Beyond ARMA Models: Nonstationarity & Seasonality

- now know how to handle time series in a certain 'comfort zone'
  - no worrisome trends or seasonal patterns
  - sample ACF and PACF decay fairly rapidly
  - simple ARMA model quantifies correlation structure ('simple' meaning $p + q$ is relatively small)
- can handle certain 'out of zone' time series through manipulations yielding series that are back in comfort zone
  1. differencing – leads to ARIMA models for handling certain types of nonstationary time series (next topic for discussion)
  2. seasonal differencing – leads to SARIMA models for handling certain types of seasonality
  3. regression analysis with ARMA errors – designed to handle certain types of trends

BD: 158, 177, 184

Intrinsically Stationary Processes

- stochastic process $\{X_t\}$ said to be intrinsically stationary of integer order $d > 0$ if $\{X_t\}$, $\{\nabla X_t\}$, ..., $\{\nabla^{d-1} X_t\}$ are non-stationary, but $\{\nabla^d X_t\}$ is a stationary process, where

$$
\begin{aligned}
\nabla X_t &\stackrel{\text{def}}{=} (1 - B)X_t = X_t - X_{t-1} \\
\nabla^2 X_t &\stackrel{\text{def}}{=} \nabla(\nabla X_t) = (1 - B)^2 X_t \\
&= (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) \\
&= X_t - 2X_{t-1} + X_{t-2} \\
&\vdots \\
\nabla^d X_t &\stackrel{\text{def}}{=} (1 - B)^d X_t = \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k}
\end{aligned}
$$

- convenient to refer to stationary process as being intrinsically stationary of order $d = 0$

ARIMA(p, d, q) Processes

- process $\{X_t\}$ is an ARIMA(p, d, q) process if
  1. $\{X_t\}$ is intrinsically stationary of order d and
  2. $\{\nabla^d X_t\}$ is an ARMA(p, q) process
- with $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, can express model as

$$\phi(B)(1 - B)^d X_t = \theta(B) Z_t$$

  or, equivalently, as

$$\phi^*(B) X_t = \theta(B) Z_t,$$

  where $\phi^*(B) = \phi(B)(1 - B)^d$
- example: for ARIMA(1,1,0) model, have

$$\phi^*(B) = (1 - \phi B)(1 - B) = 1 - (1 + \phi)B + \phi B^2$$

ARIMA(0,1,0) Process: I

- simplest example of ARIMA process is ARIMA(0,1,0):

$$(1 - B)X_t = X_t - X_{t-1} = Z_t,$$

for which, assuming existence of X_0 and assuming $t \geq 1$,

$$\begin{aligned} X_1 &= X_0 + Z_1 \\ X_2 &= X_1 + Z_2 = X_0 + Z_1 + Z_2 \\ X_3 &= X_2 + Z_3 = X_0 + Z_1 + Z_2 + Z_3 \\ &\vdots \\ X_t &= X_0 + \sum_{u=1}^{t} Z_u, \end{aligned}$$

- above is a random walk starting from X_0

ARIMA(0,1,0) Process: II

- assuming RV X_0 is uncorrelated with Z_t's, have

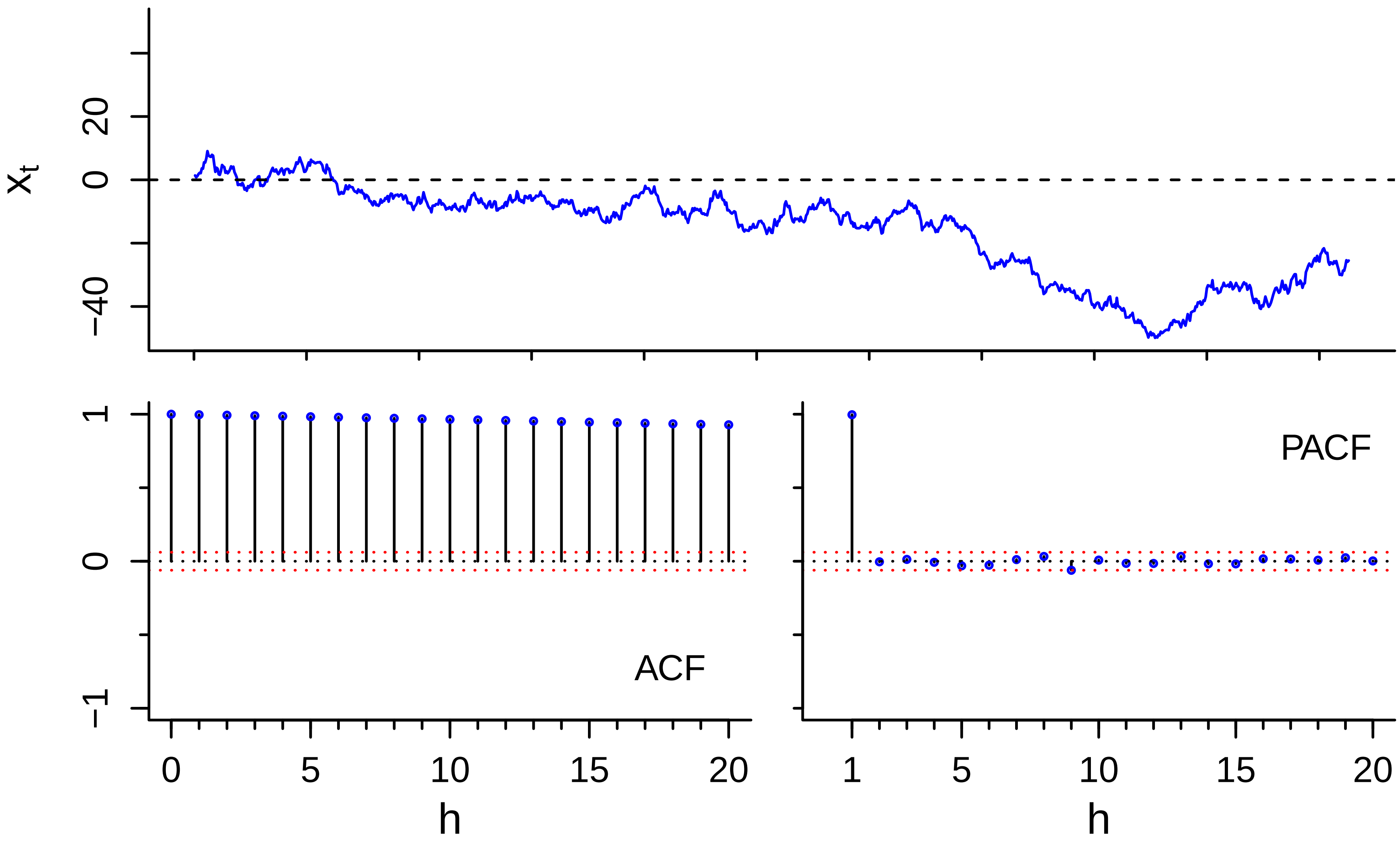
$$\begin{aligned}\operatorname{var}\{X_t\} &= \operatorname{var}\left\{X_0 + \sum_{u=1}^{t} Z_u\right\} \\ &= \operatorname{var}\{X_0\} + \sum_{u=1}^{t} \operatorname{var}\{Z_u\} \\ &= \operatorname{var}\{X_0\} + t\sigma^2,\end{aligned}$$

  which is either time-dependent or infinite (if $\operatorname{var}\{X_0\} = \infty$)

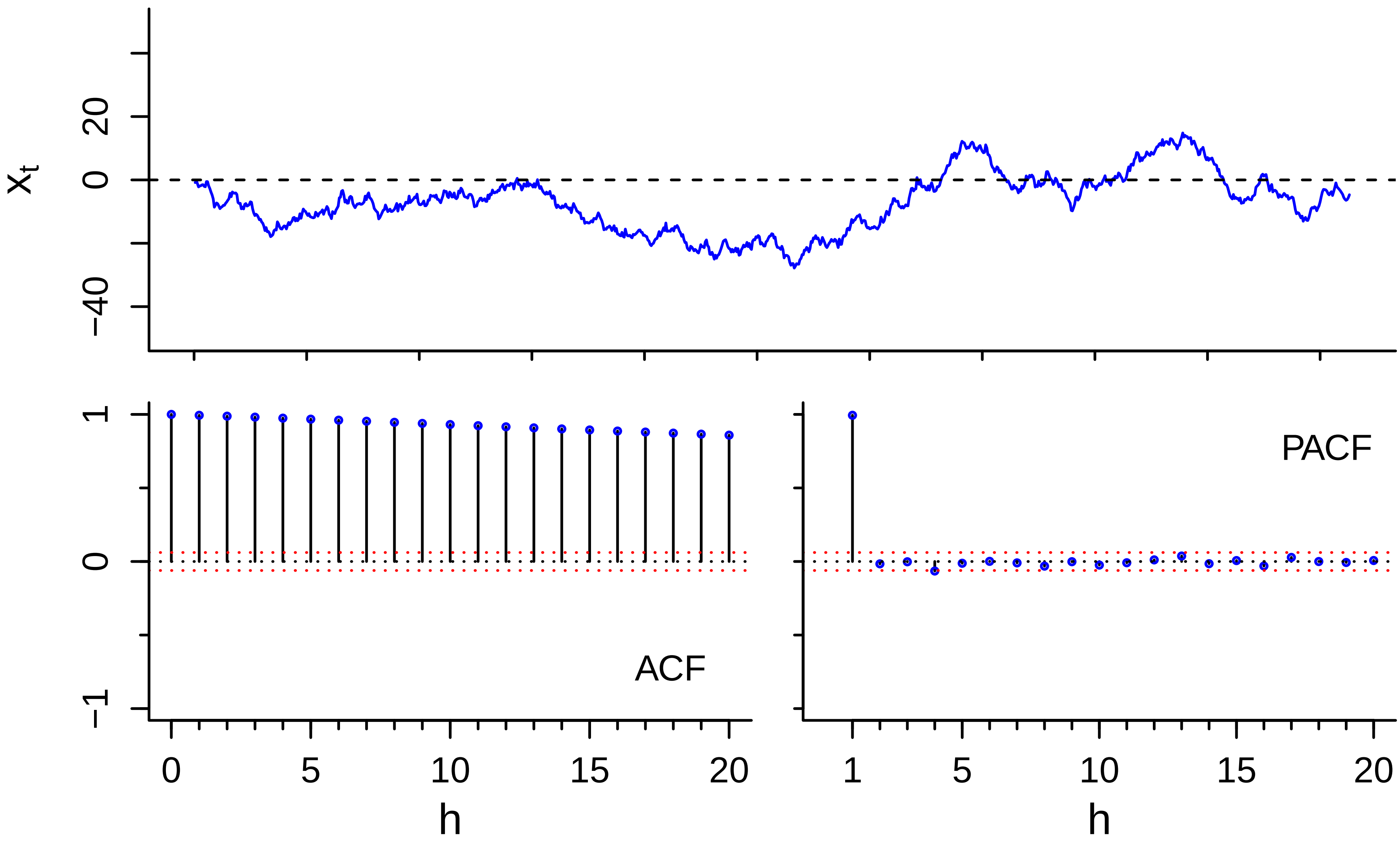
- ARIMA(0,1,0) is thus a nonstationary process (same true for all ARIMA(p, d, q) processes when d is a positive integer)

- let's look at three realizations of a random walk ($n = 1026$)

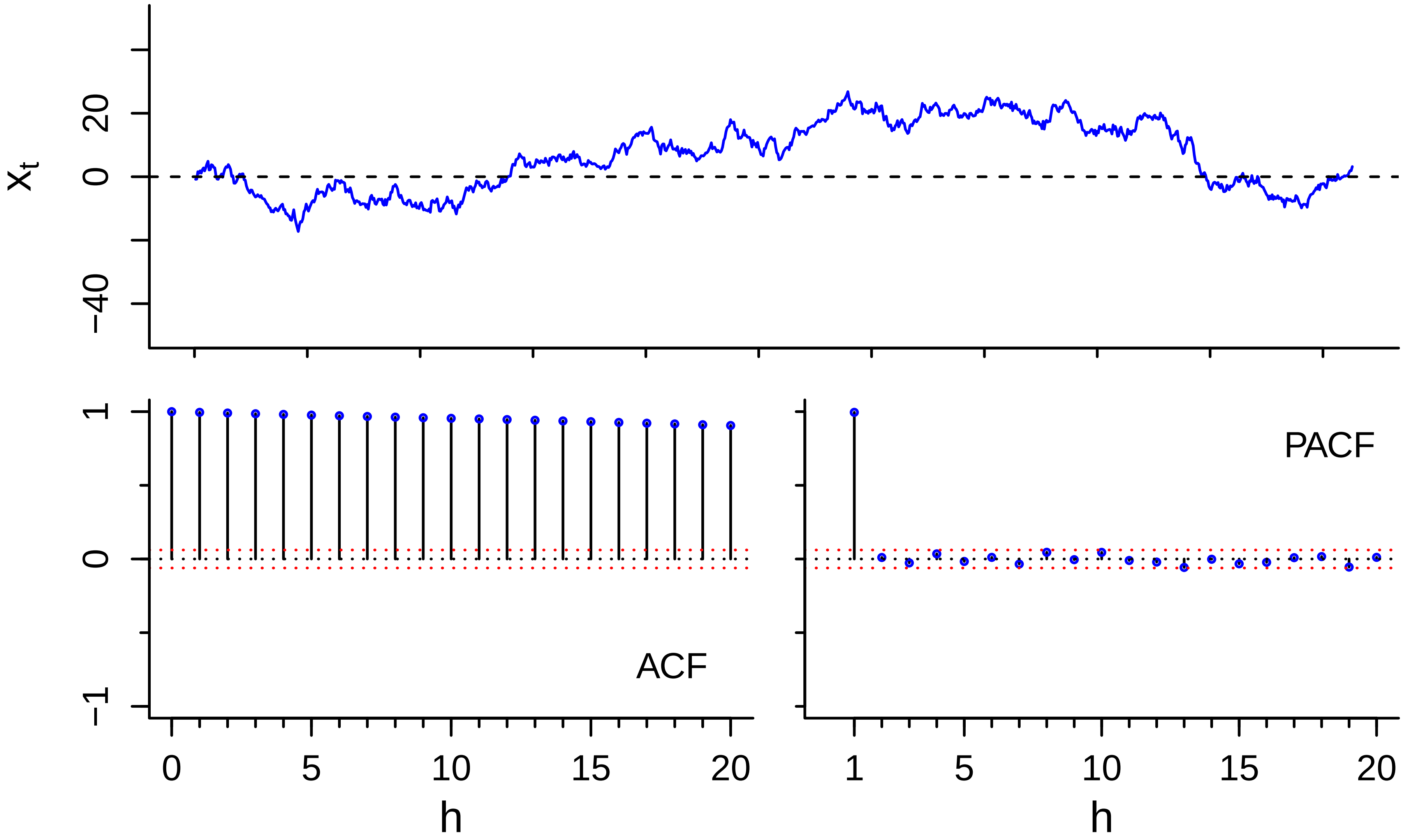
Random Walk Time Series: I



Random Walk Time Series: II



Random Walk Time Series: III



ARIMA(0,1,0) Process: III

- slowly decaying ACF is one indicator that an ARIMA model might be appropriate, but . . .
- might easily interpret sample ACF and PACF as consistent with AR(1) model with ϕ close to unity:

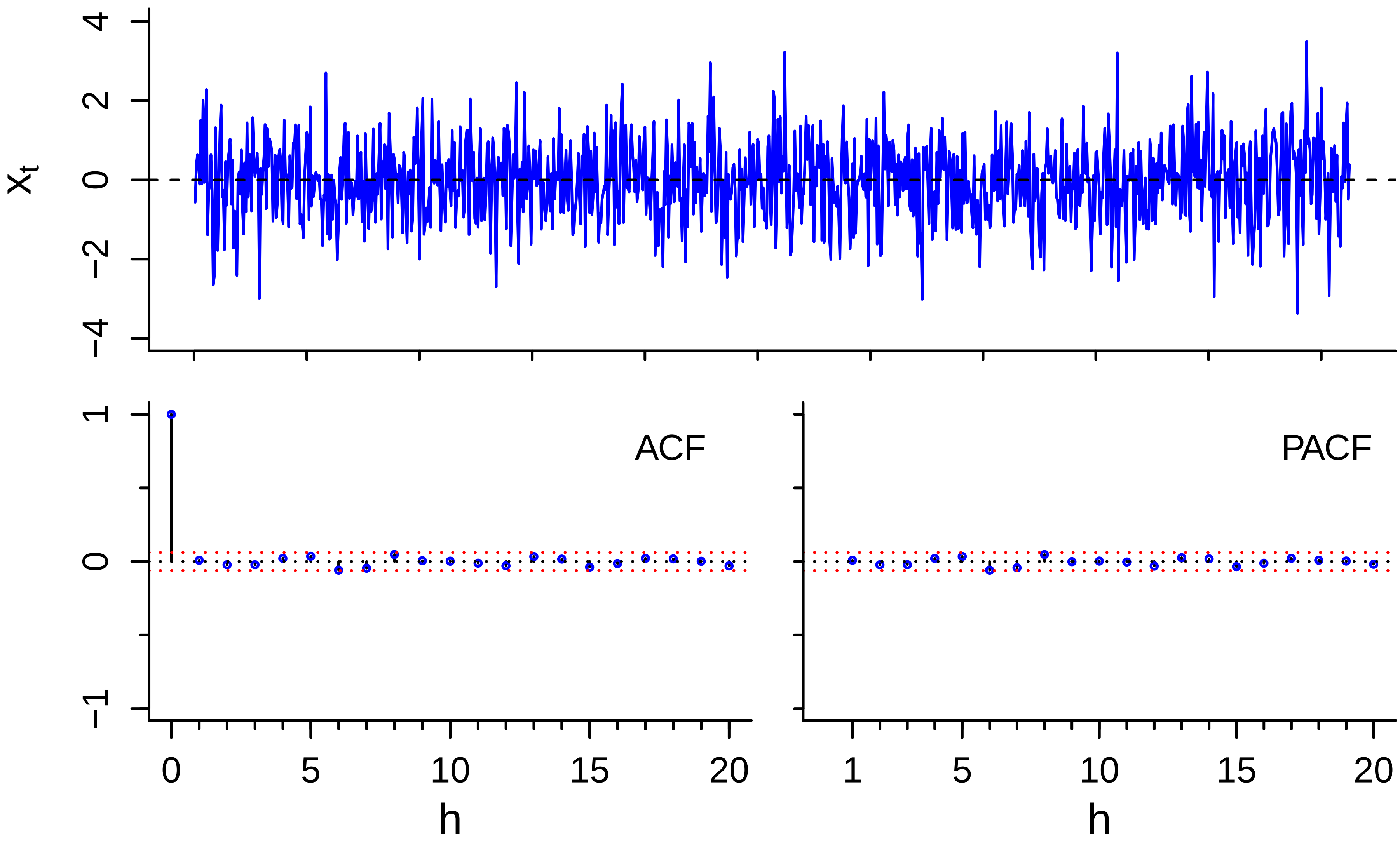
$$(1 - \phi B)X_t = X_t - \phi X_{t-1} = Z_t$$

  versus

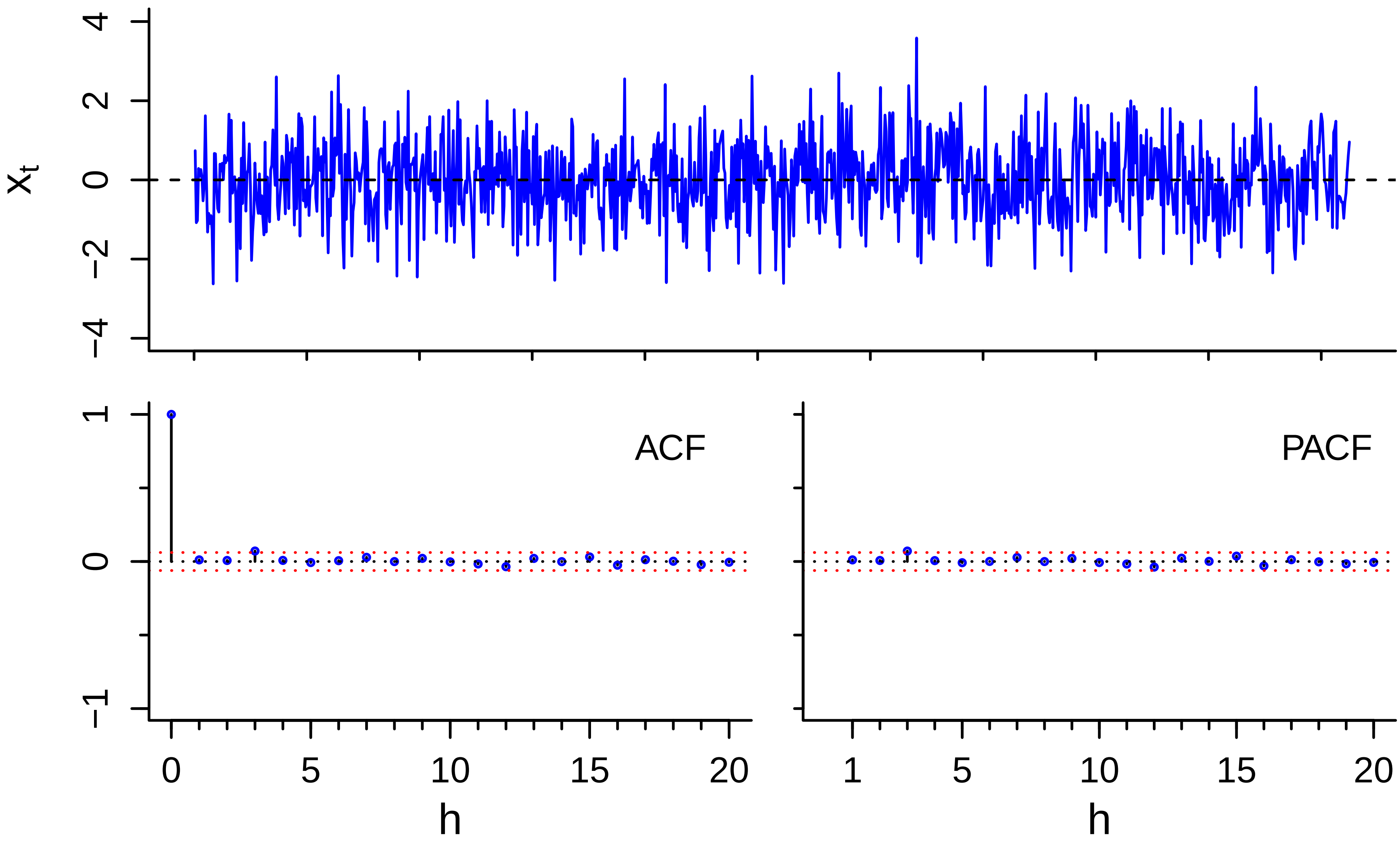
$$(1 - B)X_t = X_t - X_{t-1} = Z_t$$

- looking at ∇X_t and its sample ACF & PACF might strengthen case for ARIMA versus ARMA
- in case of random walk, ∇X_t is a white noise process

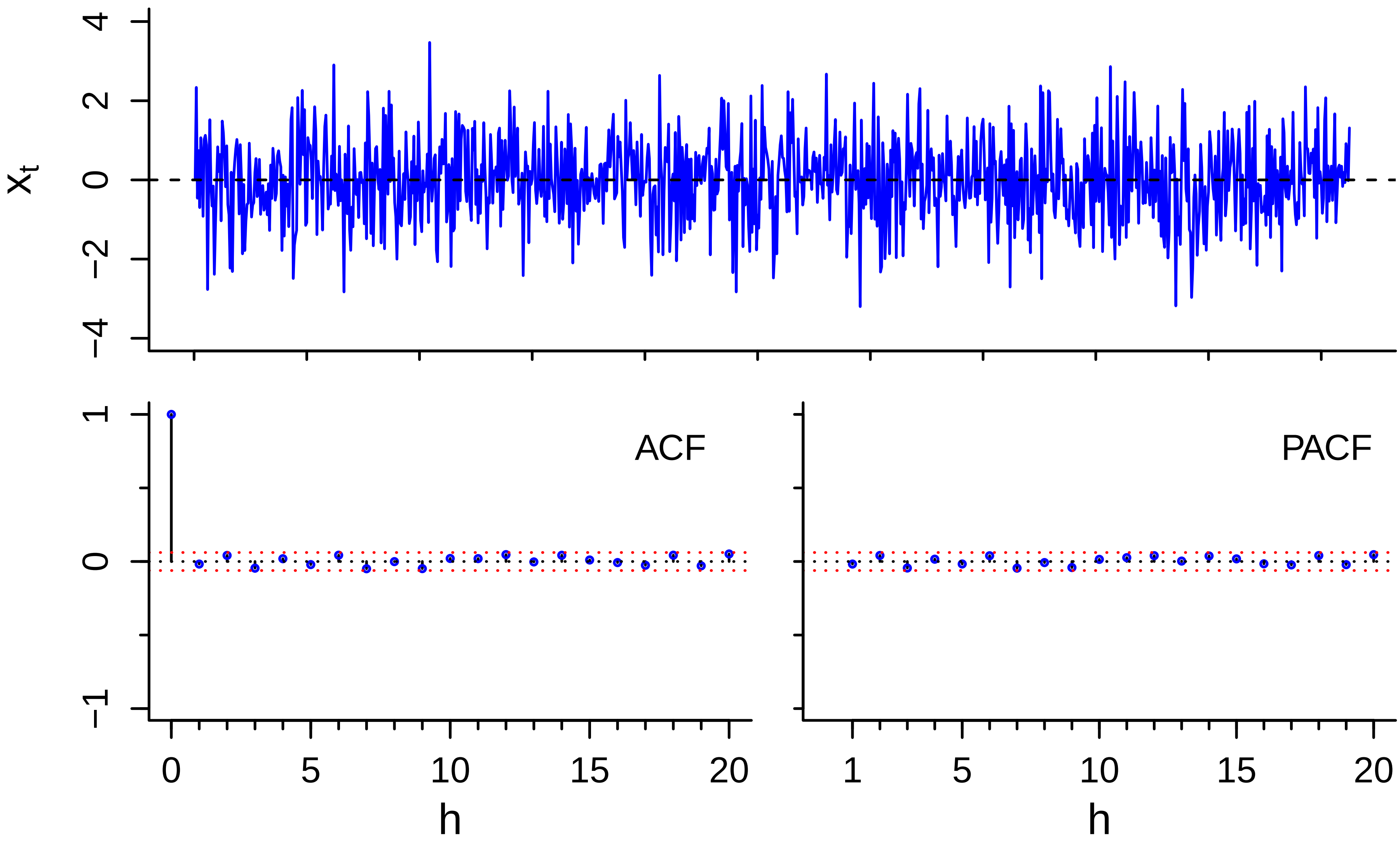
1st Difference of Random Walk Time Series: I



1st Difference of Random Walk Time Series: II



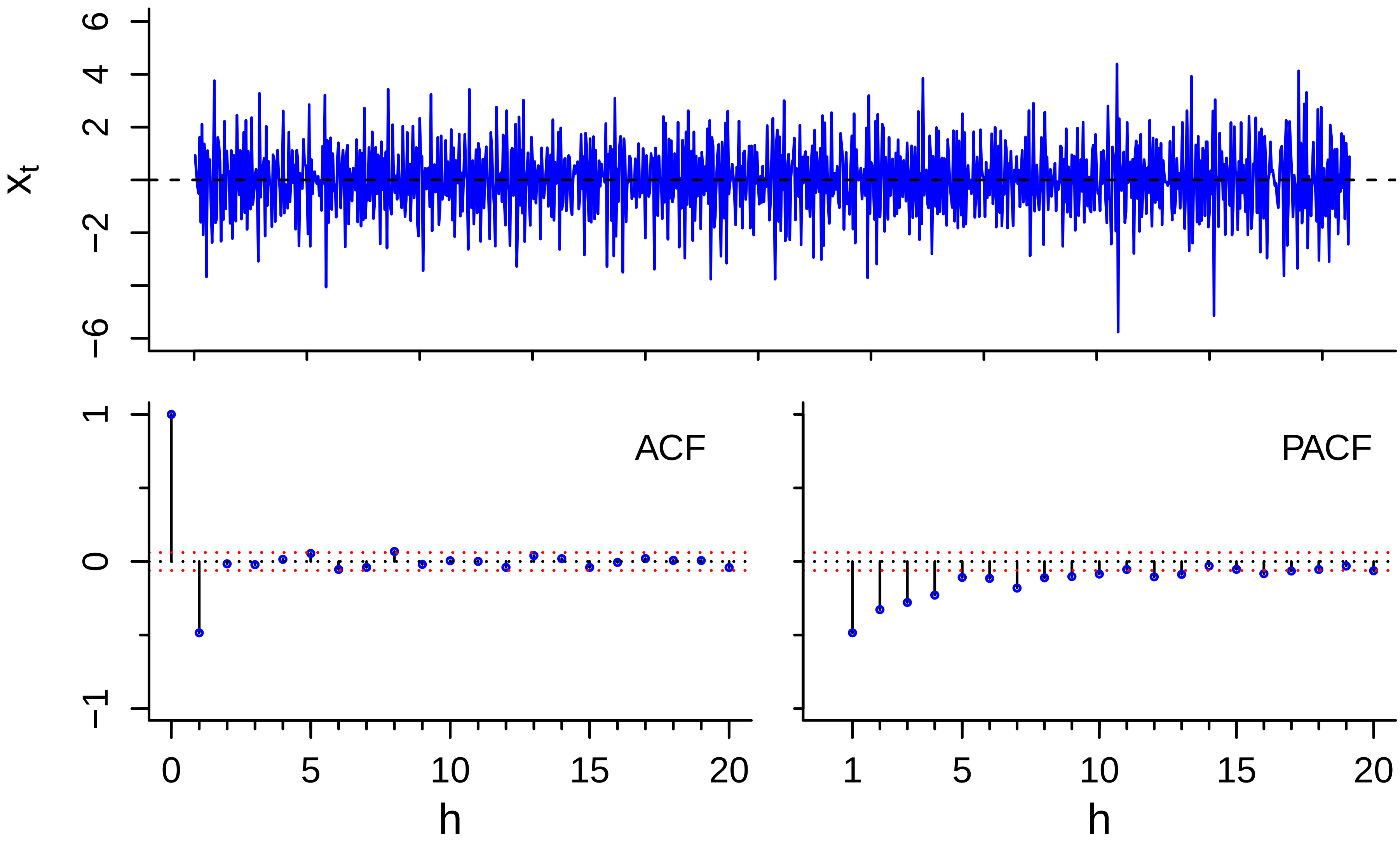
1st Difference of Random Walk Time Series: III



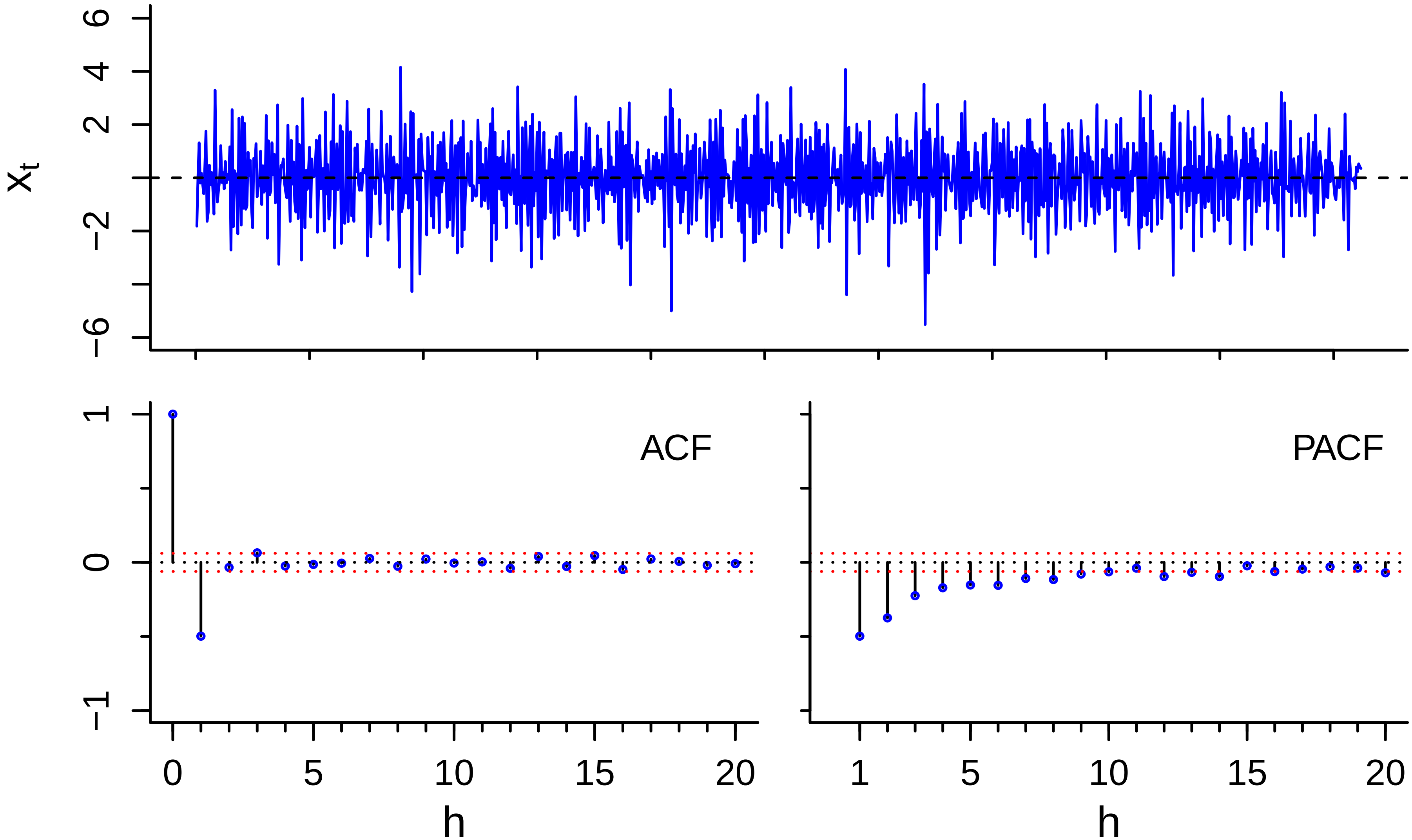
Overdifferencing: I

- while differencing a time series often seems to yield a series visually more amenable to modeling as a stationary process, overdifferencing is a danger
- 2nd difference of random walk same as 1st difference of white noise – necessarily has more complicated covariance structure (white noise takes home the prize for being process with simplest structure!)

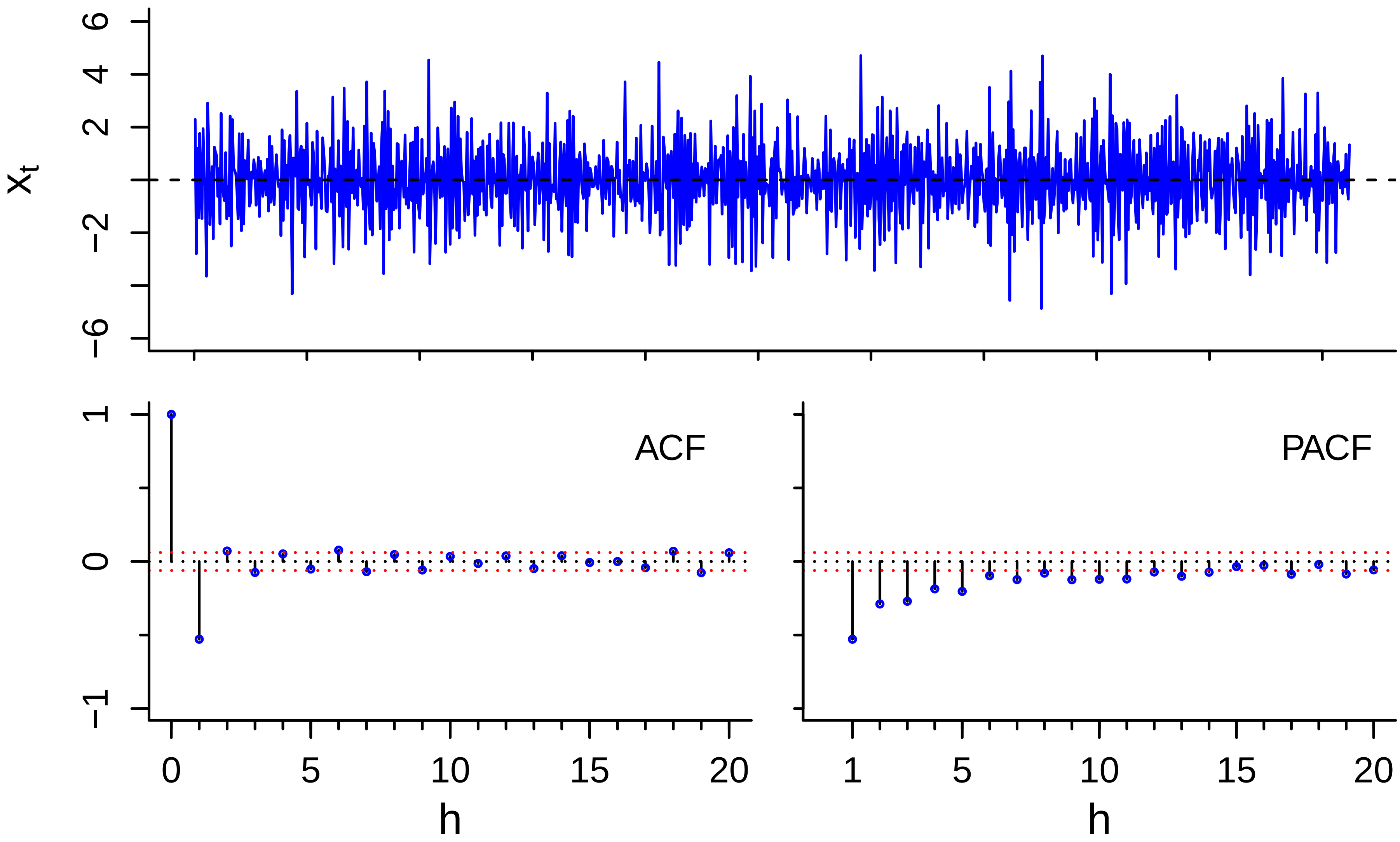
2nd Difference of Random Walk Time Series: I



2nd Difference of Random Walk Time Series: II



2nd Difference of Random Walk Time Series: III



Overdifferencing: II

- if X_t is ARMA(p, q) and hence satisfies
$$\phi(B)X_t = \theta(B)Z_t,$$
then
$$\begin{aligned}(1-B)\phi(B)X_t &= (1-B)\theta(B)Z_t \\ \phi(B)(1-B)X_t &= \theta(B)(1-B)Z_t;\end{aligned}$$
i.e., 1st difference $Y_t = \nabla X_t$ satisfies
$$\phi(B)Y_t = \theta^*(B)Z_t,$$
where $\theta^*(z) = \theta(z)(1-z)$ has a *unit root* (i.e., one of its roots is *on* the unit circle since $\theta^*(1) = 0$)

- Y_t process is thus non-invertible ARMA$(p, q+1)$, which means that π_j weights for Y_t need *not* be an absolutely summable sequence, leading to complications in 'munching' $\{Y_t\}$ into white noise (i.e., in forming one-step-ahead forecasts)

Overdifferencing: III

- evils of overdifferencing
  1. ARMA($p, q+1$) model usually has 'more complex' covariance structure than ARMA(p, q) model
  2. ARMA($p, q + 1$) model has one more parameter to estimate than ARMA(p, q) model
  3. prediction of non-invertible processes tricky (but doable)
  4. estimators other than MLEs can perform quite poorly
  5. sample size reduced by one (OK, not a big deal, but . . .)

BD: 169–170, CC: 126

Unit Root Tests: I

- unit root tests help determine if differencing is needed
- suppose X_t obeys an ARIMA$(0,1,0)$ model:
$$(1-B)X_t = X_t - X_{t-1} = Z_t$$
- above resembles the AR(1) model
$$(1-\phi_1 B)X_t = X_t - \phi_1 X_{t-1} = Z_t$$
with $\phi(z) = (1-\phi_1 z)$ having a unit root when $\phi_1 = 1$
- condition $\phi(1) = 0$ is equivalent to $\phi_1 = 1$
- want to devise a test for null hypothesis $\phi_1 = 1$
- obvious candidate for test statistic is $\hat{\phi}_1$, which is approximately $\mathcal{N}(\phi_1, \frac{1-\phi_1^2}{n})$ for $|\phi_1| < 1$ & large n (alas, not useful here)
- Dickey & Fuller (1979) devised an alternative test statistic

Unit Root Tests: II

- since $X_t = \phi_1 X_{t-1} + Z_t$ and $X_{t-1} = \phi_1 X_{t-2} + Z_{t-1}$, have

$$\begin{aligned}\nabla X_t &= \phi_1 \nabla X_{t-1} + \nabla Z_t \\ &= \phi_1 X_{t-1} - \phi_1 X_{t-2} + Z_t - Z_{t-1} \\ &= \phi_1 X_{t-1} - X_{t-1} + Z_t \\ &= (\phi_1 - 1) X_{t-1} + Z_t = \phi_1' X_{t-1} + Z_t, \ \text{ where } \phi_1' \stackrel{\text{def}}{=} \phi_1 - 1\end{aligned}$$

- unit root condition $\phi_1 = 1$ is equivalent to $\phi_1' = 0$
- Dickey–Fuller unit root test: use ordinary least squares (OLS) to regress ∇X_t on X_{t-1}, $t = 2, 3, \ldots, n$, and then test null hypothesis $\phi_1' = 0$
- for AR(1) model with $E\{X_t\} = \mu \neq 0$, model becomes

$$\nabla X_t = \phi_0' + \phi_1' X_{t-1} + Z_t,$$

where $\phi_0' = \mu(1 - \phi_1)$ and, as before, $\phi_1' = \phi_1 - 1$

Unit Root Tests: III

- in regression model $y_t = a + bx_t + e_t$, OLS estimator of b is
$$\hat{b} = \frac{\sum_t (x_t - \bar{x})(y_t - \bar{y})}{\sum_t (x_t - \bar{x})^2} = \frac{\sum_t (x_t - \bar{x}) y_t}{\sum_t (x_t - \bar{x})^2},$$
where $\bar{x}$ & $\bar{y}$ are sample means

- given $X_1, \ldots, X_n$ & letting $\widetilde{X}_{t-1} \stackrel{\text{def}}{=} X_{t-1} - \frac{1}{n-1} \sum_{s=1}^{n-1} X_s$, let $\hat{\phi}_1'$ be OLS estimator of ϕ_1' in model $\nabla X_t = \phi_0' + \phi_1' X_{t-1} + Z_t$:

$$\hat{\phi}_1' = \frac{\sum_{t=2}^n \widetilde{X}_{t-1} \nabla X_t}{\sum_{t=2}^n \widetilde{X}_{t-1}^2} = \frac{\sum_{t=2}^n \widetilde{X}_{t-1}(X_t - X_{t-1})}{\sum_{t=2}^n \widetilde{X}_{t-1}^2}$$
$$= \frac{\sum_{t=2}^n \widetilde{X}_{t-1}(\widetilde{X}_t - \widetilde{X}_{t-1})}{\sum_{t=2}^n \widetilde{X}_{t-1}^2} = \frac{\sum_{t=2}^n \widetilde{X}_{t-1}\widetilde{X}_t}{\sum_{t=2}^n \widetilde{X}_{t-1}^2} - 1$$

Unit Root Tests: IV

- note connection of

$$\hat{\phi}_1' = \frac{\sum_{t=2}^{n} \widetilde{X}_{t-1}\widetilde{X}_t}{\sum_{t=2}^{n} \widetilde{X}_t^2} - 1$$

$$= \frac{\sum_{t=2}^{n}(X_{t-1} - \frac{1}{n-1}\sum_{s=1}^{n-1} X_s)(X_t - \frac{1}{n-1}\sum_{s=1}^{n-1} X_s)}{\sum_{t=2}^{n}(X_t - \frac{1}{n-1}\sum_{s=1}^{n-1} X_s)^2} - 1$$

to Y–W estimator $\hat{\phi}_1$ of ϕ using $X_1, \ldots, X_n$ after centering:

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{n}(X_{t-1} - \overline{X})(X_t - \overline{X})}{\sum_{t=1}^{n}(X_t - \overline{X})^2}$$

- hence $\hat{\phi}_1' \approx \hat{\phi}_1 - 1$

Unit Root Tests: V

- in regression model $y_t = a + bx_t + e_t$ for m observations, usual standard error of OLS estimator $\hat{b}$ is taken to be

$$\widehat{\text{SE}}(\hat{b}) = \left(\frac{\sum_t (y_t - \hat{a} - \hat{b}x_t)^2}{(m-2)\sum_t (x_t - \bar{x})^2} \right)^{1/2}, \quad \text{where } \hat{a} = \bar{y} - \hat{b}\bar{x},$$

- for model $\nabla X_t = \phi_0' + \phi_1' X_{t-1} + Z_t$, above leads to

$$\widehat{\text{SE}}(\hat{\phi}_1') = \left(\frac{\sum_{t=2}^{n} (\nabla X_t - \hat{\phi}_0' - \hat{\phi}_1' X_{t-1})^2}{(n-3)\sum_{t=2}^{n} \left(X_{t-1} - \frac{1}{n-1}\sum_{s=1}^{n-1} X_s \right)^2} \right)^{1/2}$$

where

$$\hat{\phi}_0' = \frac{1}{n-1} \left(\sum_{t=2}^{n} \nabla X_t - \hat{\phi}_1' \sum_{t=1}^{n-1} X_t \right) = \frac{1}{n-1} \left(X_n - X_1 - \hat{\phi}_1' \sum_{t=1}^{n-1} X_t \right)$$

Unit Root Tests: VI

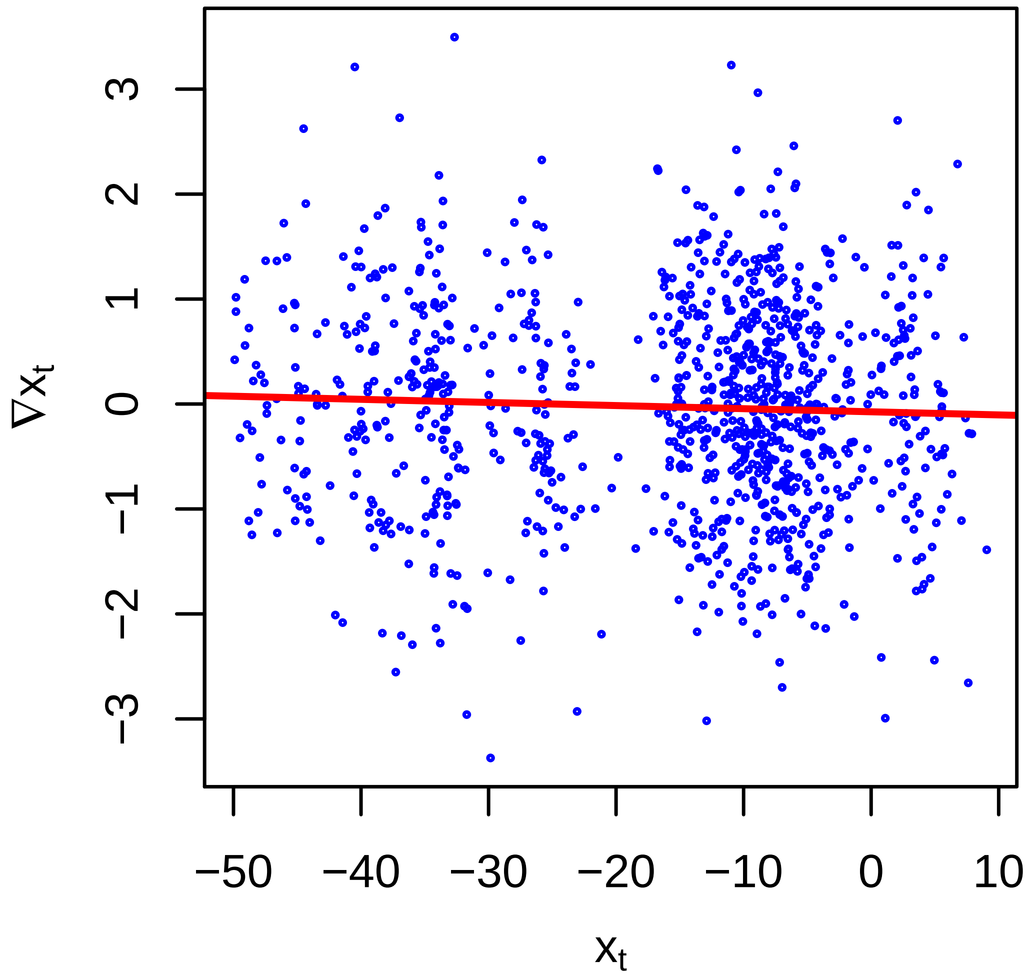
- test statistic is t-like ratio
$$t = \frac{\hat{\phi}'_1}{\widehat{\mathrm{SE}}(\hat{\phi}'_1)},$$
where denominator is standard error for slope as prescribed by OLS theory (but: t does *not* obey a t-distribution!!!)
- null hypothesis is $\phi'_1 = \phi_1 - 1 = 0$ (there is a unit root)
- $\phi'_1 > 0$ (i.e., $\phi_1 > 1$) problematic (non-causal AR(1) model), so will take alternative to be $\phi'_1 < 0$
- reject null $\phi'_1 = 0$ (have a unit root) in favor of alterative $\phi'_1 < 0$ (AR(1) is appropriate) at level, say, $\alpha = 0.05$ if t falls below 5% percentage point established for Dickey–Fuller test statistic under assumption that n is large

Unit Root Tests: VII

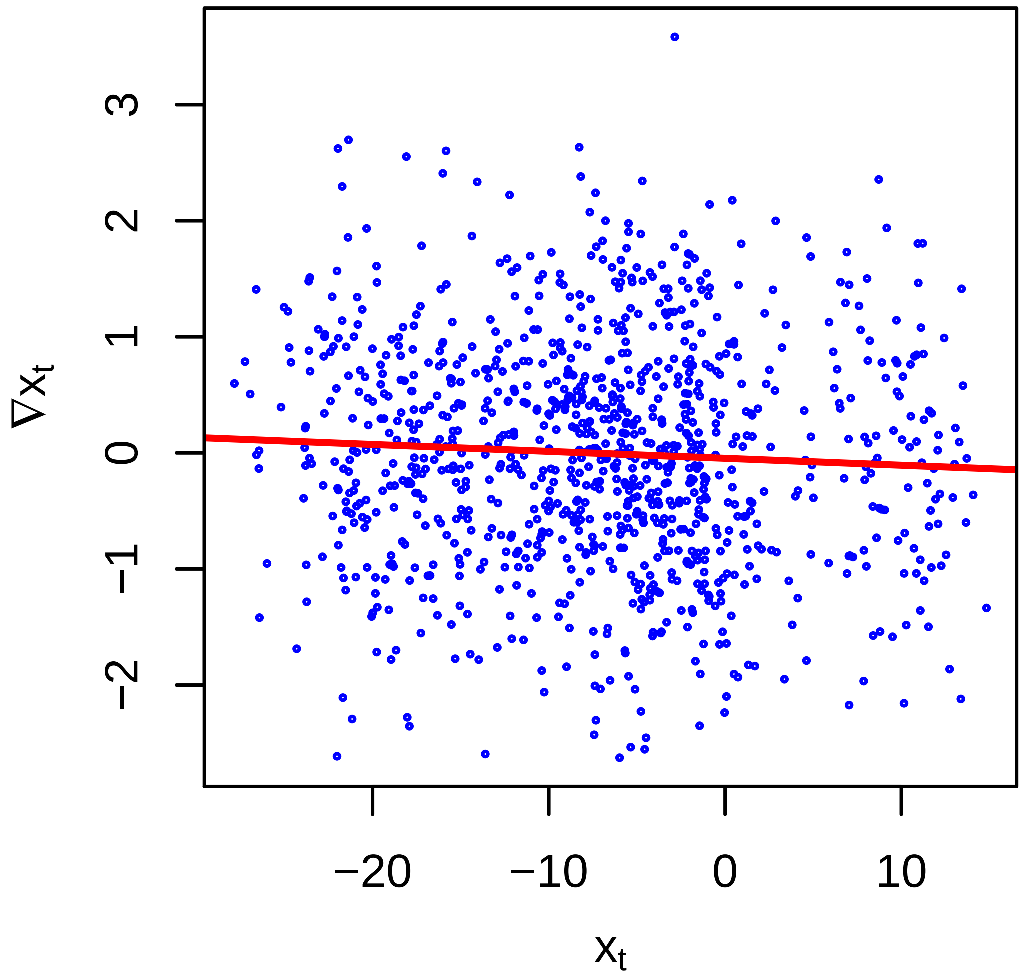
- let's demonstrate test by lobbing it some softballs: 3 random walk series (overheads XIV–6 to XIV–8) and 3 white noise series (overheads XIV–10 to XIV–12)
- 1%, 5% and 10% percentage points are -3.43, -2.86 and -2.57

	$\hat{\phi}'_1$	$\widehat{\text{SE}}(\hat{\phi}'_1)$	t	null hypothesis
random walk #1	-0.002963	0.002185	-1.357	fail to reject
random walk #2	-0.005964	0.003353	-1.779	fail to reject
random walk #3	-0.004671	0.002961	-1.577	fail to reject
white noise #1	-0.99170	0.03128	-31.71	reject
white noise #2	-0.98890	0.03129	-31.61	reject
white noise #3	-1.01723	0.03130	-32.50	reject

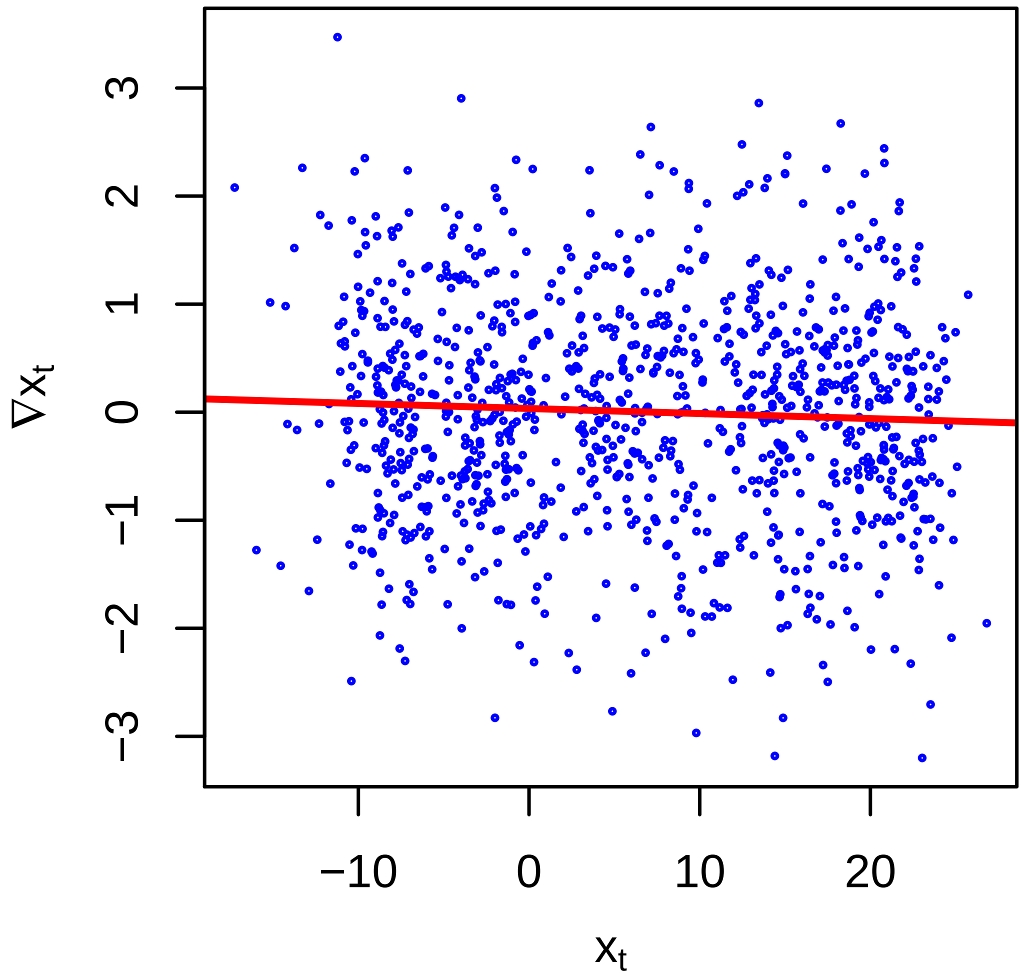
Scatterplot and Fitted Line for Random Walk: I



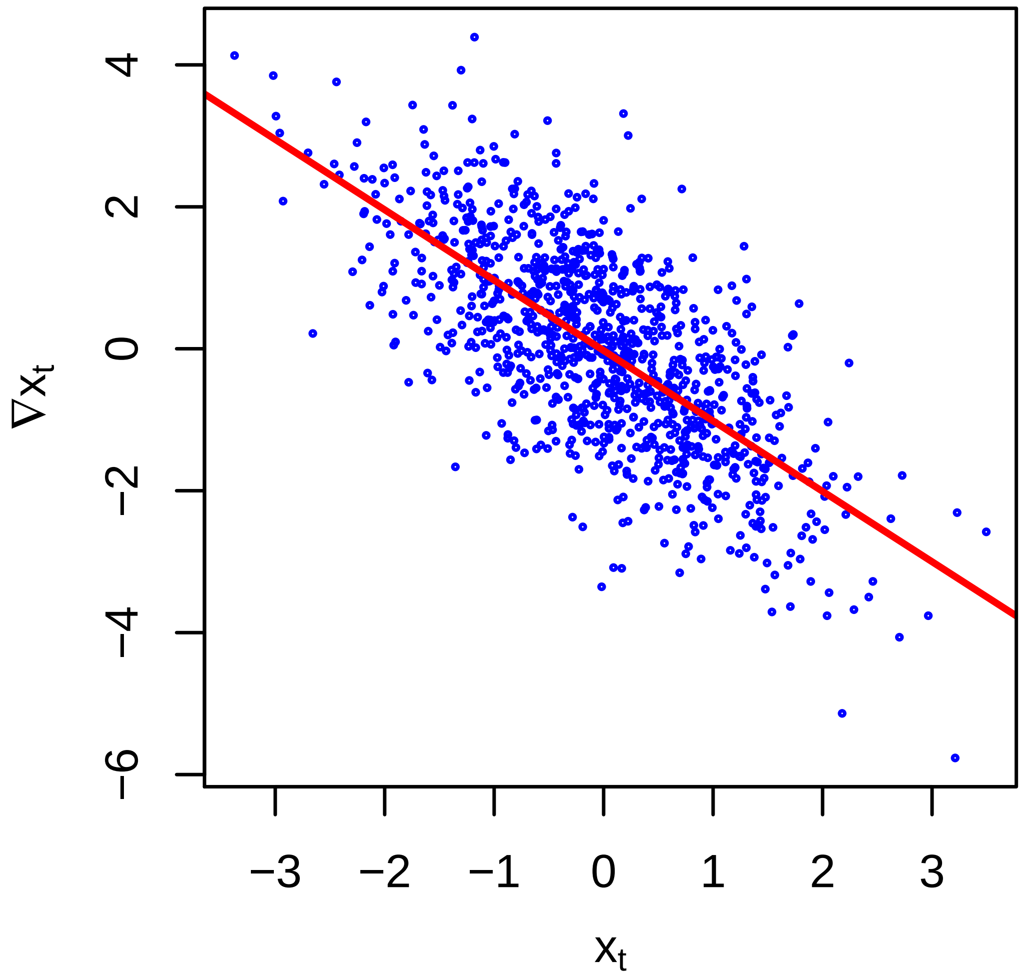
Scatterplot and Fitted Line for Random Walk: II



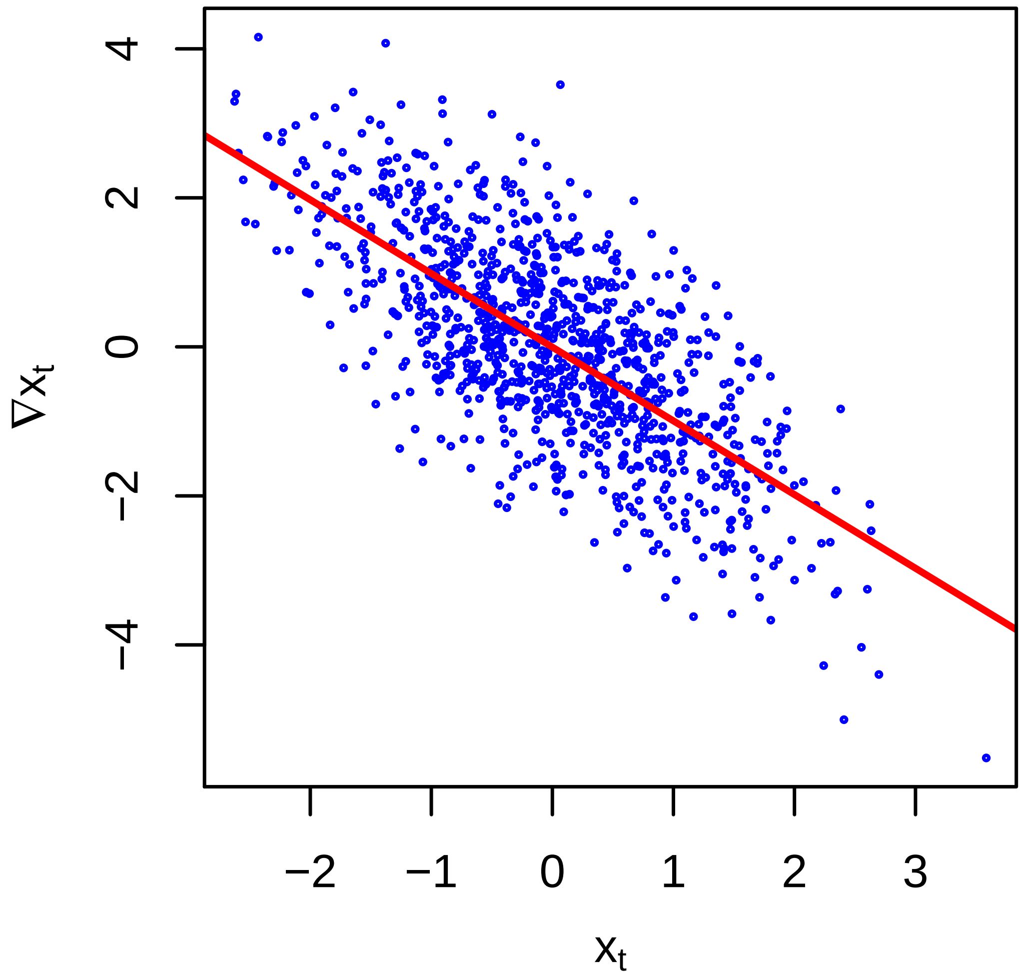
Scatterplot and Fitted Line for Random Walk: III



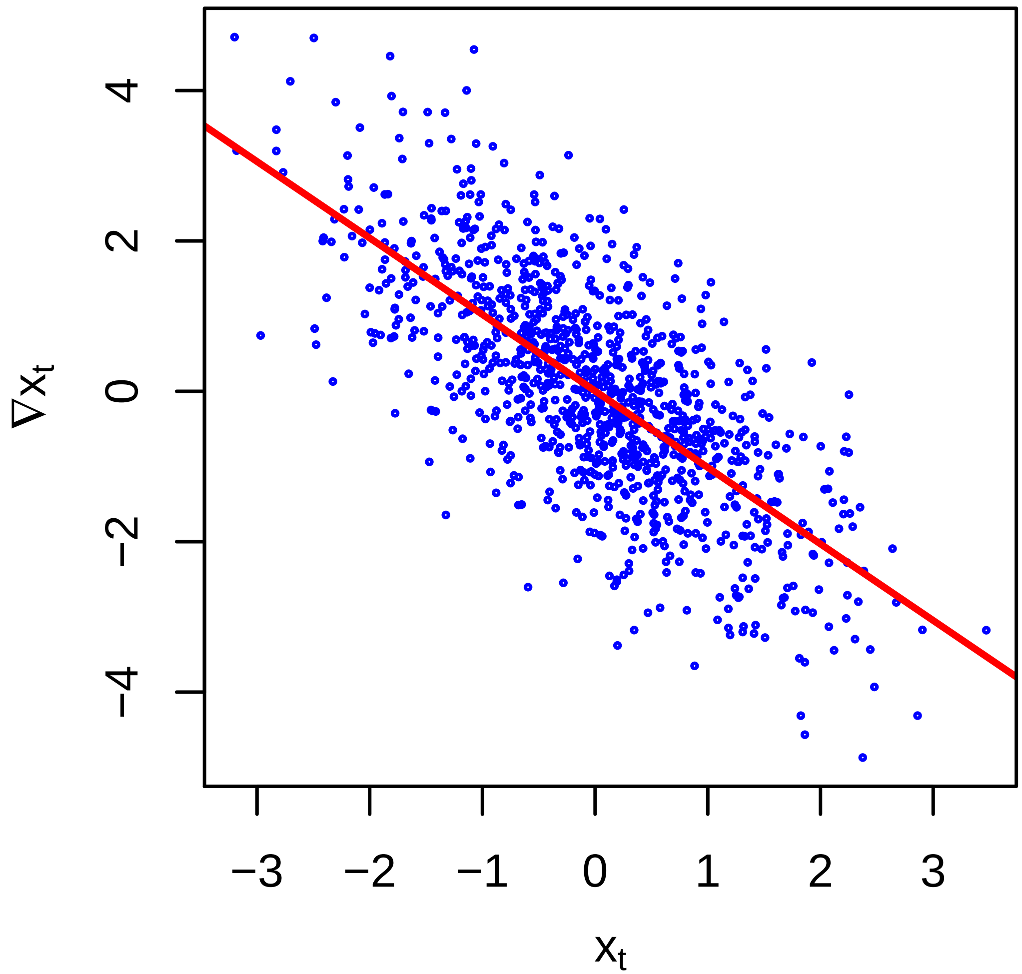
Scatterplot and Fitted Line for White Noise: I



Scatterplot and Fitted Line for White Noise: II



Scatterplot and Fitted Line for White Noise: III



Unit Root Tests: VIII

- idea for testing ARIMA(0,1,0) vs. AR(1) (i.e., ARIMA(1,0,0)) can be extended to test ARIMA($p-1$,1,0) vs. AR(p)
- regard ARIMA($p-1, 1, 0$) model
$$\phi^*(B)(1-B)X_t = Z_t,$$
where
$$\phi^*(z) = 1 - \phi_1^* z - \cdots - \phi_{p-1}^* z^{p-1},$$
as limiting case of AR(p) model
$$\phi(B)X_t = Z_t,$$
where
$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p,$$
by equating $\phi(z)$ with $\phi^*(z)(1-z)$
- unit root condition $\phi(1) = 0$ now equivalent to $1 - \sum_{i=1}^{p} \phi_i = 0$

Unit Root Tests: IX

- cleverness (Dickey & Fuller, 1979) says

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$$

can be rewritten as

$$\nabla X_t = \phi_1' X_{t-1} + \phi_2' \nabla X_{t-1} + \cdots + \phi_p' \nabla X_{t-(p-1)} + Z_t, \quad (*)$$

where $\phi_1' = \sum_{i=1}^{p} \phi_i - 1$ and $\phi_j' = -\sum_{i=j}^{p} \phi_i - 1$, $j = 2, \ldots, p$

- adding intercept term ϕ_0' to $(*)$ to allow for AR(p) model with nonzero mean yields multiple regression model

$$\nabla X_t = \phi_0' + \phi_1' X_{t-1} + \phi_2' \nabla X_{t-1} + \cdots + \phi_p' \nabla X_{t-(p-1)} + Z_t,$$

where $\phi_1' = 0$ is equivalent to unit root condition

Unit Root Tests: X

- as before, use OLS to get estimator $\hat{\phi}_1'$ and form t-like ratio to test null hypothesis $\phi_1' = 0$
- above called *augmented* Dickey–Fuller (ADF) unit root test
- note: unit root tests have also been devised for $\theta(z)$ to assess possible noninvertibility
- Chapter 4 of Zivot & Wang (2006) has additional discussion

Nonstationarity Due to Polynomial Trend: I

- consider time series given by

$$Y_t = a + bt + X_t \text{ with } b \neq 0,$$

  where X_t is ARMA(p,q) with zero mean: $\phi(B)X_t = \theta(B)Z_t$

- Y_t is nonstationary since $E\{Y_t\} = a + bt$
- first difference of Y_t is stationary:

$$\nabla Y_t = Y_t - Y_{t-1} = a + bt + X_t - (a + b(t-1) + X_{t-1}) = b + \nabla X_t,$$

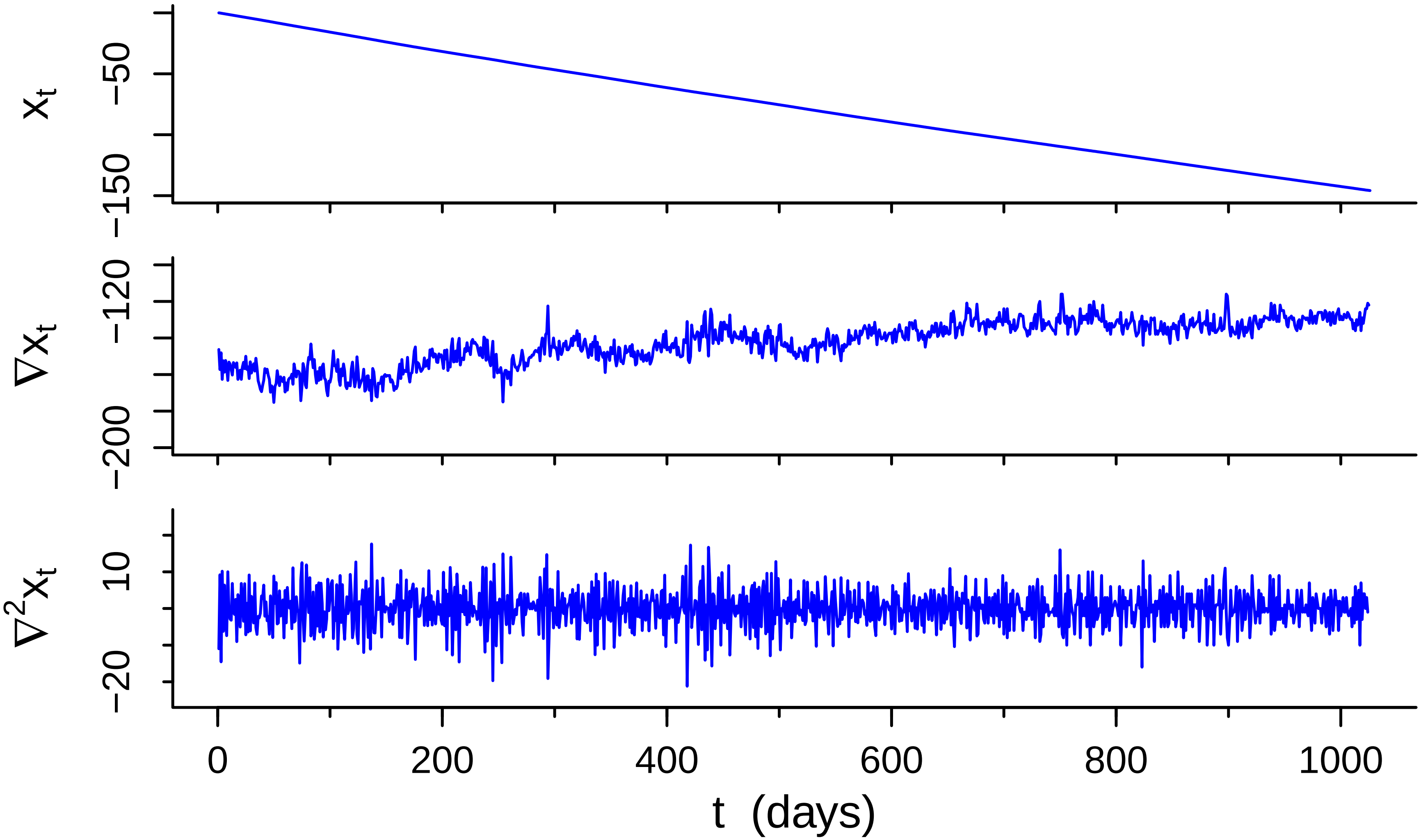
  where ∇X_t is necessarily a stationary process

- since $(1 - B)\phi(B)X_t = (1 - B)\theta(B)Z_t$, can be reexpressed as $\phi(B)\nabla X_t = (1 - B)\theta(B)Z_t$, it follows that ∇X_t is a noninvertible ARMA(p,$q+1$) process
- Y_t is thus an ARIMA(p,1,$q+1$) process whose first difference is a noninvertible ARMA(p,$q+1$) process with mean b

Nonstationarity Due to Polynomial Trend: II

- when X_t is ARMA(p,q), treating
$$Y_t = a + bt + X_t$$
as an ARIMA(p,1,$q+1$) process thus leads to evils of overdifferencing (see overhead XIV–18)
- suppose, however, that time series satisfies
$$Y_t = a + bt + X_t \text{ with } b \neq 0,$$
where X_t is an ARIMA(p,1,q) process whose first difference is an invertible ARMA(p,q) process with zero mean
- now Y_t becomes an ARIMA(p,1,q) whose first difference is an invertible ARMA(p,q) process with mean b

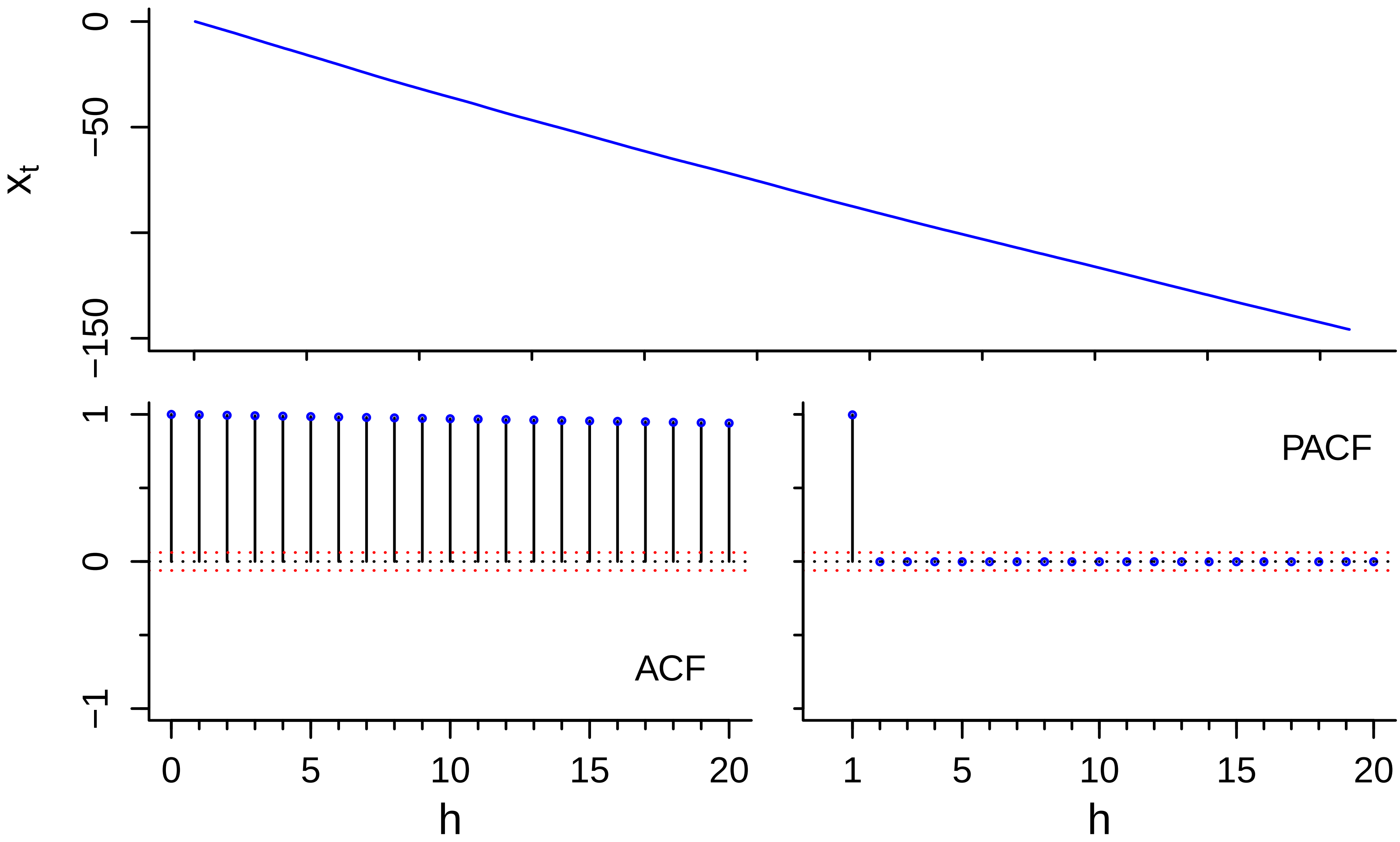
Atomic Clock Deviates: I



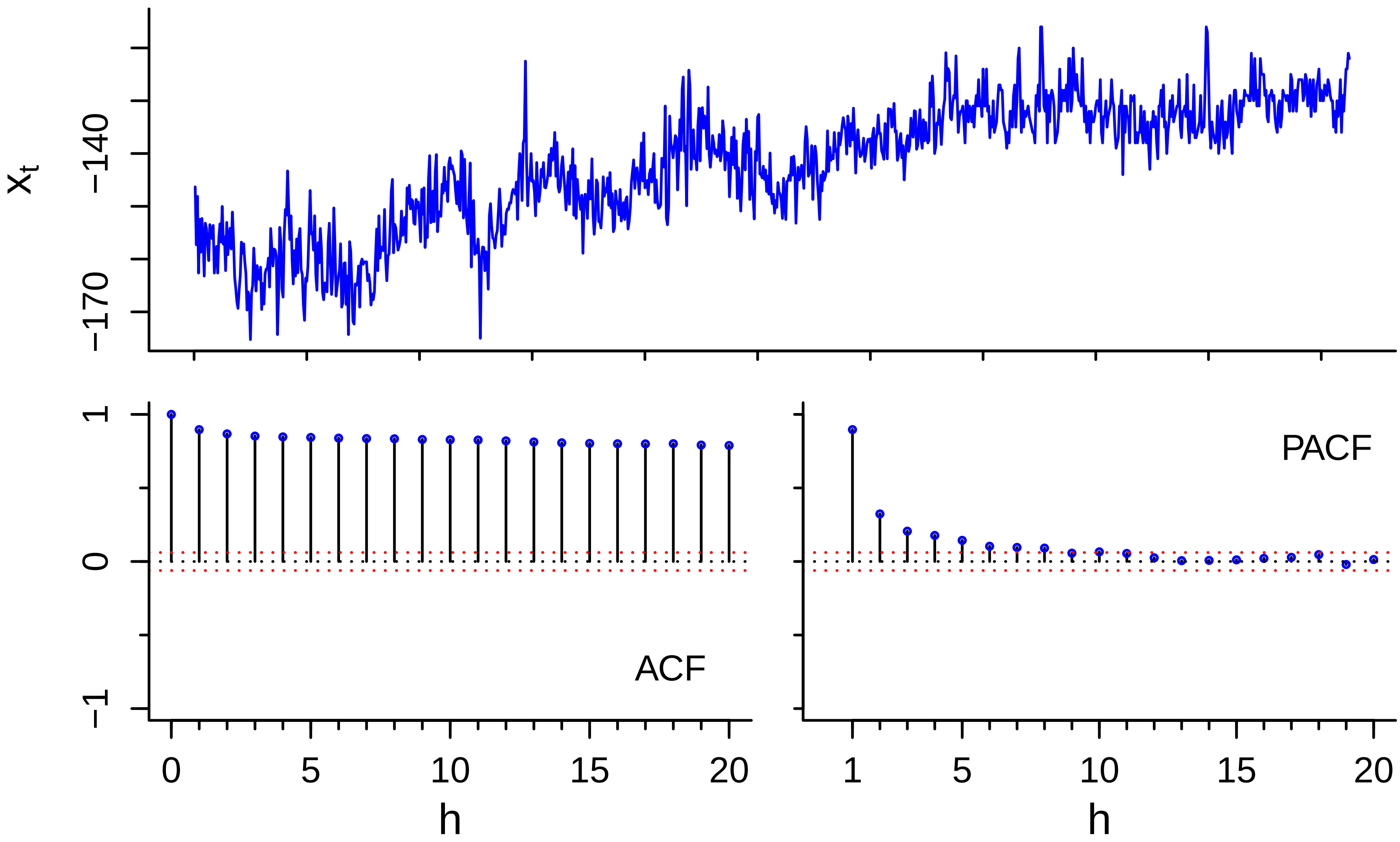
Atomic Clock Deviates: II

- top plot: errors X_t in time kept by atomic clock 571 as compared to time kept at Naval Observatory (measured in microseconds, where 1,000,000 microseconds = 1 second)
- middle: first backward differences ∇X_t in nanoseconds (1000 nanoseconds = 1 microsecond) – can be related to frequency mechanism driving clock
- bottom: second backward differences $\nabla^2 X_t$, also in nanoseconds – can be related to changes in frequency
- note: possible linear trend in ∇X_t would result in nonzero mean for $\nabla^2 X_t$,

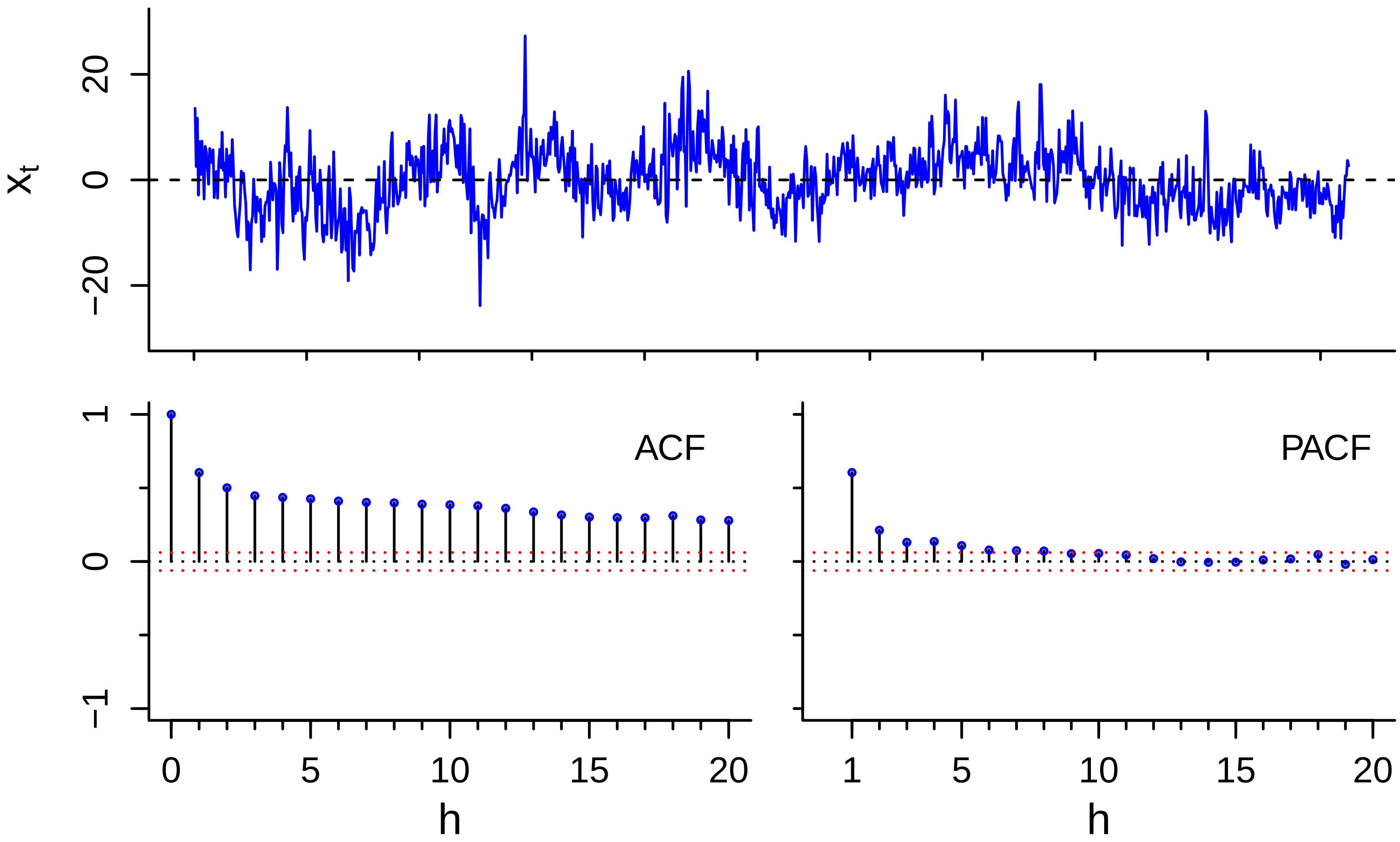
Atomic Clock Time Series



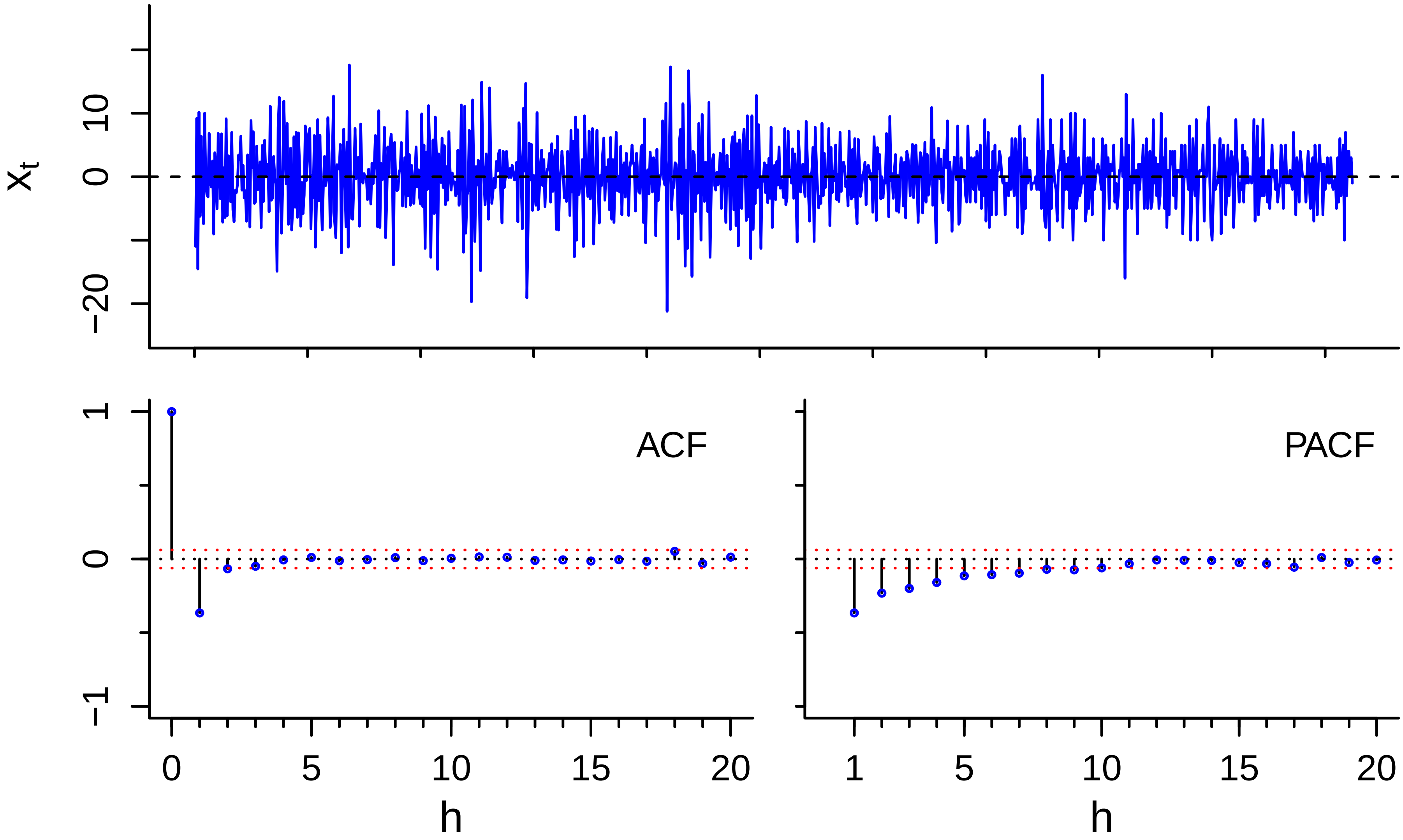
1st Difference of Atomic Clock Time Series



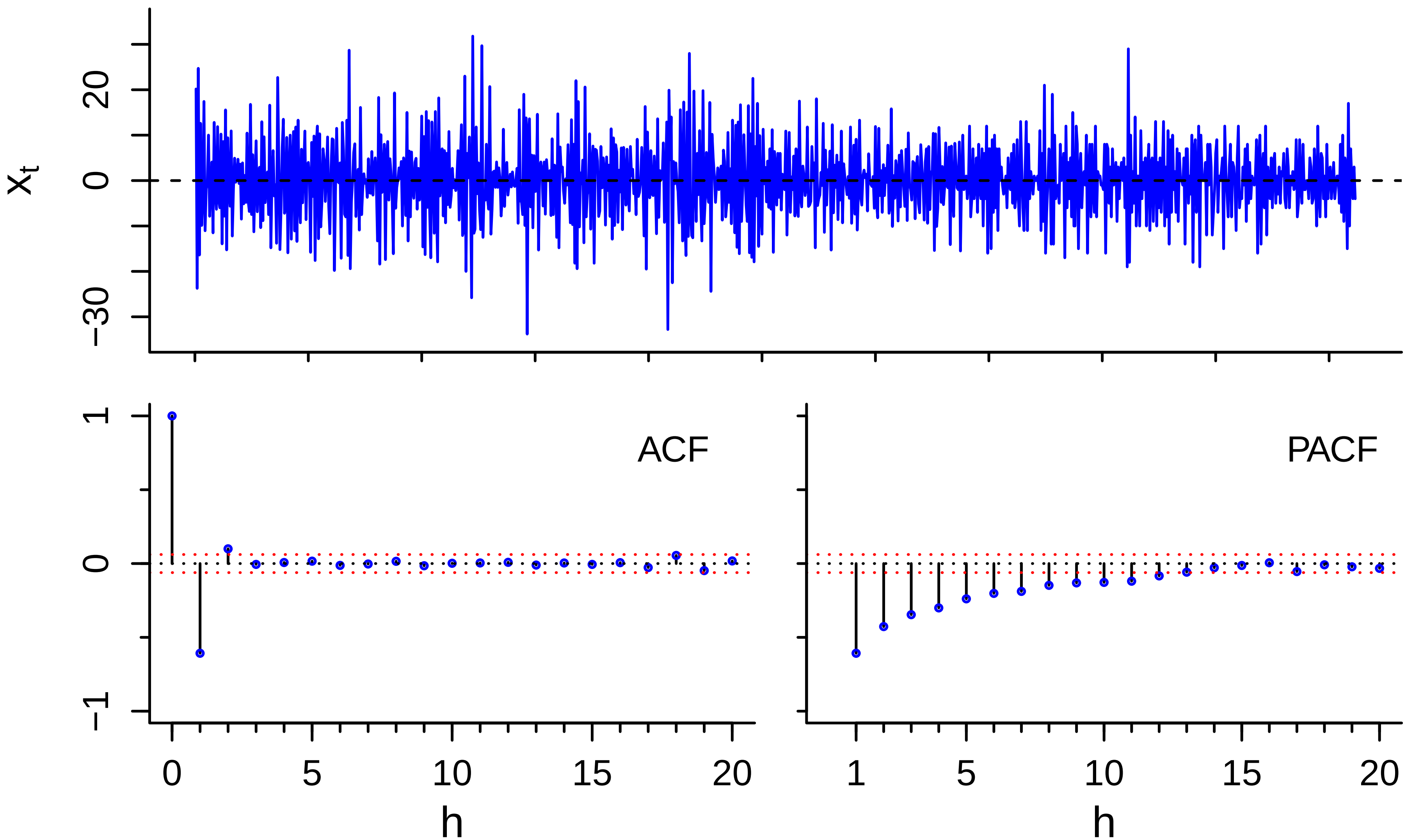
Detrended 1st Difference



2nd Difference of Atomic Clock Time Series



3rd Difference of Atomic Clock Time Series



Atomic Clock Deviates: III

- 1%, 5% and 10% percentage points are -3.43, -2.86 and -2.57

	test	$\hat{\phi}_1'$	$\widehat{\text{SE}}(\hat{\phi}_1')$	t	null hypothesis
1st difference	DF	-0.10067	0.01378	-7.307	reject
1st difference	ADF	-0.02465	0.01296	-1.902	fail to reject
detrended 1st difference	DF	-0.39494	0.02481	-15.918	reject
detrended 1st difference	ADF	-0.16494	0.03173	-5.198	reject

- presuming that ADF with $p = 8$ is more appropriate than DF, test suggests $\nabla^2 X_t$ is to be preferred over ∇X_t
- found ARMA(1,1) model to adequately describe $\nabla^2 X_t$, implying ARIMA(1,2,1) model for original data:

$$\phi(B)(1 - B)^2 X_t = \theta(B) Z_t$$

Atomic Clock Deviates: IV

- note: actual model needs to include term for mean since apparent linear drift in ∇X_t would imply nonzero mean in $\nabla^2 X_t$
- unit root tests on linearly detrended series ∇X_t suggest competing model

$$X_t = a + bt + ct^2 + W_t \text{ so that } \nabla X_t = \alpha + \beta t + \nabla W_t$$

with $\alpha \stackrel{\text{def}}{=} b - c$ and $\beta \stackrel{\text{def}}{=} 2c$, where ∇W_t is an ARMA process (or possibly another stationary process)

References

- D. A. Dickey and W. A. Fuller (1979), 'Distribution of the Estimators for Autoregressive Time Series with a Unit Root,' *Journal of the American Statistical Association*, **74**, pp. 427–431
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