Uniqueness of $\widehat{X}_{n+h}$: I

- overhead X–13 states that $\widehat{X}_{n+h}$ is *unique* even if $\Gamma_n \boldsymbol{a}_n = \boldsymbol{\gamma}_n(h)$ has multiple solutions
- as an example, consider stationary process of Problem 2(b):

$$X_t = Z_2 \cos(\omega t) + Z_1 \sin(\omega t),$$

  where Z_1 and Z_2 are independent $\mathcal{N}(0,1)$ RVs
- ACVF for $\{X_t\}$ is $\gamma(h) = \cos(\omega h)$
- to forecast X_4 based on $\boldsymbol{X}_3 \stackrel{\text{def}}{=} [X_3, X_2, X_1]$, consider
  1. Γ_3, the 3×3 matrix whose (i,j)th element is $\gamma(i-j)$;
  2. $\boldsymbol{\gamma}_3(1) = [\gamma(1), \gamma(2), \gamma(3)]'$; and
  3. $\boldsymbol{a} = [a_1, a_2, a_3]'$, a solution to $\Gamma_3 \boldsymbol{a} = \boldsymbol{\gamma}_3(1)$

Uniqueness of $\widehat{X}_{n+h}$: II

- since $\gamma(h) = \cos(\omega h)$,

$$\Gamma_3 = \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{bmatrix} = \begin{bmatrix} 1 & \cos(\omega) & \cos(2\omega) \\ \cos(\omega) & 1 & \cos(\omega) \\ \cos(2\omega) & \cos(\omega) & 1 \end{bmatrix}$$

- making use of the trig relationship $\cos(2\omega) = 2\cos^2(\omega) - 1$ (see, e.g., Wikipedia), it follows that a linear combination of the first two columns of Γ_3 is equal to the third:

$$-\begin{bmatrix} 1 \\ \cos(\omega) \\ \cos(2\omega) \end{bmatrix} + 2\cos(\omega) \begin{bmatrix} \cos(\omega) \\ 1 \\ \cos(\omega) \end{bmatrix} = \begin{bmatrix} \cos(2\omega) \\ \cos(\omega) \\ 1 \end{bmatrix}$$

- hence Γ_3 is rank deficient and does not possess a unique inverse

Uniqueness of $\widehat{X}_{n+h}$: III

- as a result, there are multiple solutions to $\Gamma_3 \boldsymbol{a} = \boldsymbol{\gamma}_3(1)$, where $\boldsymbol{\gamma}_3(1) = [\cos(\omega), \cos(2\omega), \cos(3\omega)]'$
- one is $\boldsymbol{a}_1$, defined to be $[2\cos(\omega), -1, 0]'$:
$$\begin{bmatrix} 1 & \cos(\omega) & \cos(2\omega) \\ \cos(\omega) & 1 & \cos(\omega) \\ \cos(2\omega) & \cos(\omega) & 1 \end{bmatrix} \begin{bmatrix} 2\cos(\omega) \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\omega) \\ \cos(2\omega) \\ \cos(3\omega) \end{bmatrix},$$
deducable from $2\cos^2(\omega) - 1 = \cos(2\omega)$
and $4\cos^3(\omega) - 3\cos(\omega) = \cos(3\omega)$ (see, e.g., Wikipedia)
- another is $\boldsymbol{a}_2$, defined to be $[0, 4\cos^2(\omega) - 1, -2\cos(\omega)]'$:
$$\begin{bmatrix} 1 & \cos(\omega) & \cos(2\omega) \\ \cos(\omega) & 1 & \cos(\omega) \\ \cos(2\omega) & \cos(\omega) & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4\cos^2(\omega) - 1 \\ -2\cos(\omega) \end{bmatrix} = \begin{bmatrix} \cos(\omega) \\ \cos(2\omega) \\ \cos(3\omega) \end{bmatrix},$$
deducible from the same two trig relationships shown above

Uniqueness of $\widehat{X}_{n+h}$: IV

- the first solution yields the prediction

$$\widehat{X}_4 = 2\cos(\omega)X_3 - X_2,$$

which predicts X_4 perfectly since $X_t = 2\cos(\omega)X_{t-1} - X_{t-2}$ for all t (the fact that $X_t = Z_2\cos(\omega t) + Z_1\sin(\omega t)$ obeys $X_t = 2\cos(\omega)X_{t-1} - X_{t-2}$ is stated on overhead VIII–35 and is the subject of an exercise)

- now

$$\begin{aligned}
\widehat{X}_4 &= 2\cos(\omega)X_3 - X_2 \\
&= 2\cos(\omega)\left(2\cos(\omega)X_2 - X_1\right) - X_2 \\
&= \left(4\cos^2(\omega) - 1\right)X_2 - 2\cos(\omega)X_1,
\end{aligned}$$

which shows that the second solution yields the same prediction

Uniqueness of $\widehat{X}_{n+h}$: V

- amongst other possible solutions are $\alpha \boldsymbol{a}_1 + (1-\alpha)\boldsymbol{a}_2$ since

$$\begin{aligned}\Gamma_3\left(\alpha \boldsymbol{a}_1 + (1-\alpha)\boldsymbol{a}_2\right) &= \alpha\Gamma_3\boldsymbol{a}_1 + (1-\alpha)\Gamma_3\boldsymbol{a}_2 \\ &= \alpha\boldsymbol{\gamma}_3(1) + (1-\alpha)\boldsymbol{\gamma}_3(1) = \boldsymbol{\gamma}_3(1)\end{aligned}$$

- if $\omega \neq k\pi$ for some $k \in \mathbb{Z}$, setting $\alpha = 1 - 1/(4\cos^2(\omega))$ yields

$$\begin{aligned}\boldsymbol{a}_3 &\stackrel{\text{def}}{=} \left(1 - \frac{1}{4\cos^2(\omega)}\right)\begin{bmatrix} 2\cos(\omega) \\ -1 \\ 0 \end{bmatrix} + \frac{1}{4\cos^2(\omega)}\begin{bmatrix} 0 \\ 4\cos^2(\omega) - 1 \\ -2\cos(\omega) \end{bmatrix} \\ &= \begin{bmatrix} 2\cos(\omega) - \frac{1}{2\cos(\omega)} \\ 0 \\ -\frac{1}{2\cos(\omega)} \end{bmatrix},\end{aligned}$$

which also predicts X_4 perfectly, but based on just X_3 and X_1 (the one based on $\boldsymbol{a}_1$ uses just X_3 and X_2, whereas the one for $\boldsymbol{a}_2$, just X_2 and X_1)