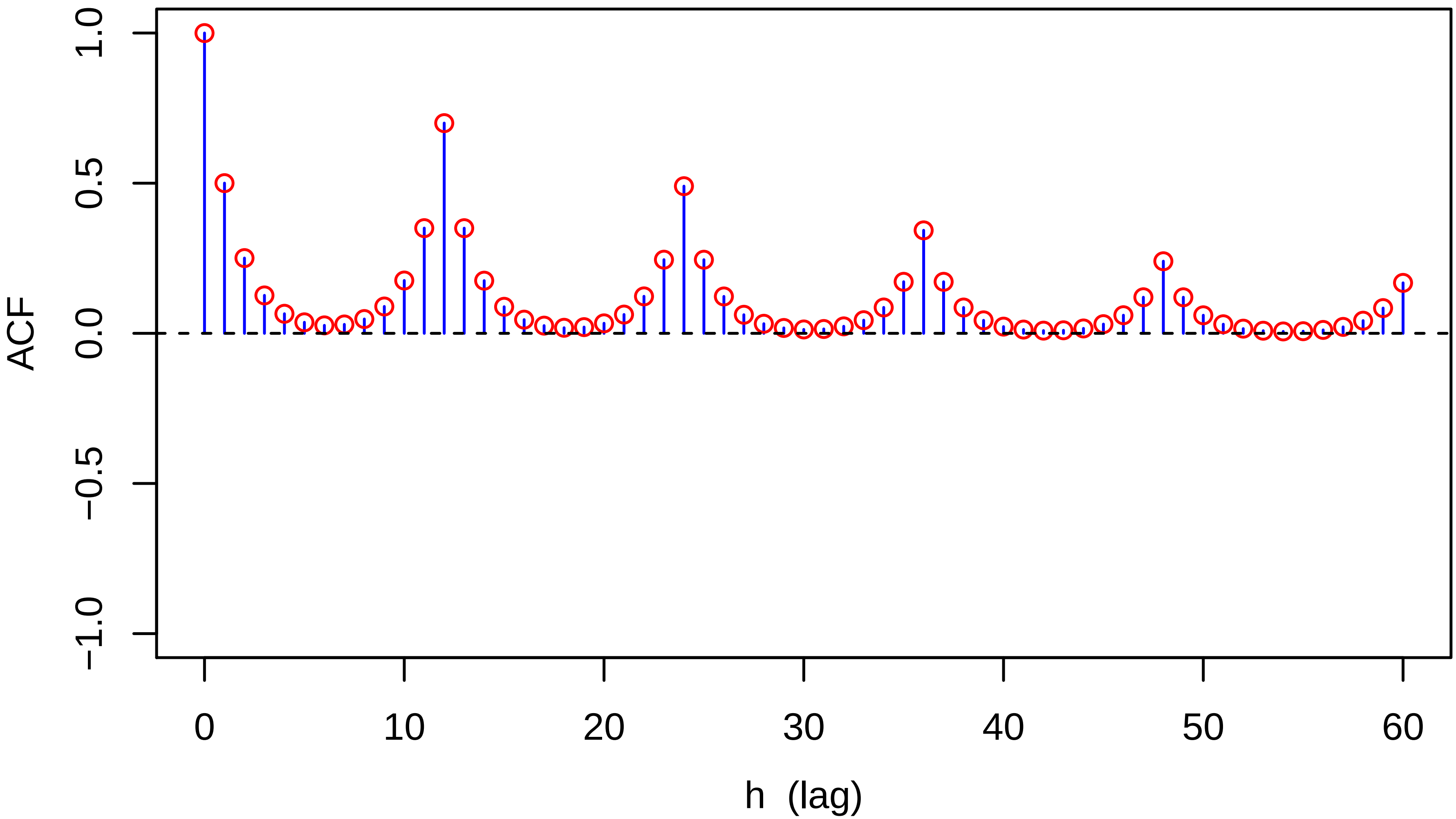
PACF for 2nd SARMA$(0,1) \times (0,1)_{12}$ Model



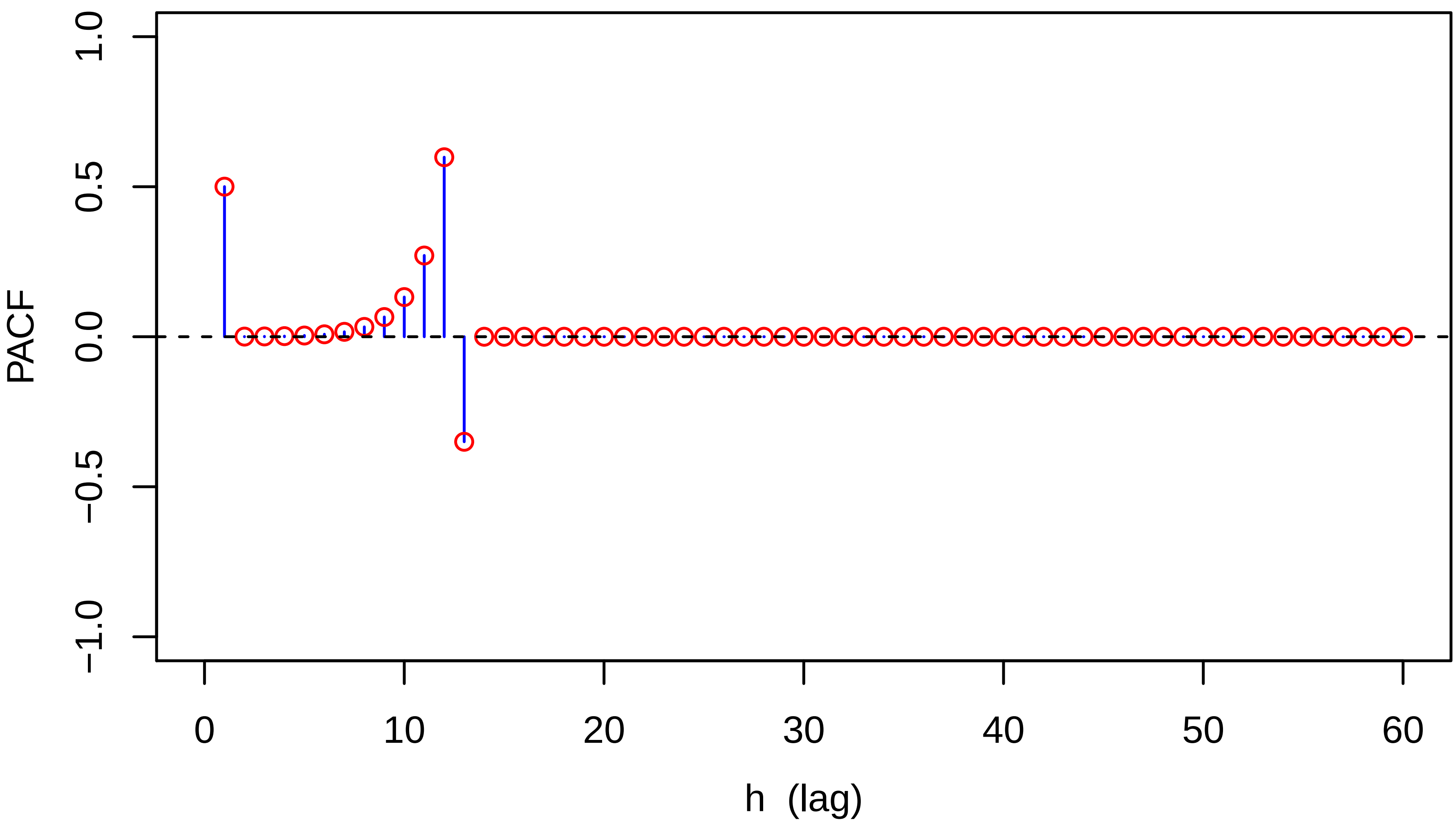
Simulated Series for 2nd SARMA$(0,1)\times(0,1)_{12}$ Model



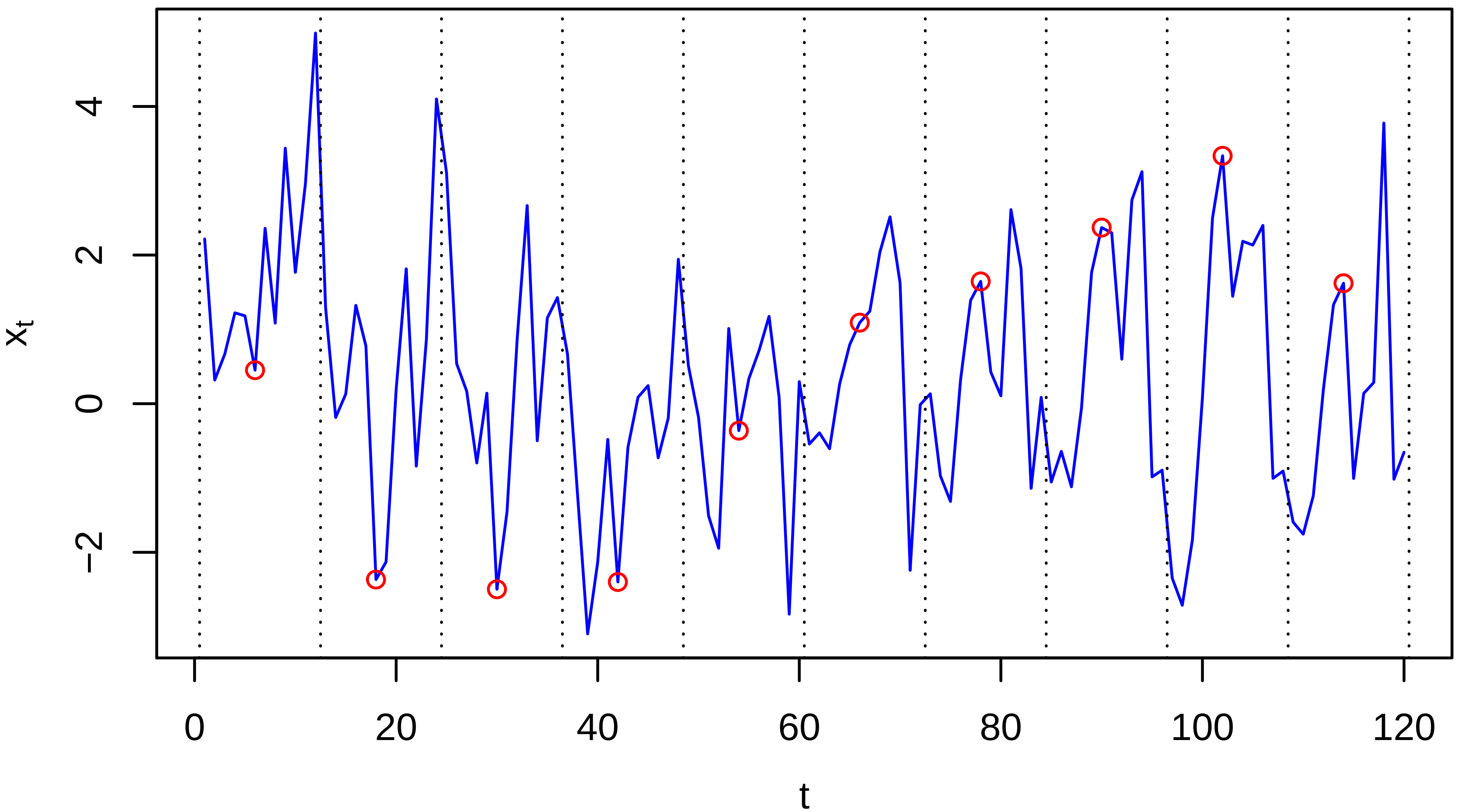
ACF for 1st SARMA$(1,0) \times (1,0)_{12}$ Model



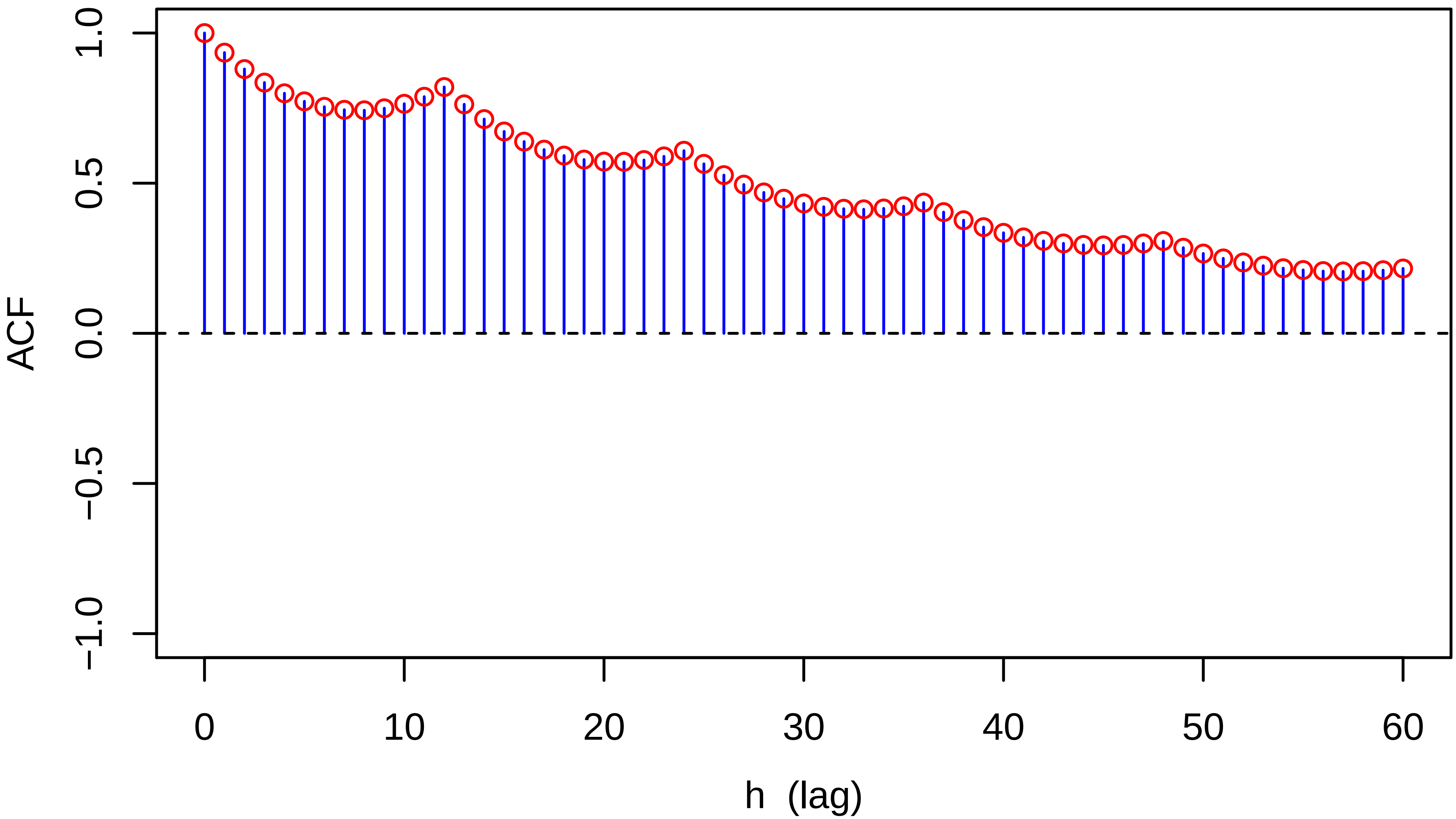
PACF for 1st SARMA$(1,0) \times (1,0)_{12}$ Model



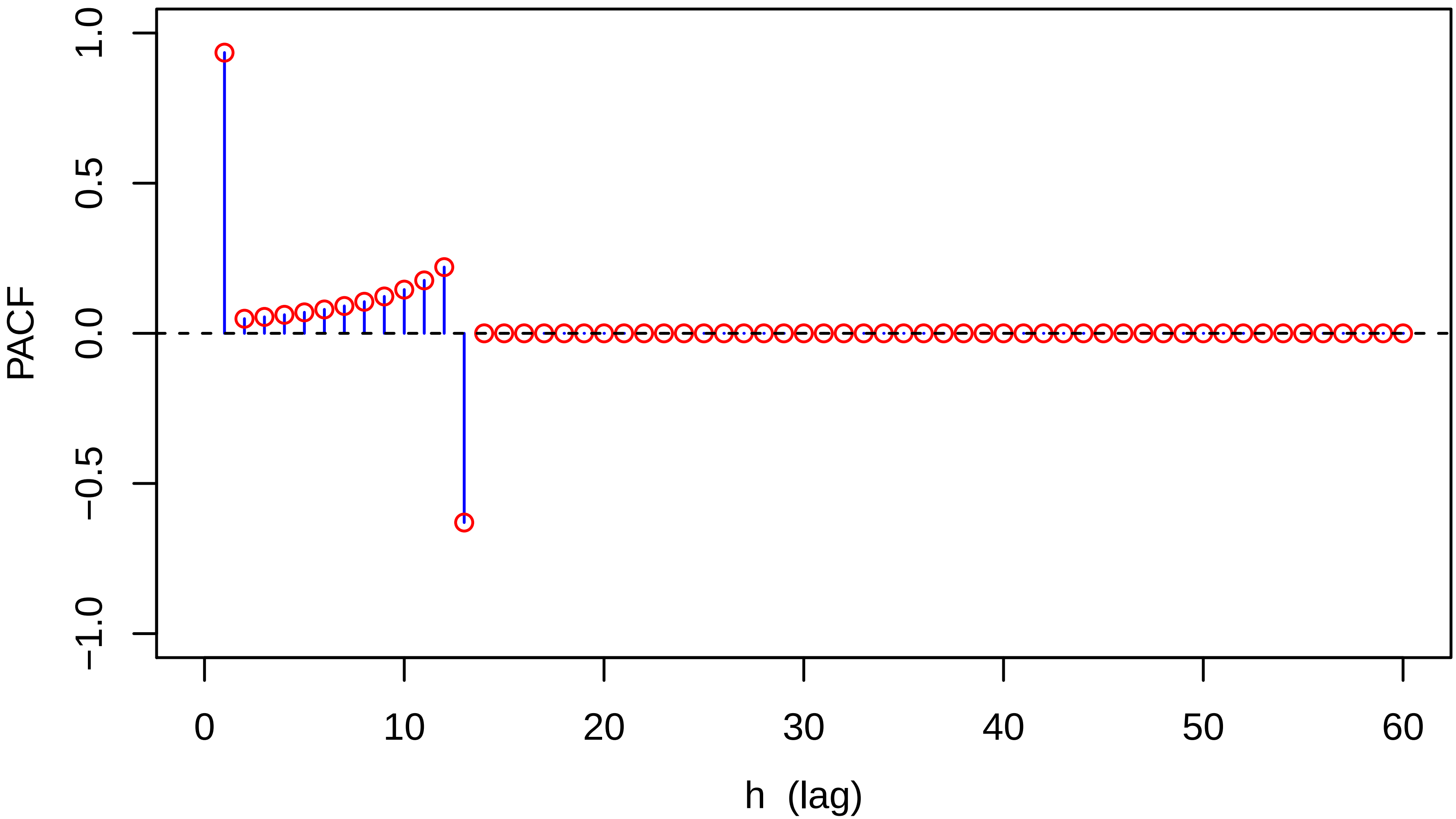
Simulated Series for 1st SARMA$(1,0) \times (1,0)_{12}$ Model



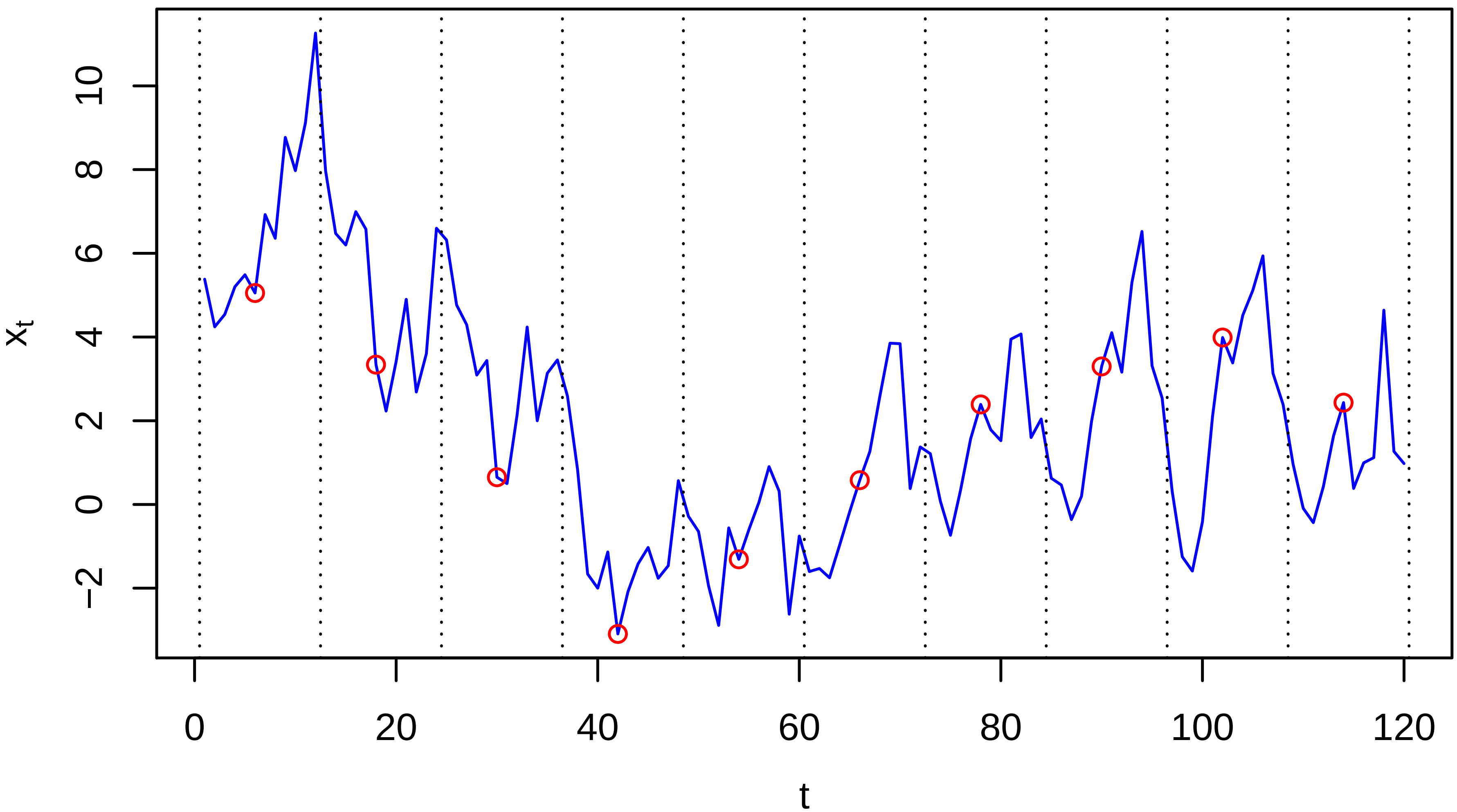
ACF for 2nd SARMA$(1,0) \times (1,0)_{12}$ Model



PACF for 2nd SARMA$(1,0) \times (1,0)_{12}$ Model



Simulated Series for 2nd SARMA$(1,0)\times(1,0)_{12}$ Model



Seasonal ARIMA Models: III

- recall definition of seasonal differencing:

$$\nabla_s X_t = X_t - X_{t-s} = (1 - B^s) X_t$$

- in model $X_t = m_t + s_t + W_t$, where $s_t = s_{t \pm s}$ is periodic with period s, seasonal differencing eliminates s_t completely
- seasonal differencing also of interest if, rather than deterministic s_t, X_t has a stochastic component that is quasi-periodic
- with monthly data in mind, consider model

$$X_t = S_t + Z_t \text{ with } \{Z_t\} \sim \mathrm{WN}(0, \sigma_Z^2),$$

where S_t is a stochastic component that is slowly varying from one January to the next, with similar patterns holding for the other 11 months

Seasonal ARIMA Models: IV

- within $X_t = S_t + Z_t$, suppose we assume

$$S_t = S_{t-12} + V_t \text{ with } \{V_t\} \sim \text{WN}(0, \sigma_V^2),$$

  where all RVs in $\{V_t\}$ are uncorrelated with those in $\{Z_t\}$

- series for each month obeys a random walk, with RVs in, e.g., random walk for May being uncorrelated with those for June

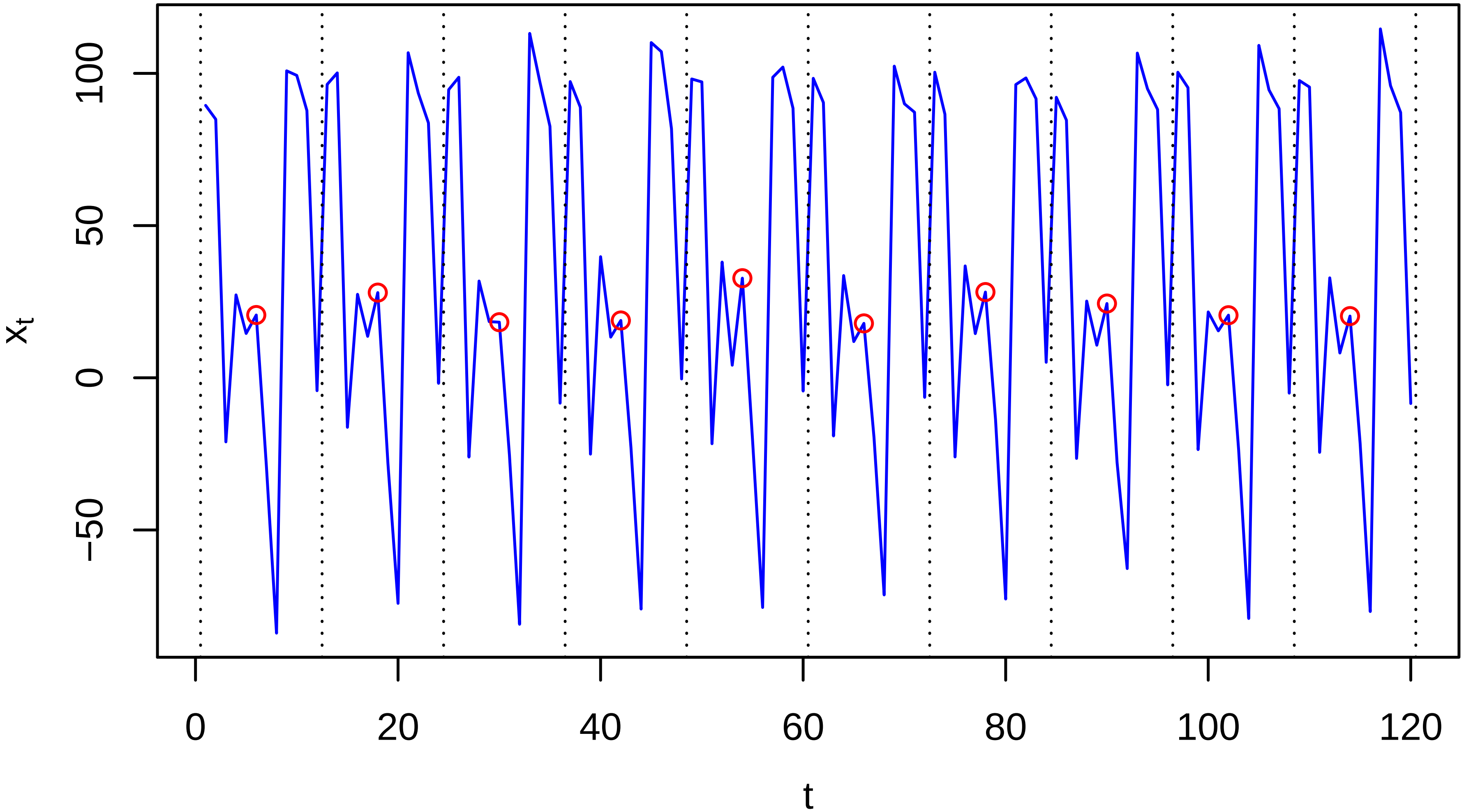
- application of seasonal differencing yields

$$\nabla_s X_t = X_t - X_{t-s} = S_t - S_{t-12} + Z_t - Z_{t-12} = V_t + Z_t - Z_{t-12} \stackrel{\text{def}}{=} W_t$$

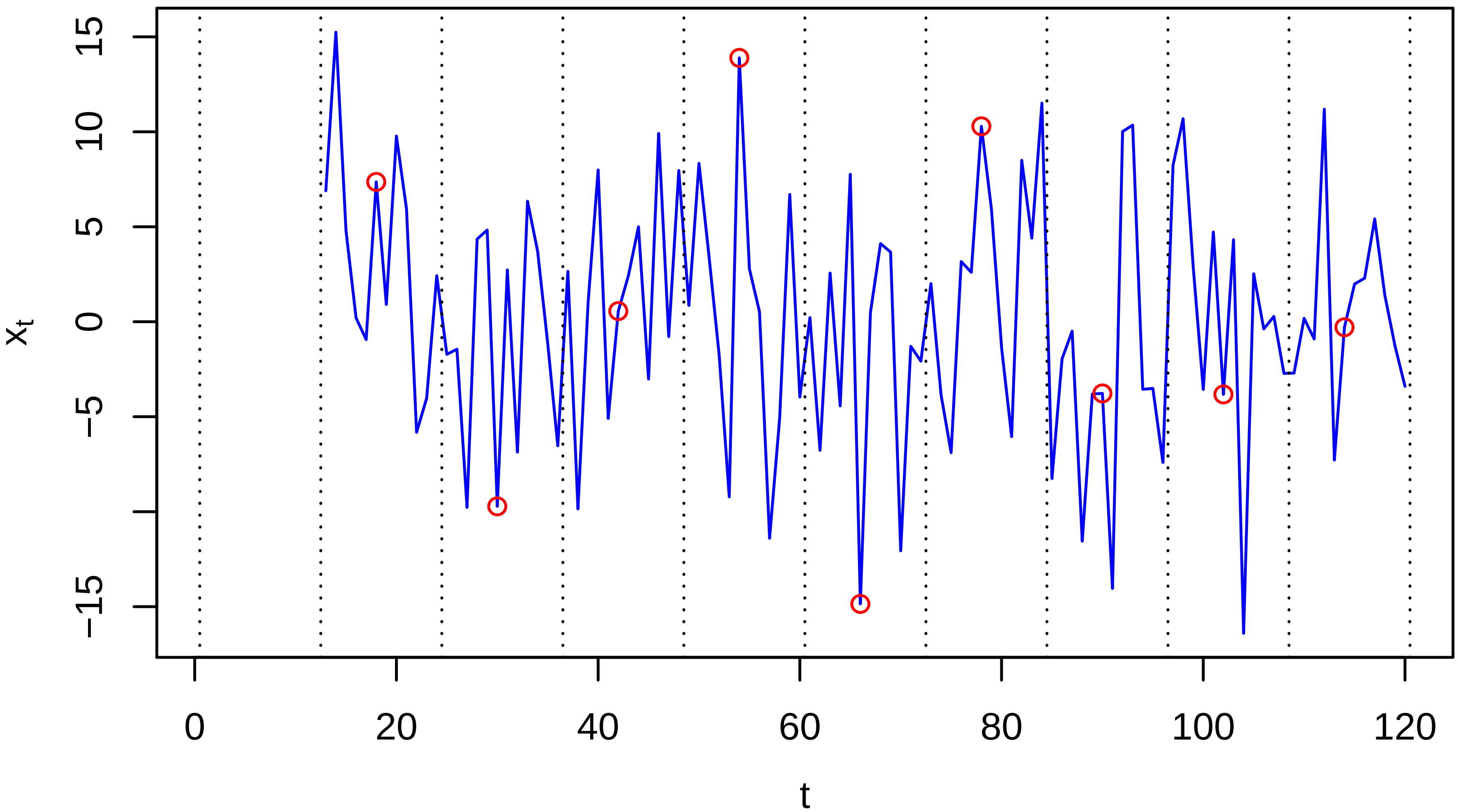
- ACVF for W_t is $\gamma_W(0) = \sigma_V^2 + 2\sigma_Z^2$, $\gamma_W(\pm 12) = -\sigma_Z^2$ and $\gamma_W(h) = 0$, otherwise, i.e., an SARMA$(0,0) \times (0,1)_{12}$ model (cf. overhead XV–9)

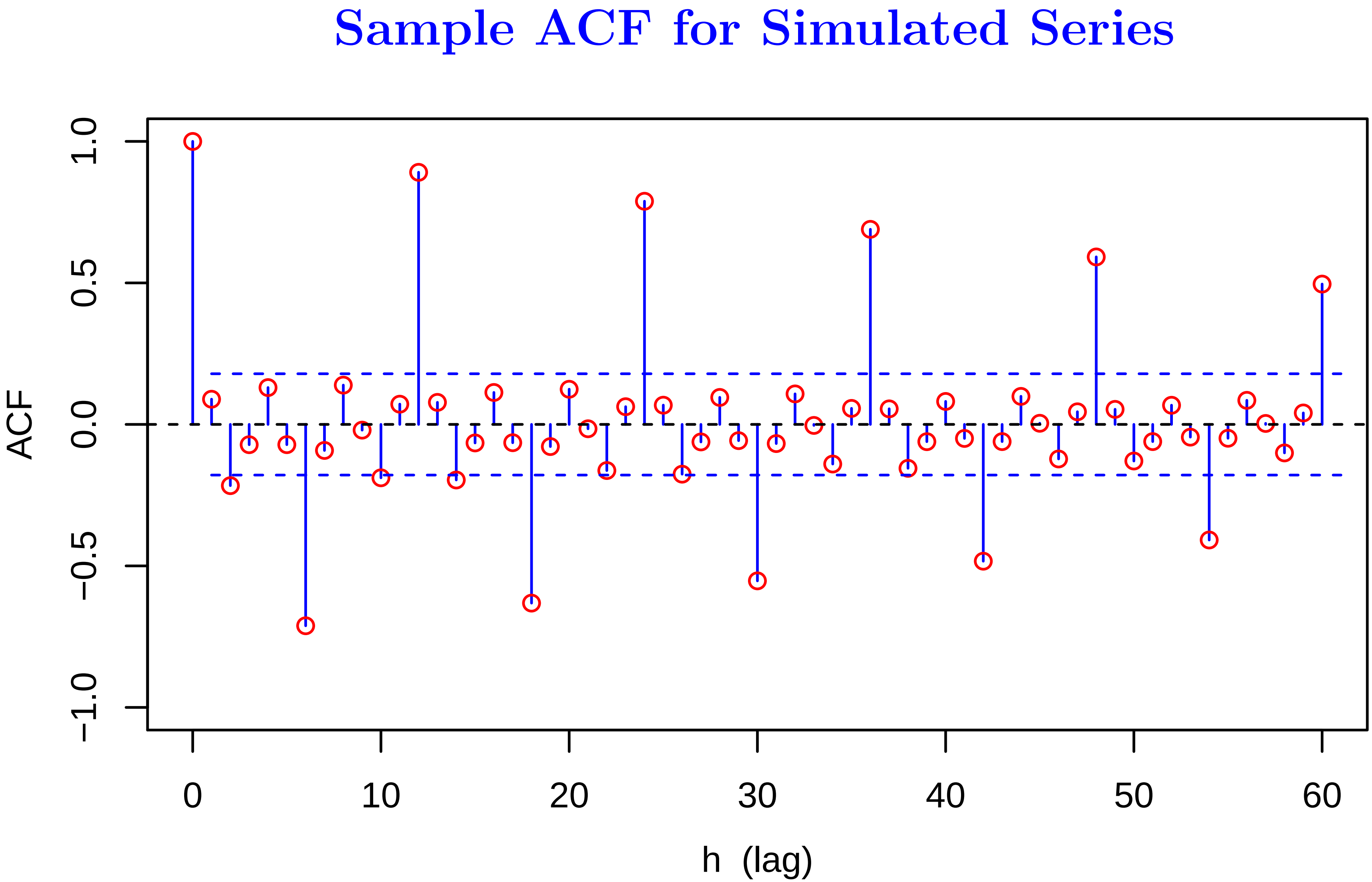
- hence X_t is a SARIMA$(0,0,0) \times (0,1,1)_{12}$ model

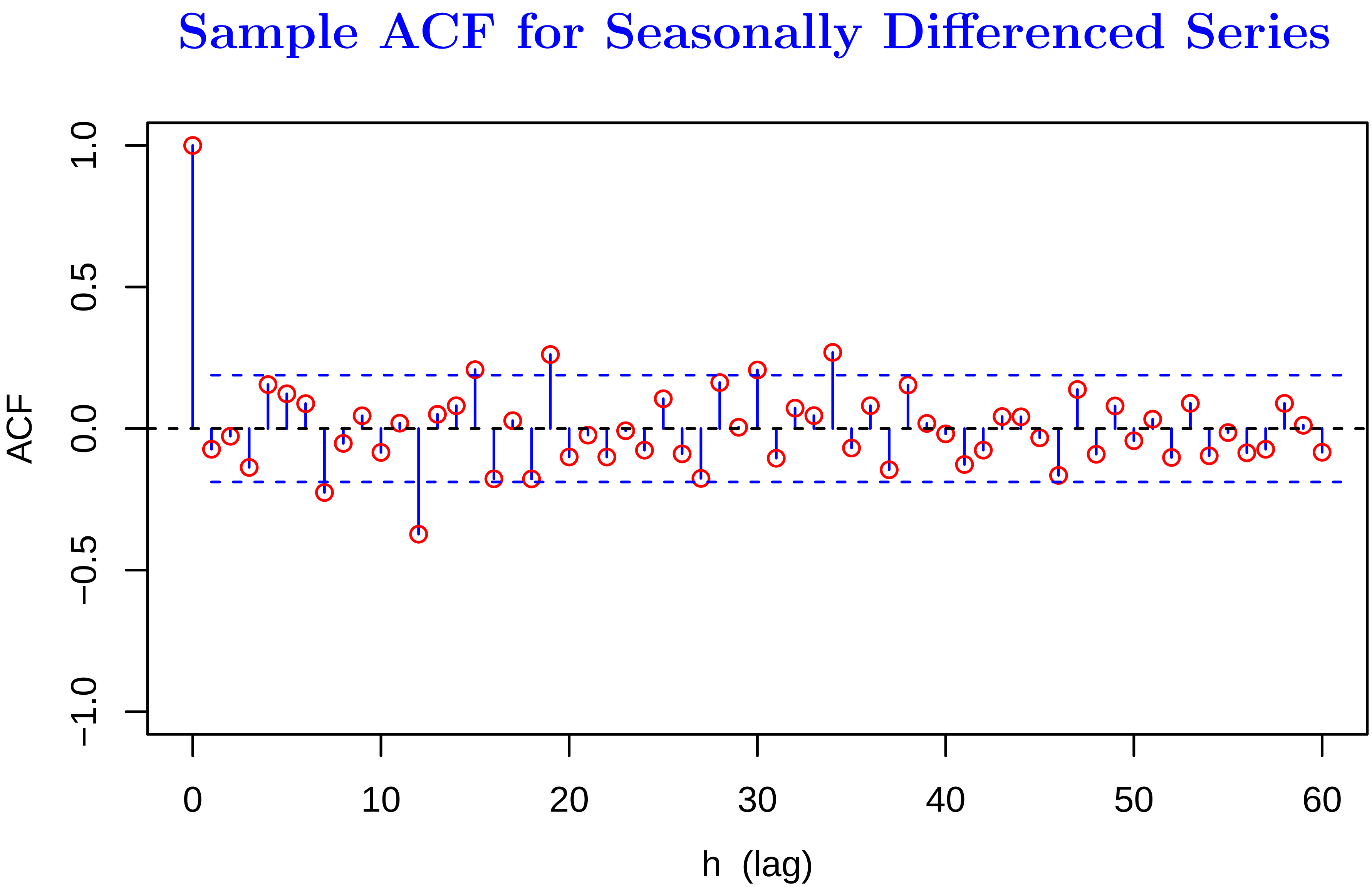
Simulation of SARIMA$(0,0,0) \times (0,1,1)_{12}$ Process



Seasonal Difference of Simulated Series



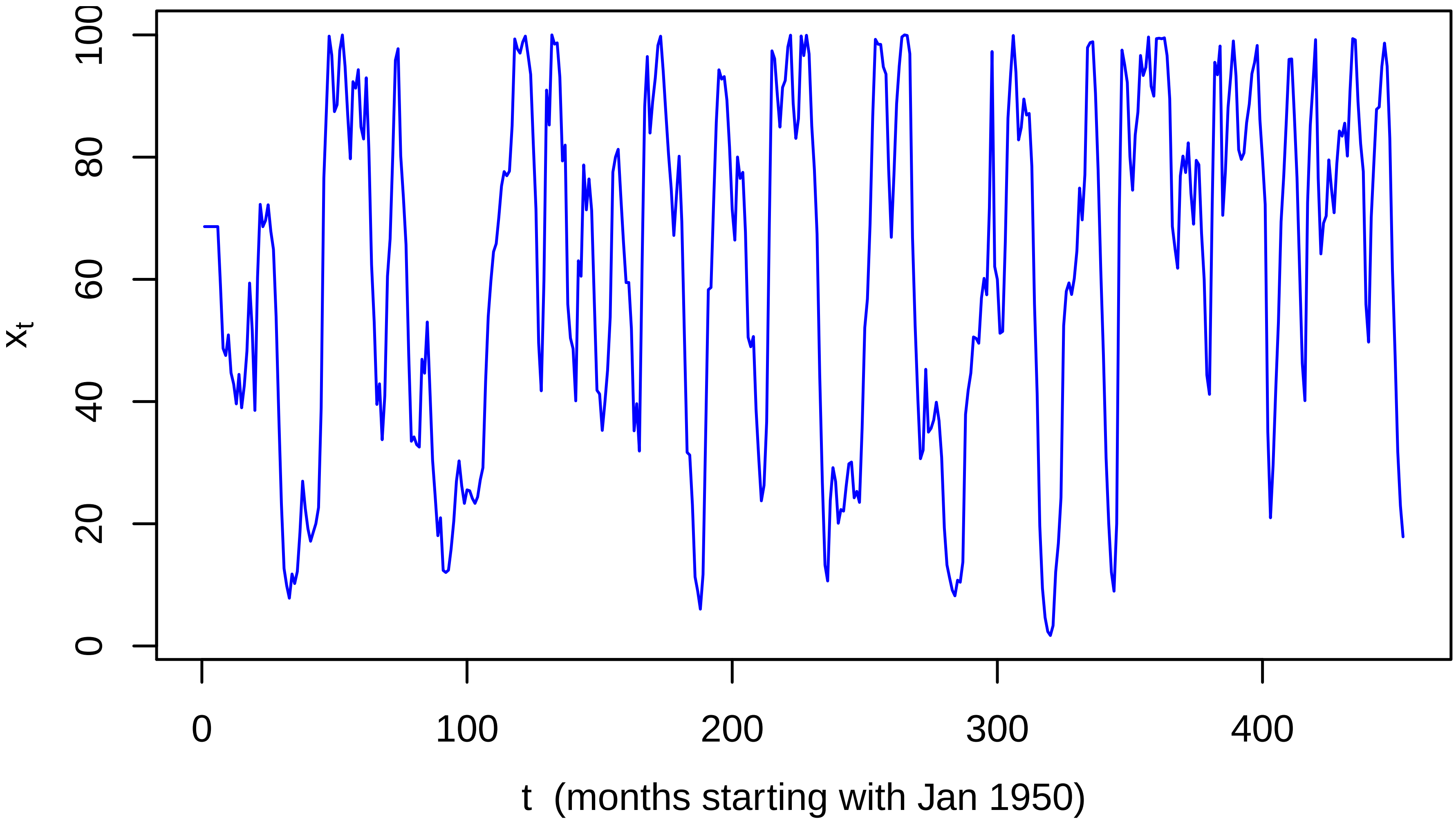

Sample ACF for Simulated Series
ACF
1.0
0.5
0.0
−0.5
−1.0
0
10
20
30
40
50
60
h (lag)


Sample ACF for Seasonally Differenced Series
ACF
1.0
0.5
0.0
−0.5
−1.0
0
10
20
30
40
50
60
h (lag)

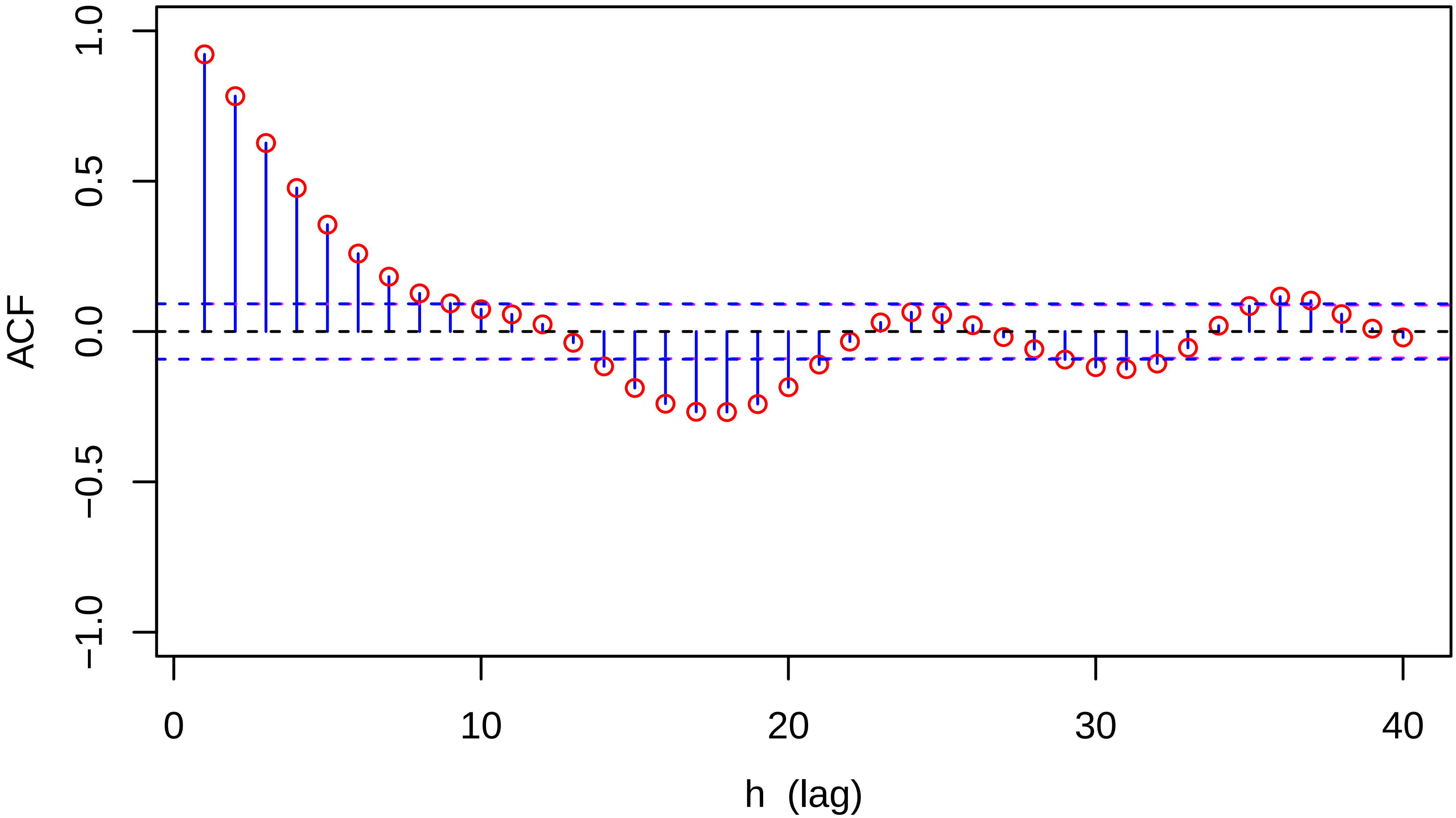
Seasonal ARIMA Models: V

- identification of SARIMA model for series X_t with known s
  1. find d and D such that
  $$Y_t = (1 - B)^d (1 - B^s)^D X_t$$
  looks amenable to modeling by a stationary process
  2. examine sample ACF and PACF for Y_t at lags s, $2s$, $3s$, $\ldots$, to select P and Q such that ARMA(P,Q) model is compatible with $\hat{\rho}(ks)$ and $\hat{\phi}_{ks,ks}$, $k = 1, 2, 3, \ldots$
  3. examine sample ACF and PACF at lags $1, 2, \ldots, s - 1$ to select p and q such that ARMA(p,q) model is compatible with $\hat{\rho}(h)$ and $\hat{\phi}_{h,h}$, $h = 1, 2, \ldots, s - 1$
- as example, reconsider recruitment time series

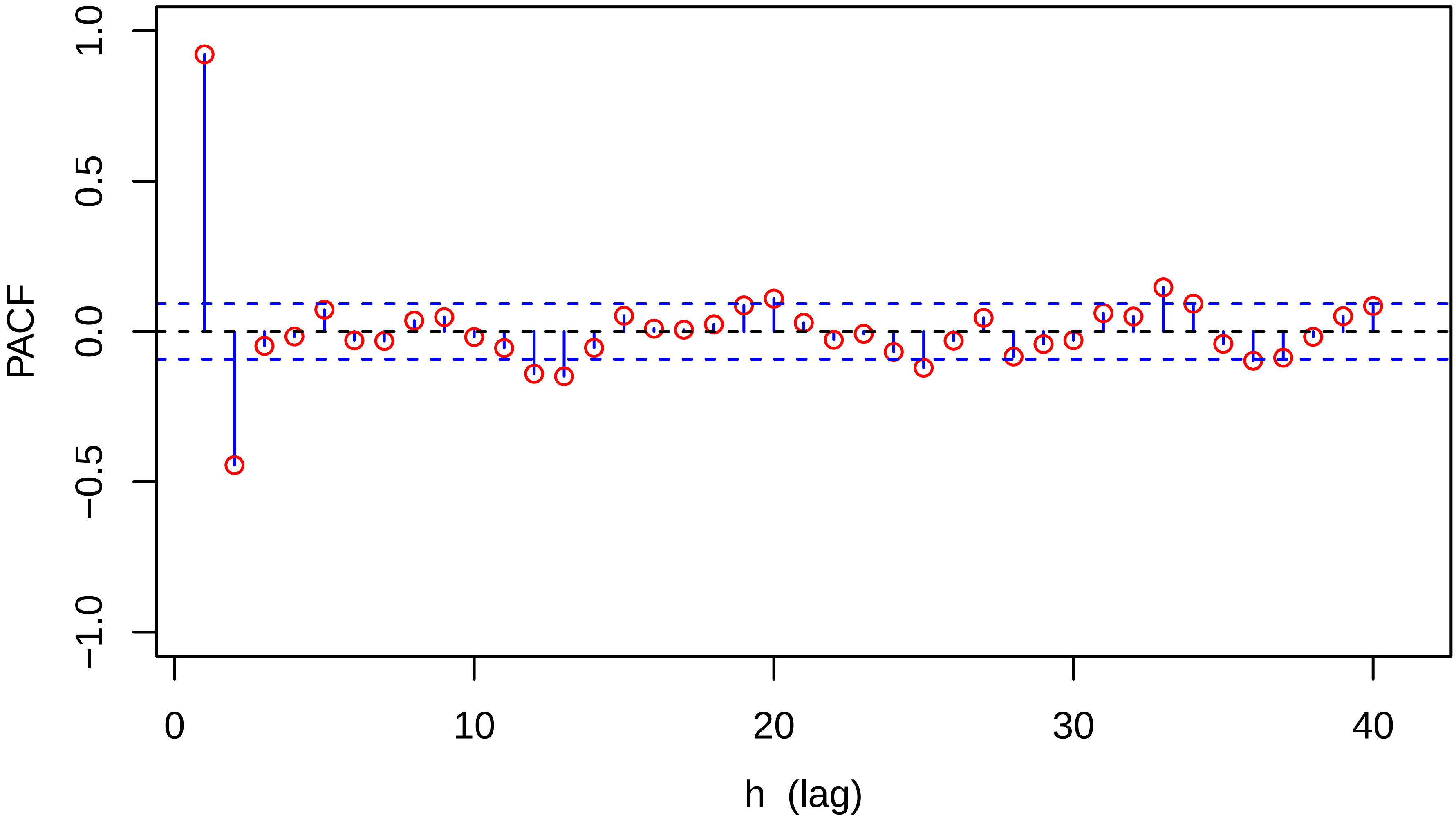
Recruitment Time Series (1950–1987)

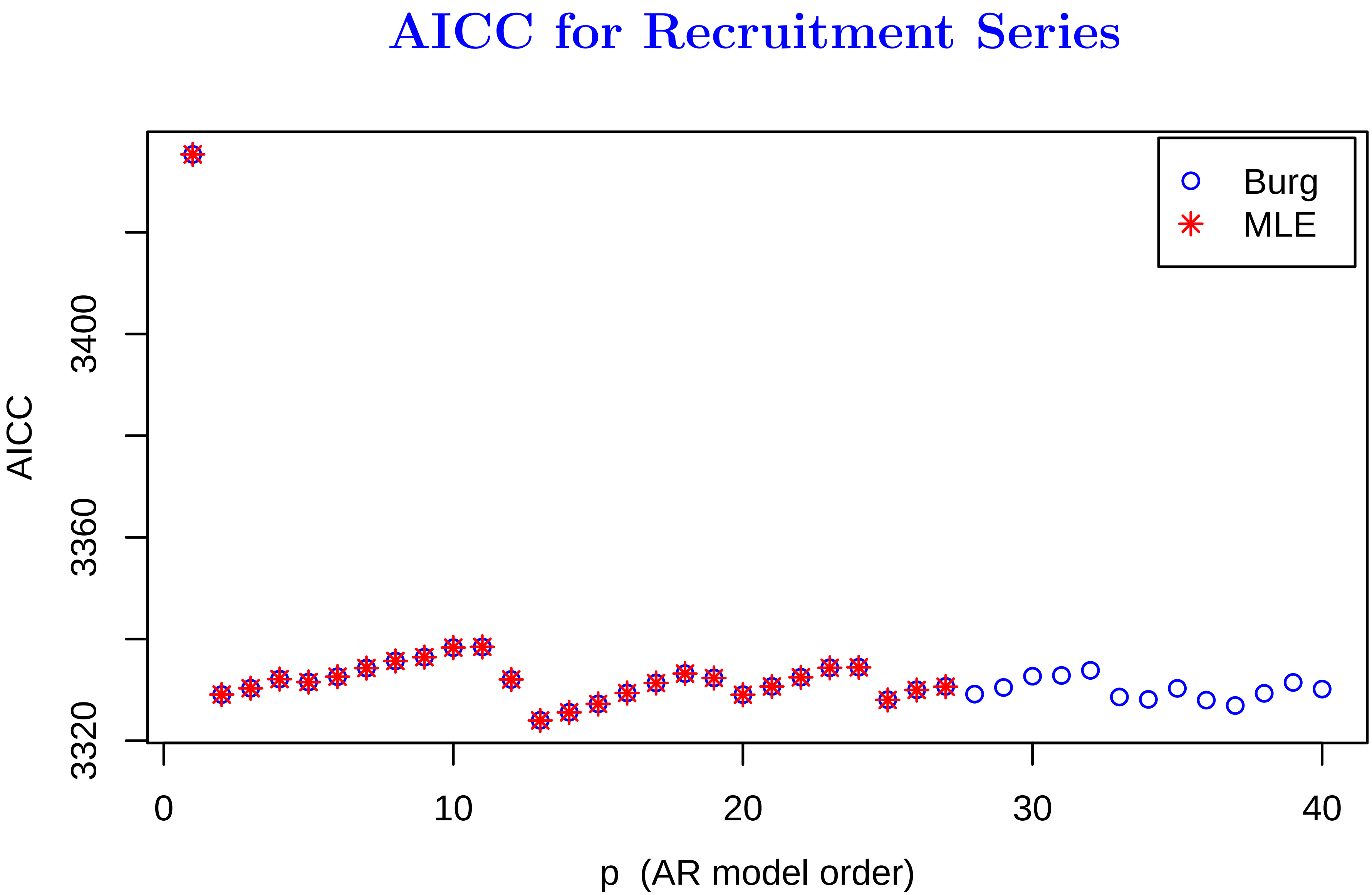


Sample ACF for Recruitment Series



Sample PACF for Recruitment Series




AICC for Recruitment Series
AICC
3320
3360
3400
0
10
20
30
40
p (AR model order)
Burg
MLE

SARIMA Model for Recruitment Series: I

- identification of SARIMA model, $s = 12$
  1. nonzero values for d and D seemed to make sample ACF and PACF more complicated, so let $d = D = 0$ and $Y_t = X_t$
  2. sample PACF has significant nonzero value at lag $s = 12$, so set $P = 1$ and $Q = 0$
  3. sample PACF has large values at lags $h = 1$ and 2, so set $p = 2$ and $q = 0$
- model is thus $\phi(B)\Phi(B)Y_t = Z_t$, $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, where

$$\begin{aligned}\phi(z)\Phi(z) &= (1 - \phi_1 z - \phi_2 z^2)(1 - \Phi z^{12}) \\ &= 1 - \phi_1 z - \phi_2 z^2 - \Phi z^{12} + \Phi\phi_1 z^{13} + \Phi\phi_2 z^{14} \stackrel{\text{def}}{=} \phi^*(z)\end{aligned}$$

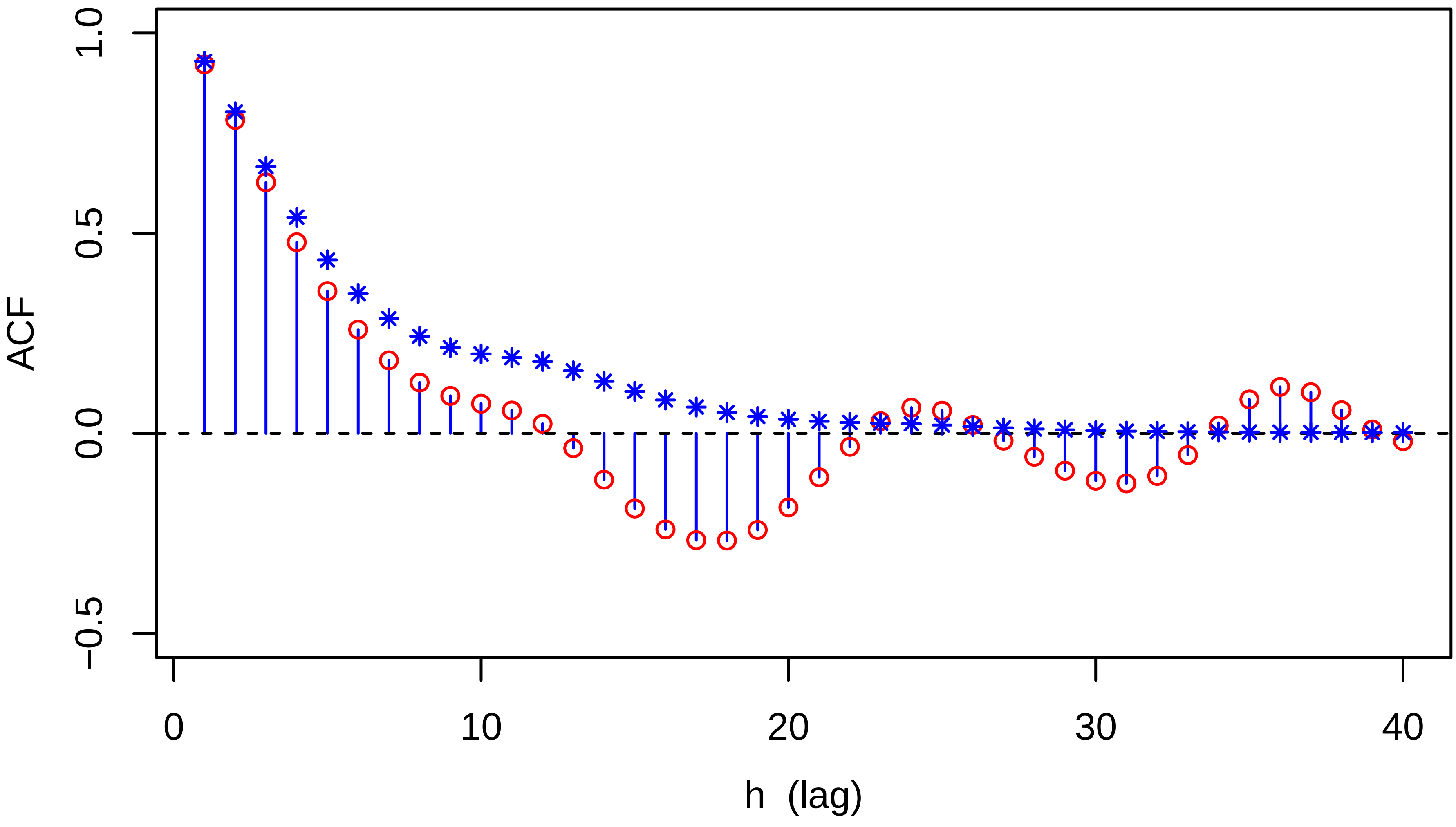
- $\phi^*(z)$ is polynomial for AR(14) process, but with $\phi_3^* = \phi_4^* = \cdots = \phi_{11}^* = 0$ (so-called subset AR(p) with dependencies amongst nonzero coefficients)

SARIMA Model for Recruitment Series: II

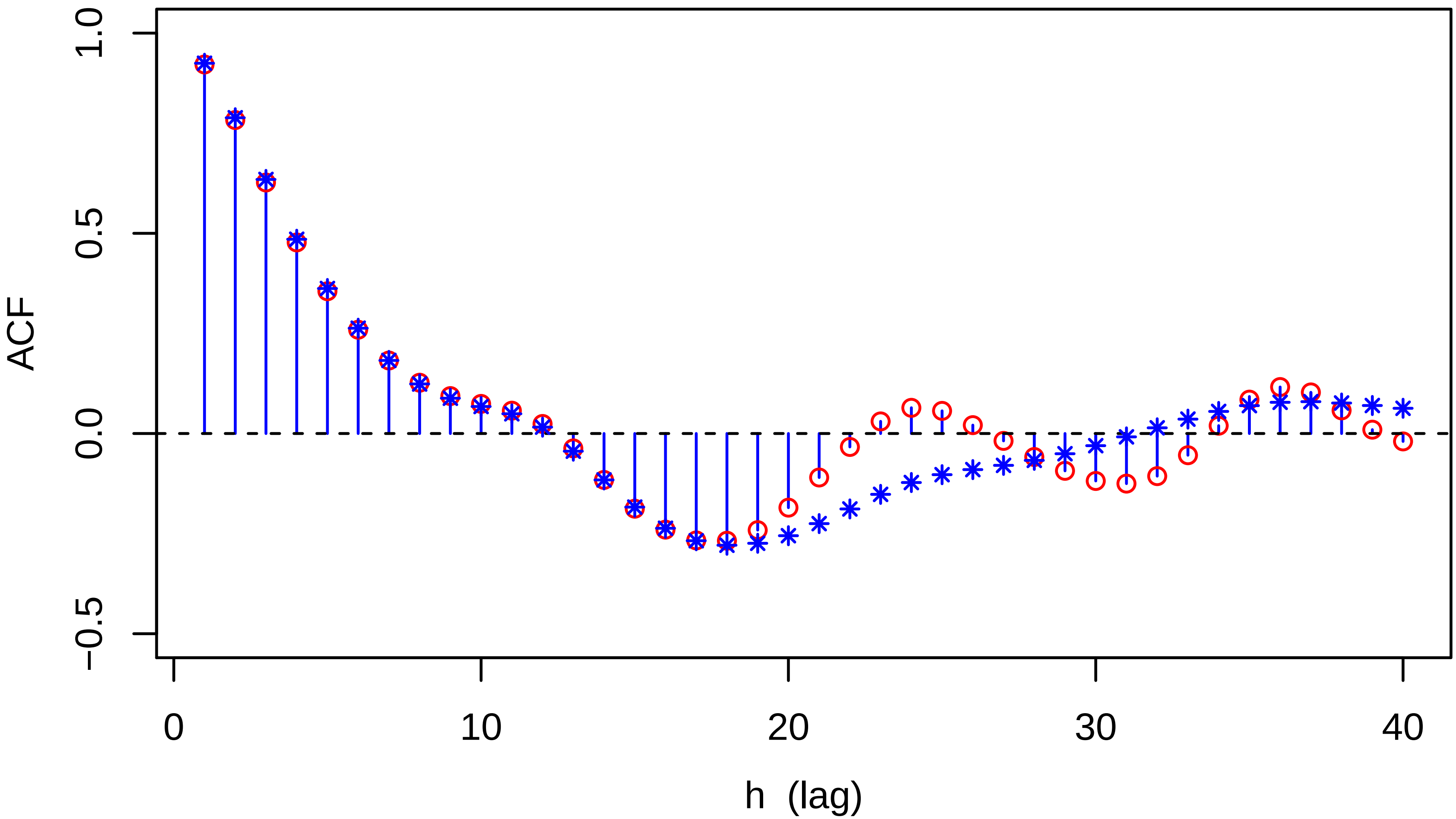
- AICC for AR(13) model is 3324.0
- AICC for SARMA$(2,0) \times (1,0)_{12}$ model is 3323.8
- tiny improvement, but # of estimated parameters now just 4
- examination of residuals gives similar results for both models (cannot reject white noise hypothesis), with exception of
  - p-value close to 0.05 for $h = 5$ portmanteau test for SARMA
  - p-value of 0.015 from rank test for AR(13) and
  - p-value of 0.047 from runs test for SARMA

  (for details, see `R` code for this overhead)
- comparison of sample ACF and PACF with corresponding theoretical functions shows better visual agreement with AR(13) (but appearances can be deceiving!)

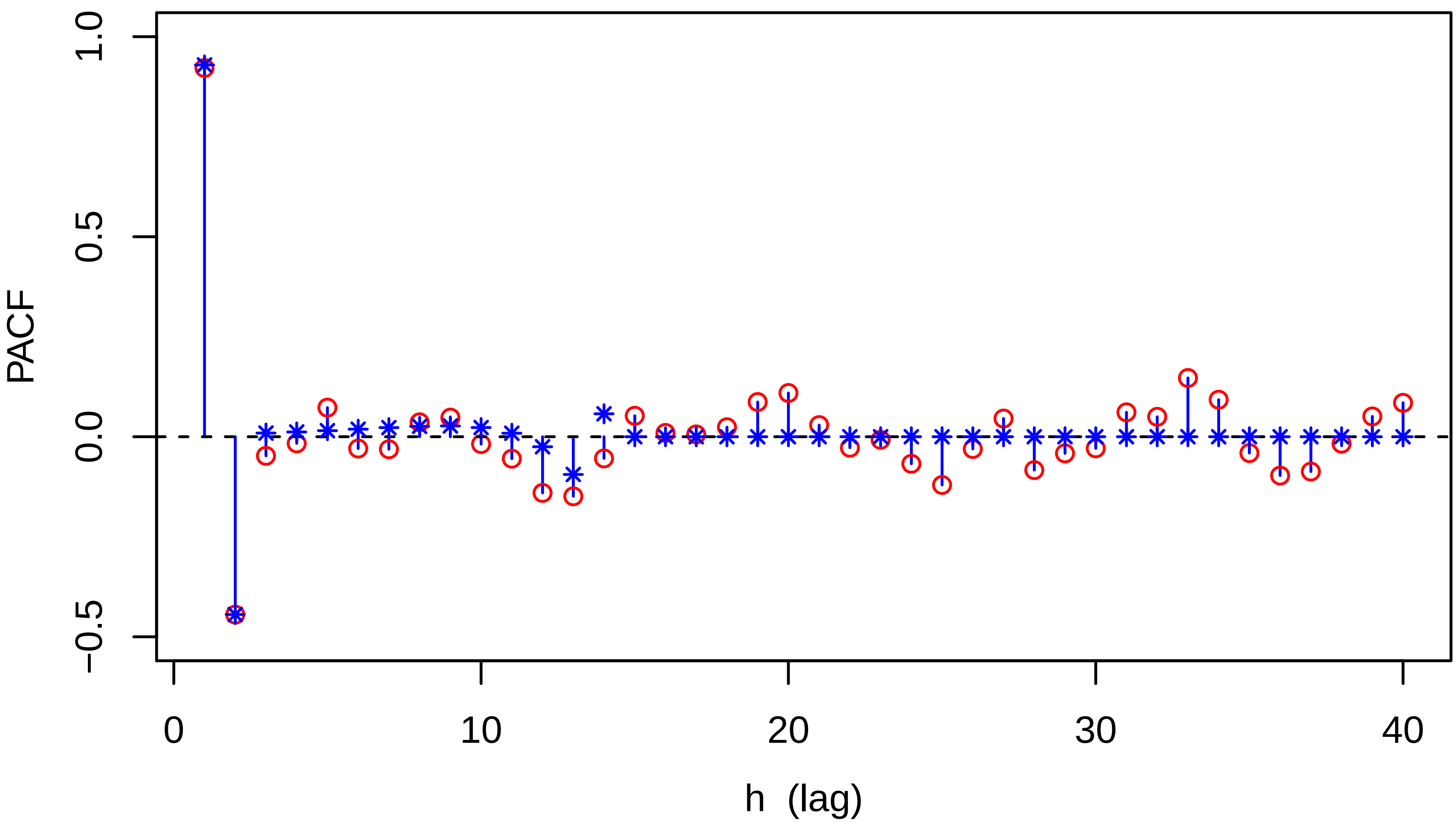
Sample & SARMA$(2,0) \times (1,0)_{12}$ ACFs



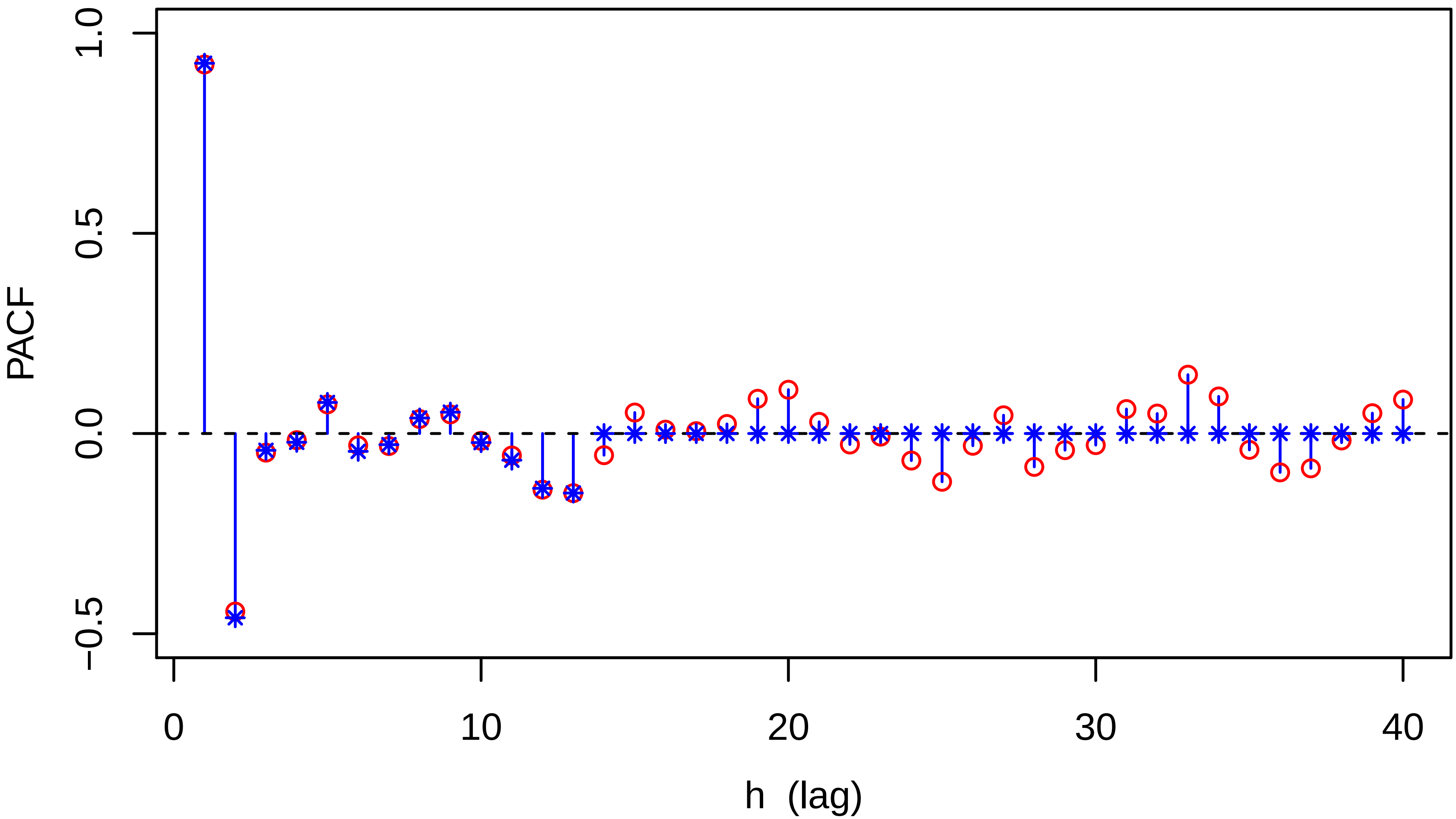
Sample & AR(13) ACFs



Sample & SARMA$(2,0) \times (1,0)_{12}$ PACFs

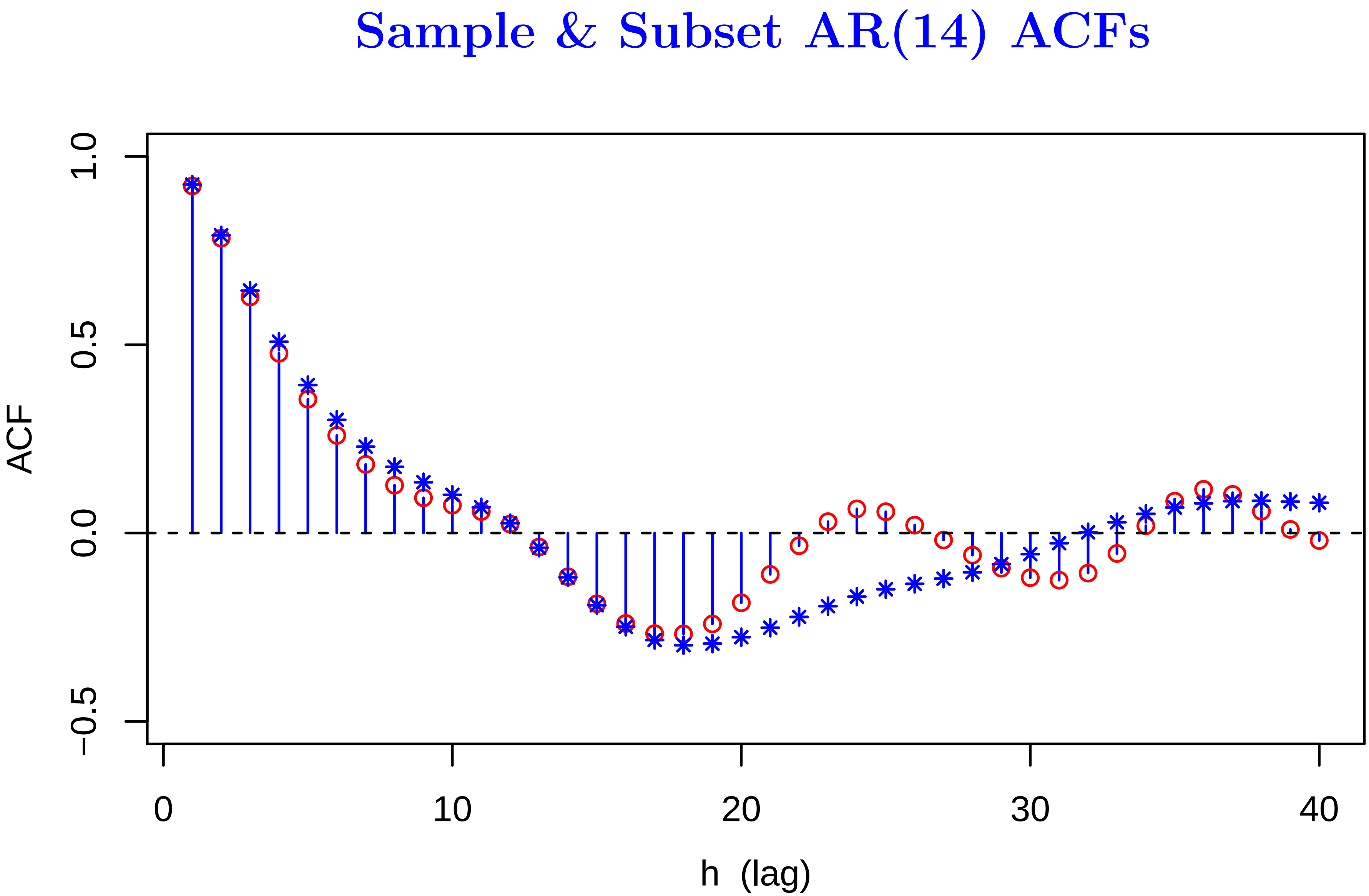


Sample & AR(13) PACFs

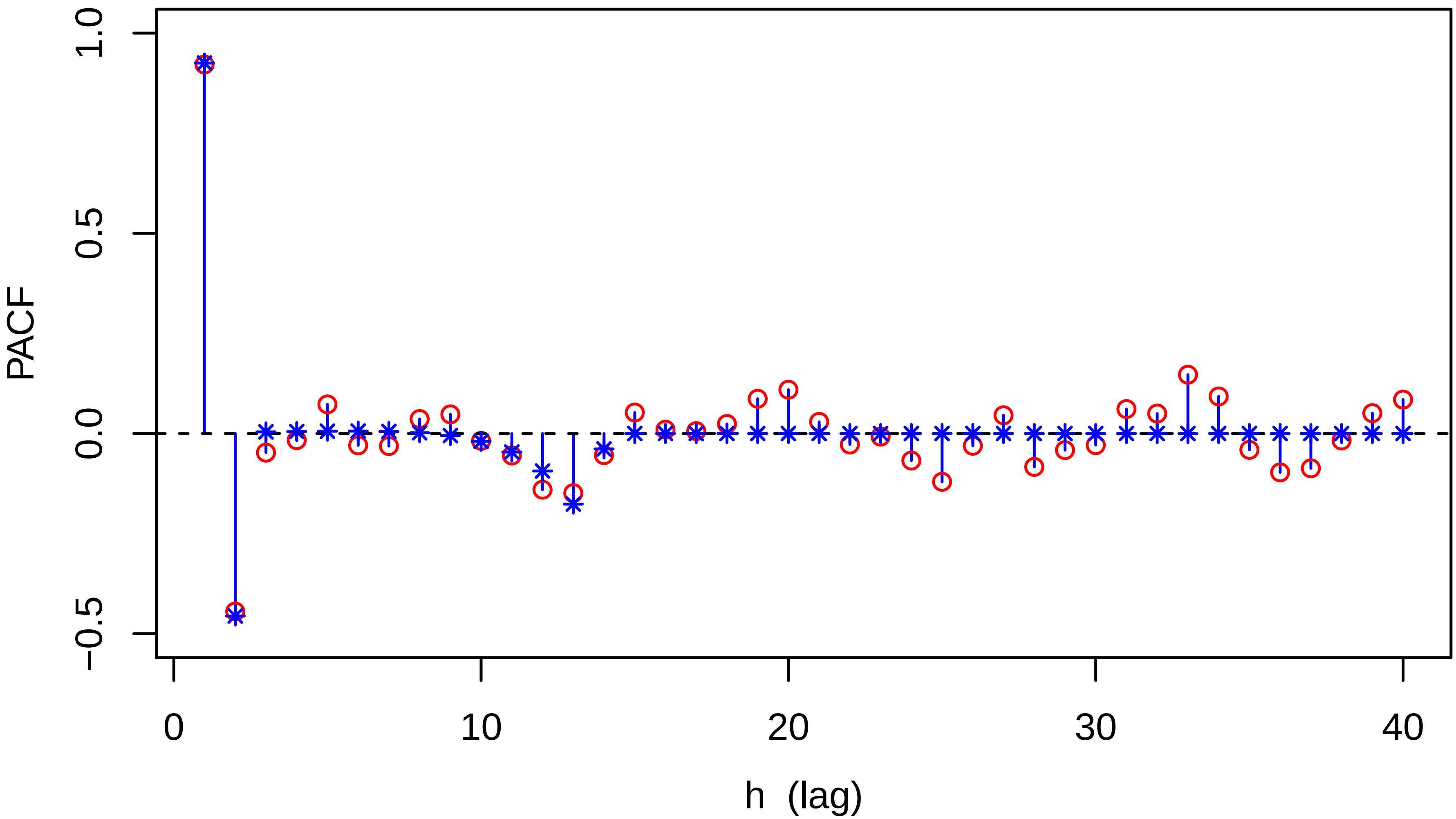


Subset AR Models for Recruitment Series: I

- SARMA$(2,0)\times(1,0)_{12}$ model is essentially AR(14) model with coefficients ϕ_i^* such that $\phi_3^* = \phi_4^* = \cdots = \phi_{11}^* = 0$
- remaining 5 coefficients ϕ_1^*, ϕ_2^*, ϕ_{12}^*, ϕ_{13}^* and ϕ_{14}^* are determined by 3 free parameters ϕ_1, ϕ_2 and Φ
- as compromise between this model and AR(13) model with 13 free coefficients, consider AR(14) model with coefficients $\phi_i^\dagger$ such that $\phi_3^\dagger = \phi_4^\dagger = \cdots = \phi_{11}^\dagger = 0$
- remaining 5 coefficients $\phi_1^\dagger$, $\phi_2^\dagger$, $\phi_{12}^\dagger$, $\phi_{13}^\dagger$ and $\phi_{14}^\dagger$ are allowed to vary without constraints being imposed
- AICC for this model is 3315.1, as compared to 3324.0 for AR(13) and 3323.8 for SARMA$(2,0)\times(1,0)_{12}$


Sample & Subset AR(14) ACFs
ACF
1.0
0.5
0.0
−0.5
0
10
20
30
40
h (lag)

Sample & Subset AR(14) PACFs



Subset AR Models for Recruitment Series: II

- another approach to subsetting is to look at variability in $\hat{\phi}_i$'s in fitted AR(13)
- formulate subset model with coefficients corresponding to just those $\hat{\phi}_i$'s that are significantly different from zero at, say, 0.05 level of significance
- only $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\phi}_{13}$ are such
- AICC for this subset AR(13) model is 3323.6, a tiny improvement over AICCs for AR(13) and SARMA$(2,0)\times(1,0)_{12}$ models (3324.0 and 3323.8), but higher than AICC for subset 14 model (3315.1)
- conclusion: subset AR(14) model with 5 non-zero coefficients best amongst models considered