Calculation of ACVF for ARMA Process: I

- consider causal ARMA(p, q) defined by
$$\phi(B)X_t = \theta(B)Z_t, \quad \{Z_t\} \sim \text{WN}(0, \sigma^2)$$
- want to determine ACVF $\{\gamma(h)\}$ for this process, which can be done using four complementary methods
- 1st method is based on MA(∞) representation
$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \psi(B)Z_t,$$
where $\psi(B) = \phi^{-1}(B)\theta(B)$
- have noted (overhead VII–8) that ACVF can be expressed as
$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

Calculation of ACVF for ARMA Process: II

- can use recursive scheme to compute ψ_j's (overhead VIII–16):

$$\psi_j = \sum_{k=1}^{p} \phi_k \psi_{j-k} + \theta_j, \quad j = 0, 1, 2, \ldots,$$

  but in general need ψ_j's for infinite number of integers j

- since $\psi_j \to 0$ as $j \to \infty$, could compute ψ_j's out to, say, $j = J + |h|$ and use

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} \approx \sigma^2 \sum_{j=0}^{J} \psi_j \psi_{j+|h|},$$

  with approximation getting better with increasing J

- if we have a manageable expression for ψ_j's (true for some processes), can get analytic expression for $\gamma(h)$

Example – ARMA(1,1) Process: I

- for an ARMA(1,1) process, overhead VII–26 says that

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j} \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

so $\psi_0 = 1$ and $\psi_j = (\phi + \theta)\phi^{j-1}$ for $j \geq 1$

- armed with $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ (valid for $|x| < 1$), away we go:

$$\frac{\gamma(0)}{\sigma^2} = \sum_{j=0}^{\infty} \psi_j^2 = 1 + (\phi + \theta)^2 \sum_{j=1}^{\infty} \phi^{2j-2}$$

$$= 1 + (\phi + \theta)^2 \sum_{j=0}^{\infty} \phi^{2j} = 1 + \frac{(\phi + \theta)^2}{1 - \phi^2}$$

Example – ARMA(1,1) Process: II

- for $h > 0$, with $\psi_j = (\phi + \theta)\phi^{j-1}$ for $j \geq 1$, have

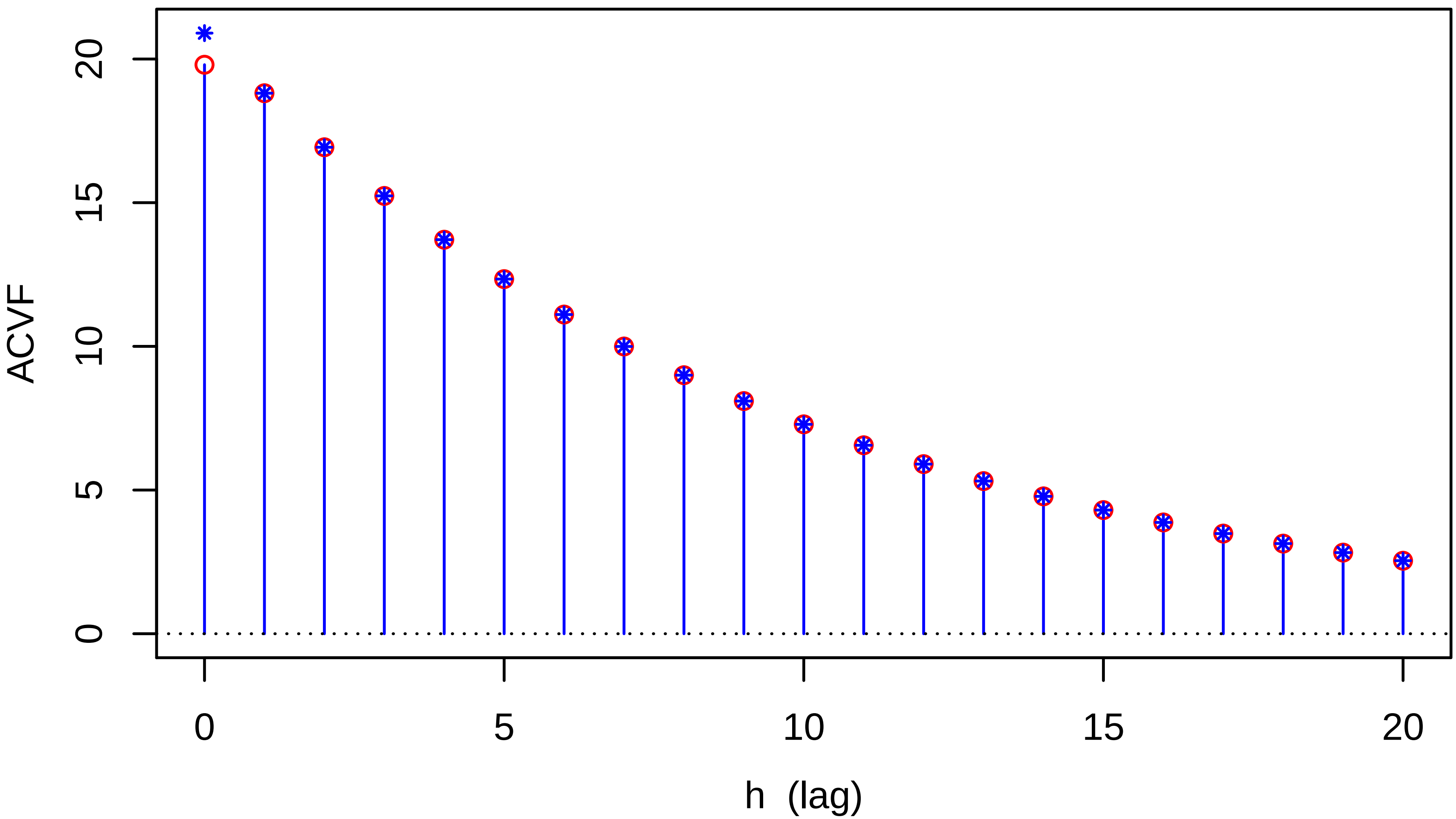
$$\begin{aligned}\frac{\gamma(h)}{\sigma^2} = \sum_{j=0}^{\infty} \psi_j \psi_{j+h} &= \psi_h + \sum_{j=1}^{\infty} \psi_j \psi_{j+h} \\ &= (\phi + \theta)\phi^{h-1} + (\phi + \theta)^2 \sum_{j=1}^{\infty} \phi^{2j+h-2} \\ &= (\phi + \theta)\phi^{h-1} + \frac{\phi^h (\phi + \theta)^2}{1 - \phi^2} \\ &= \phi^{h-1} \left(\phi + \theta + \frac{\phi(\phi + \theta)^2}{1 - \phi^2} \right)\end{aligned}$$

- note: $\gamma(h) = \phi\gamma(h - 1)$ for $h \geq 2$

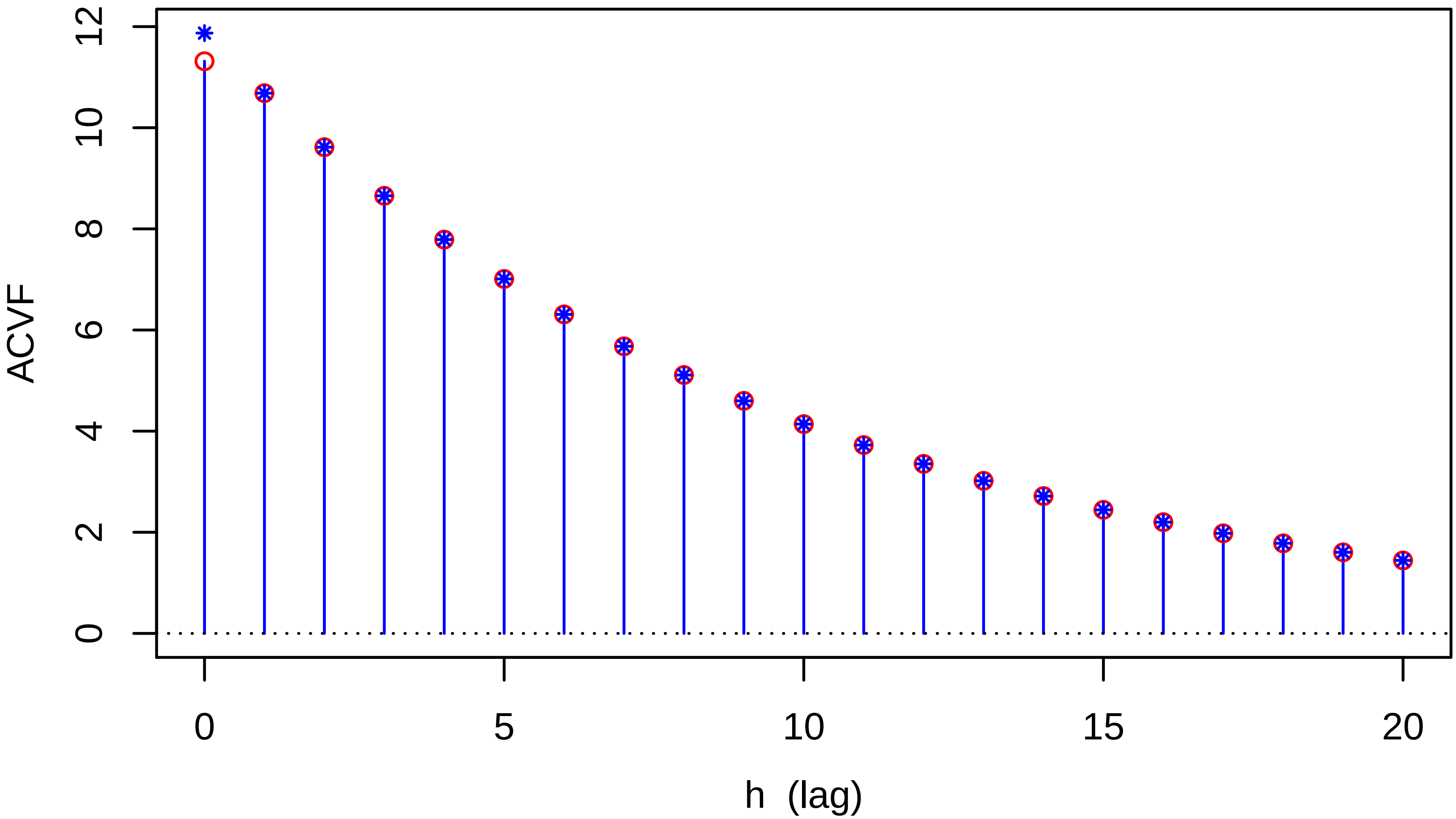
Example – ARMA(1,1) Process: III

- following overheads show ACVFs (circles) for ARMA(1,1) processes $\{X_t\}$ with
  - $\sigma^2 = 1$
  - AR parameter $\phi = 0.9$
  - MA parameter θ ranging from 0.99 down to -0.99
- have $\gamma_X(h) = \phi\gamma_X(h-1)$ for $h \geq 2$, but not for $h = 1$
- casual AR(1) process $\{Y_t\}$ has ACVF $\gamma_Y(h) = \phi^{|h|}\gamma_Y(0)$
- have $\gamma_Y(h) = \phi\gamma_Y(h-1)$ for $h \geq 1$
- overheads also show ACVFs (asterisks) for $\{Y_t\}$ with $\phi = 0.9$ and with $\gamma_Y(0)$ set such that $\gamma_Y(1) = \gamma_X(1)$
  - note: for some θ, not possible to do!

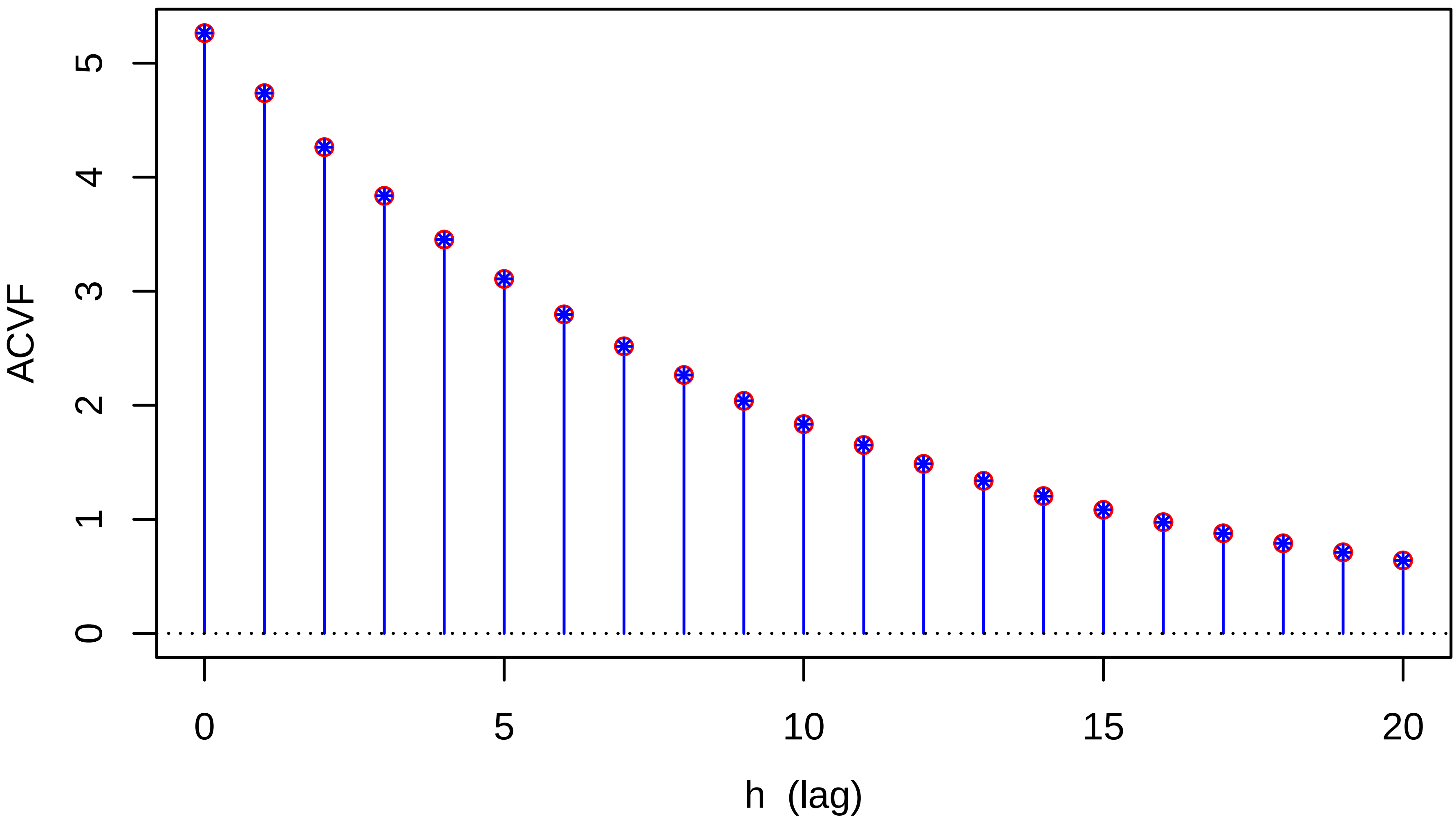
ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = 0.99$



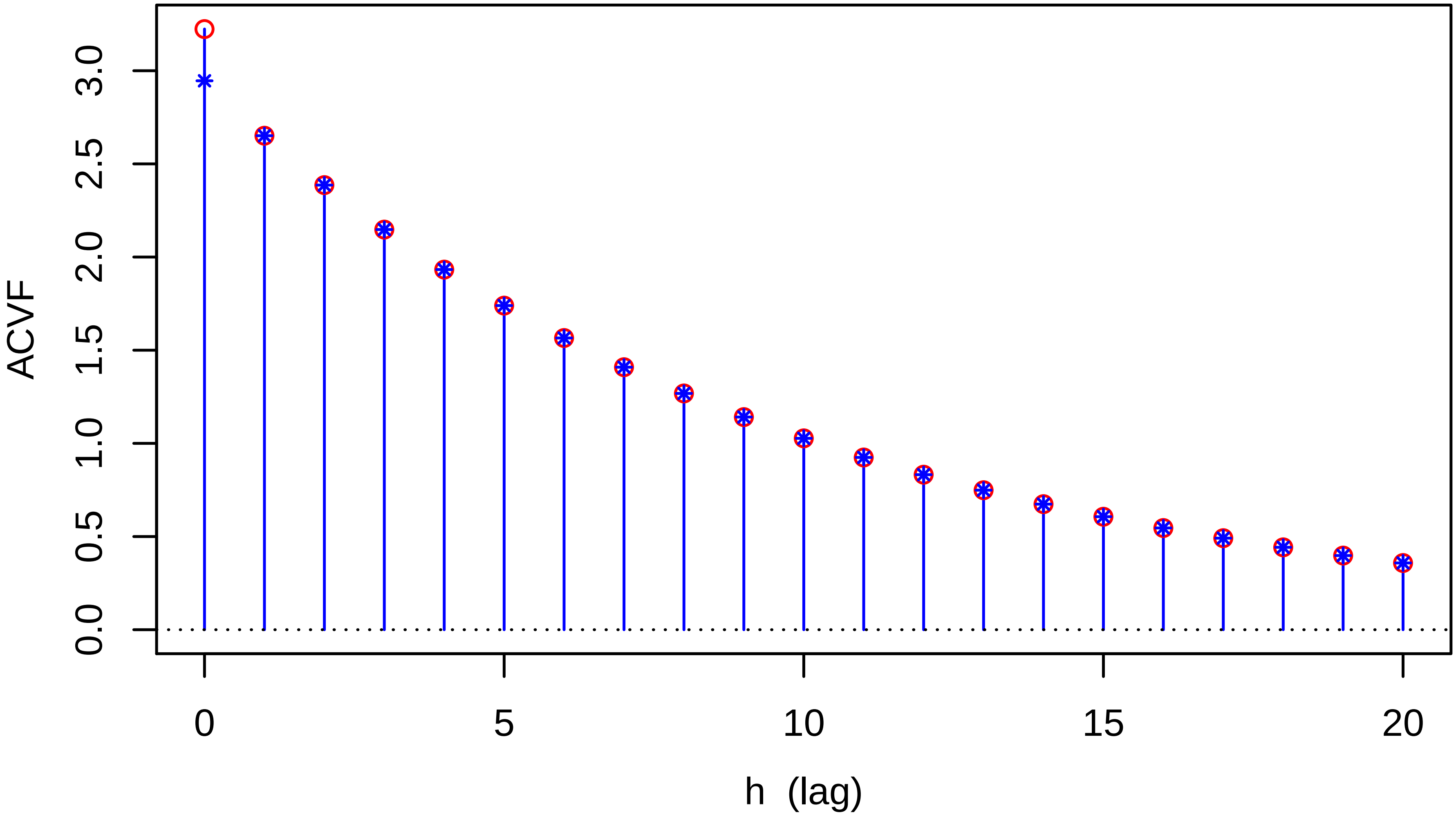
ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = 0.5$



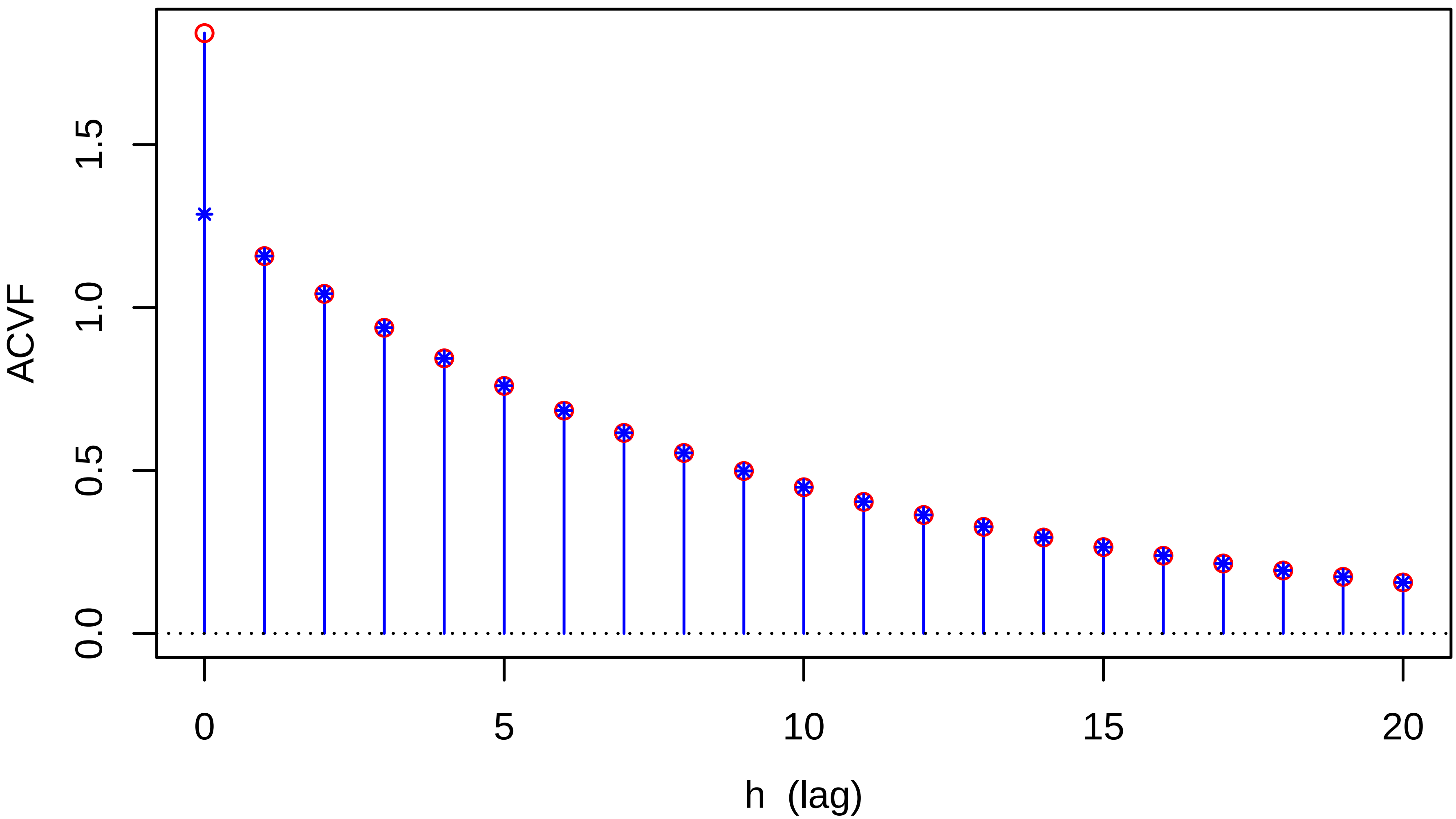
ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = 0$



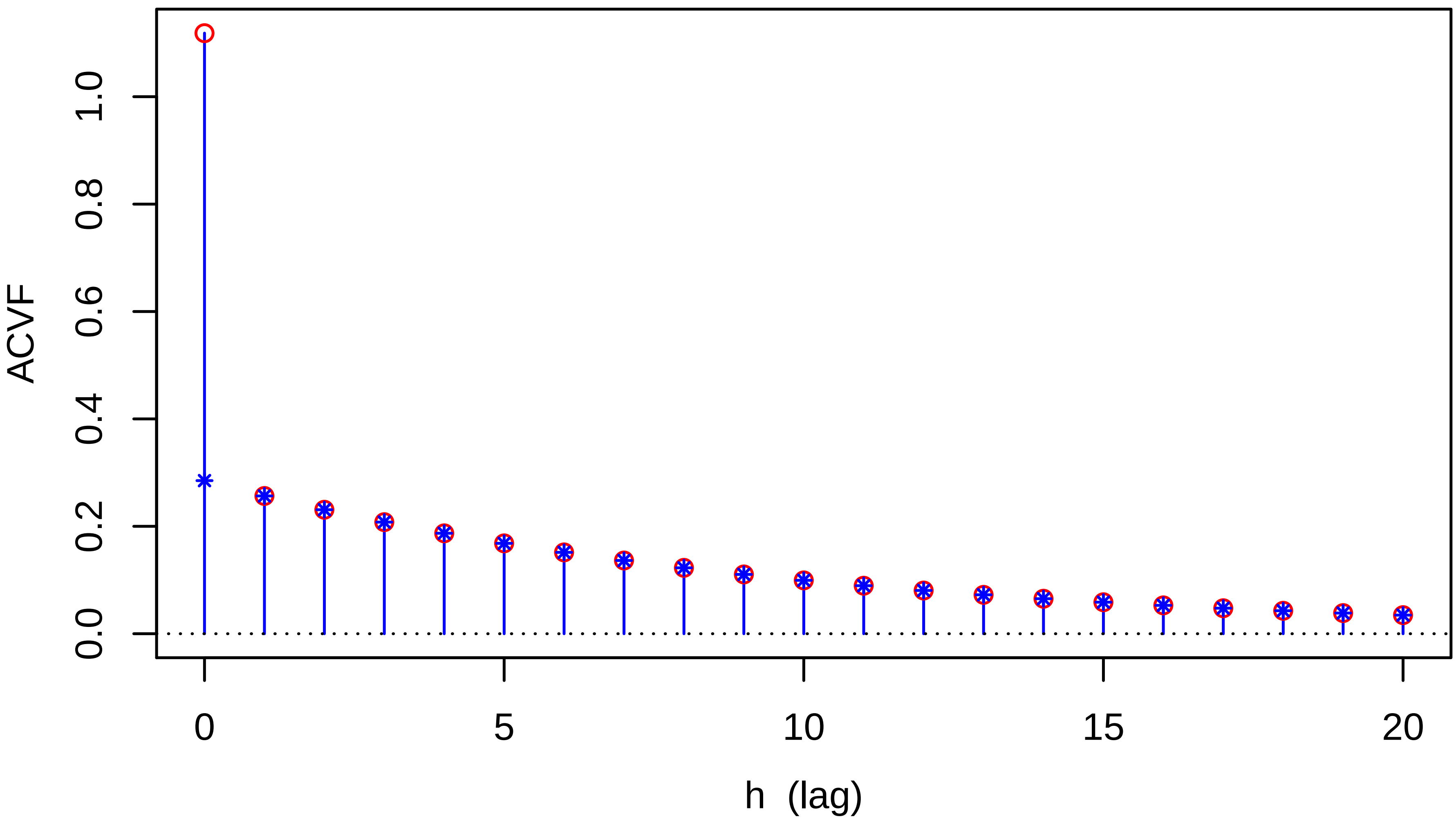
ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = -0.25$



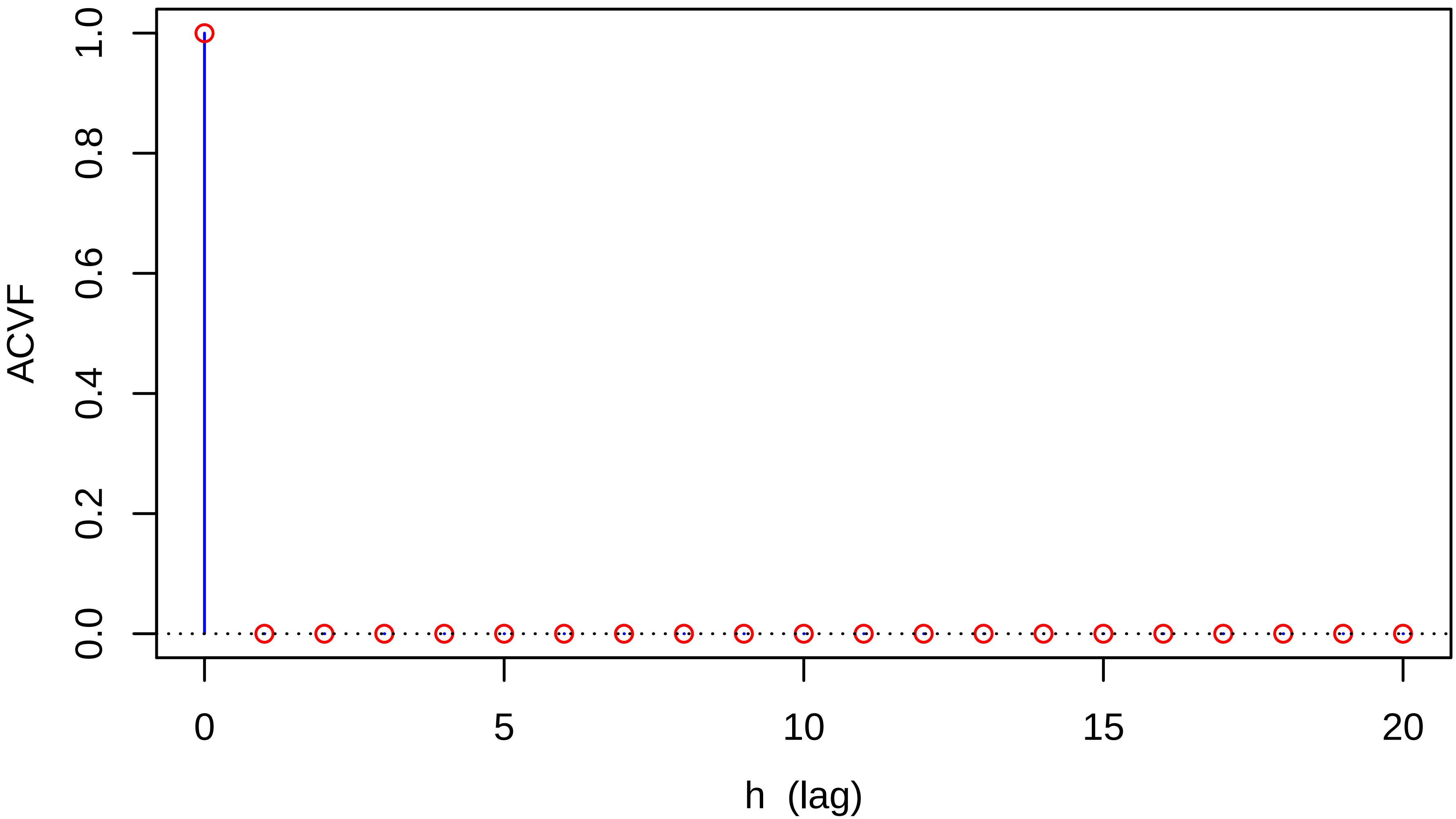
ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = -0.5$



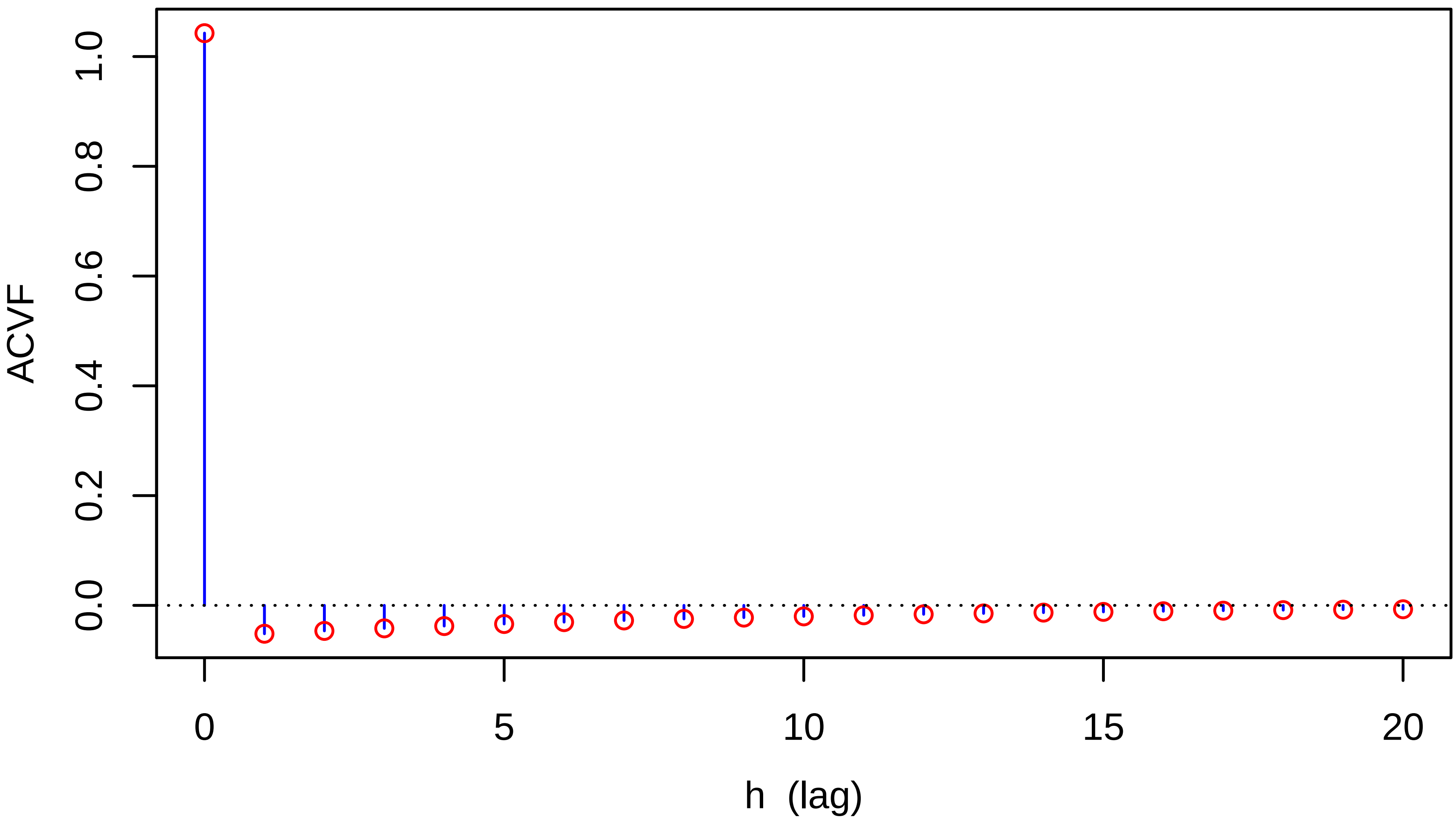
ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = -0.75$



ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = -0.9$



ACVF for ARMA(1,1) Process, $\phi = 0.9$, $\theta = -0.99$



Example – ARMA(1,1) Process: IV

- for three cases ($\theta = -0.25$, -0.5 and -0.75), ARMA(1,1) process $\{X_t\}$ has an ACVF $\gamma_X(h)$ that can be expressed as
$$\gamma_X(h) = \begin{cases} \gamma_Y(h) + C, & h = 0; \\ \gamma_Y(h), & |h| \neq 0, \end{cases}$$
where $C > 0$, and $\gamma_Y(h)$ is ACVF for AR(1) process
- called a 'nugget effect' in geological literature
- for six cases with $\theta \geq -0.75$, AR(1) ACVF emerges at lags $h \geq 1$, while $\theta = 0$ reduces ARMA(1,1) process to AR(1)
- $\theta = -0.9$ reduces ARMA(1,1) to white noise
- $\theta = -0.99$ causes slow decay of $\gamma_X(h) < 0$ toward zero
  - different from AR(1) ACVF when $\phi = \gamma_Y(1) < 0$: has $\gamma_Y(h)$ alternating between positive & negative as h increases

Example – MA(q) Process

- for an MA(q) process, have

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots \theta_q Z_{t-q} = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

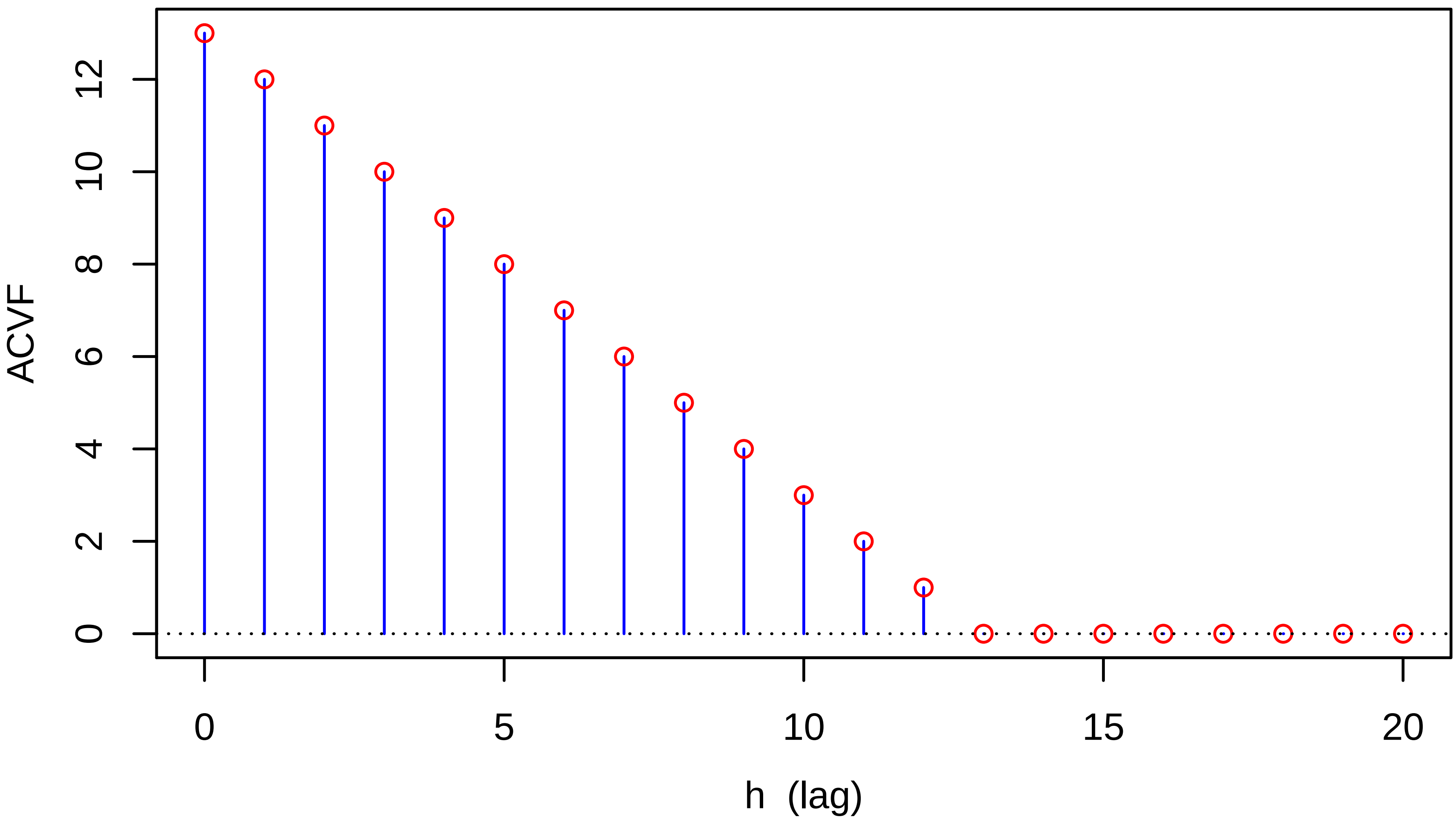
so $\psi_0 = 1$, $\psi_j = \theta_j$ for $1 \le j \le q$, and $\psi_j = 0$ for $j > q$

- letting $\theta_0 = 1$, have already noted (overhead VII–7) that

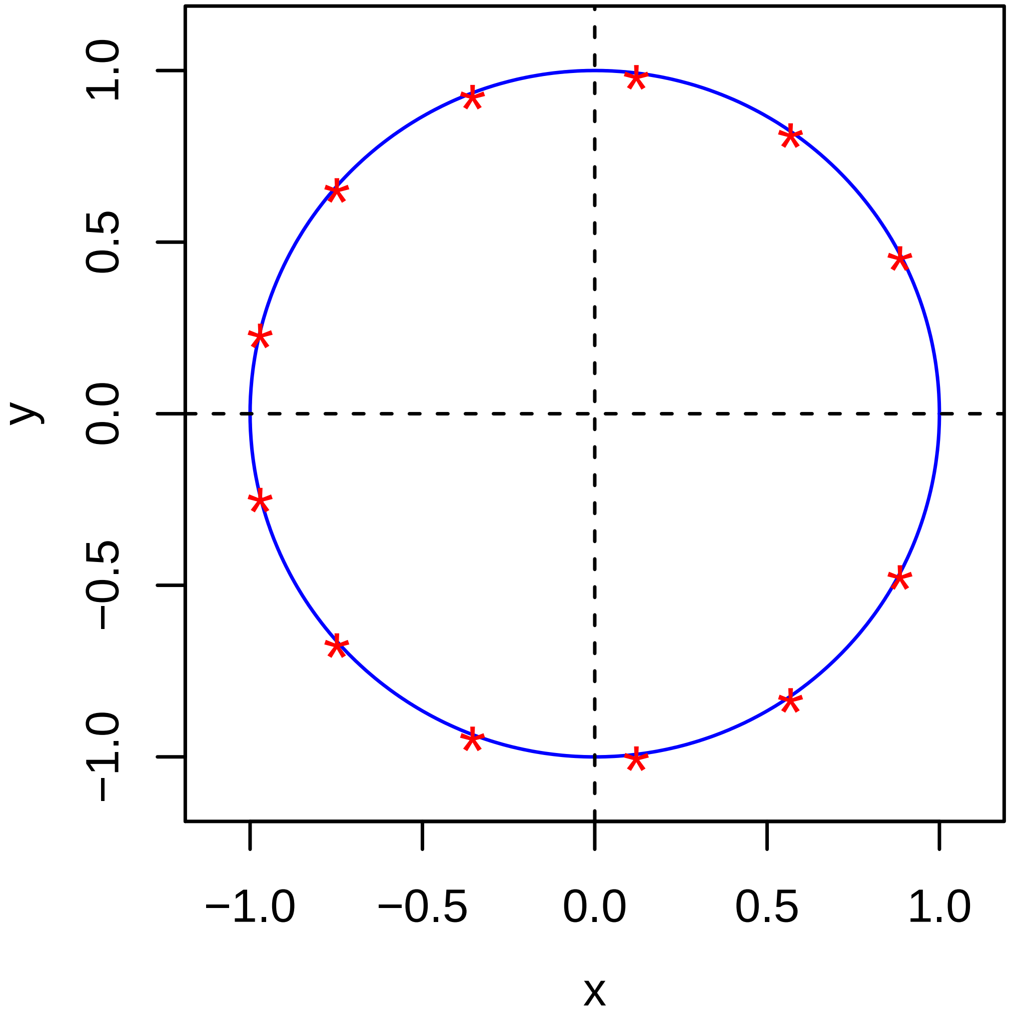
$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & |h| \le q; \\ 0, & |h| > q \end{cases}$$

- consider $q = 12$ with $\theta_0 = \theta_1 = \cdots = \theta_{12} = 1$ (13-point sums)

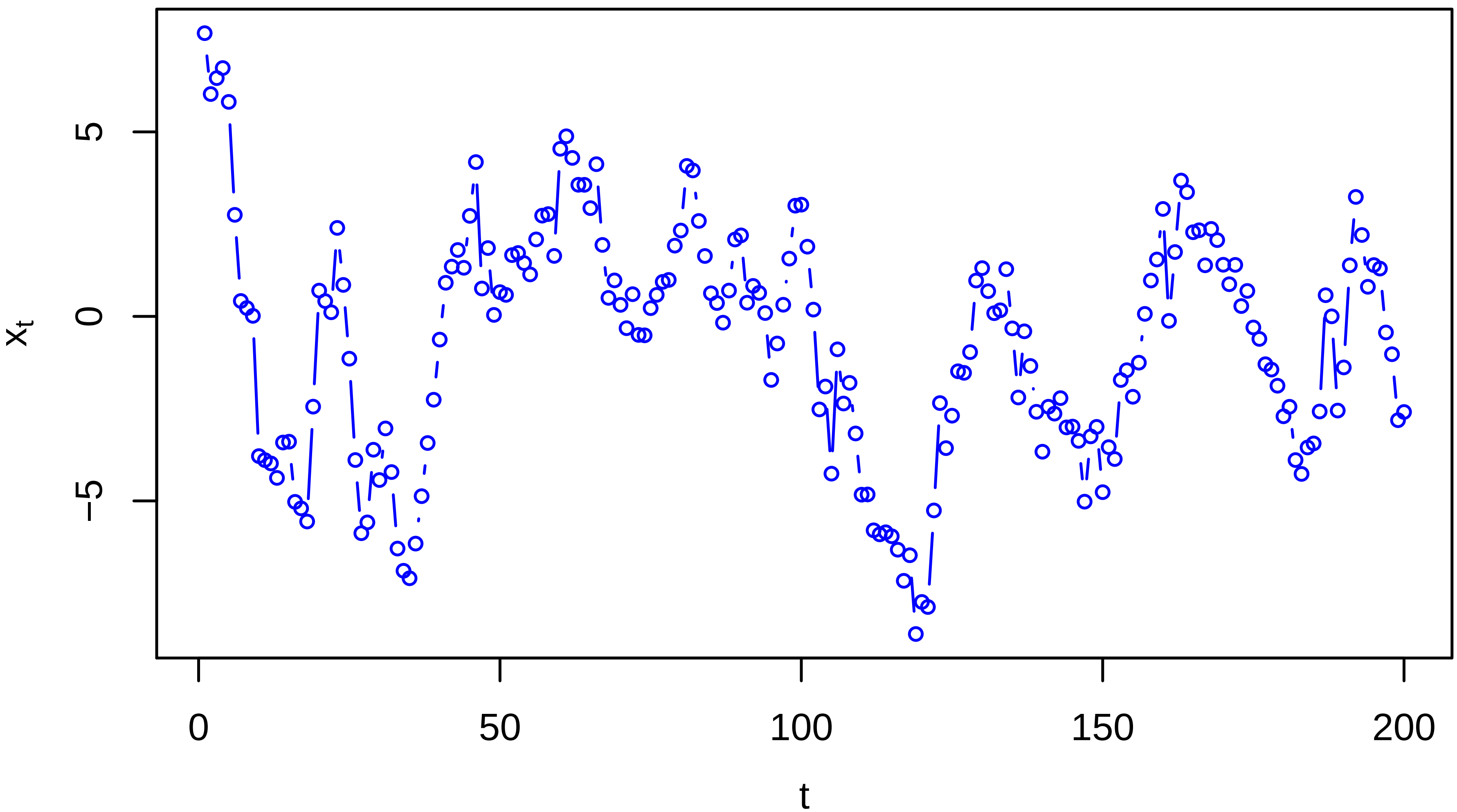
ACVF for MA(12) Process, $\theta_1 = \cdots = \theta_{12} = 1$



Roots of $\theta(z)$



Realization of MA(12) Process



Calculation of ACVF for ARMA Process: III

- 2nd method (of interest when $p \geq 1$): multiply both sides of
  $X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$
  by X_{t-k} for $k \geq 0$ and take expectations:

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = E\{Z_t X_{t-k}\} + \theta_1 E\{Z_{t-1}X_{t-k}\} + \cdots + \theta_q E\{Z_{t-q}X_{t-k}\}$$

- since

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \text{ get } E\{Z_{t-l}X_{t-k}\} = \sum_{j=0}^{\infty} \psi_j E\{Z_{t-l}Z_{t-k-j}\} = \sigma^2 \psi_{l-k}$$

(recall that $\psi_{l-k} \stackrel{\text{def}}{=} 0$ when $l - k < 0$), yielding

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}$$

Calculation of ACVF for ARMA Process: IV

- since $\psi_j = 0$ for $j < 0$, right-hand side of

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}$$

  will be 0 when $k > l$ for all $l = 0, \ldots, q$, i.e., when $k \geq q + 1$

- if $k \geq p$ so that $k - p \geq 0$, then $\gamma(k)$, $\gamma(k-1)$, ..., $\gamma(k-p)$ on left-hand side will involve $p + 1$ distinct elements of $\{\gamma(h)\}$

- thus, letting $m = \max\{q + 1, p\}$, we have

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = 0, \quad k = m, m+1, \ldots$$

- theory of homogeneous linear difference equations says that, if p roots z_j of $\phi(z) = 0$ are distinct, then

$$\gamma(h) = \alpha_1 z_1^{-h} + \alpha_2 z_2^{-h} + \cdots + \alpha_p z_p^{-h}, \quad h \geq m - p$$

Calculation of ACVF for ARMA Process: V

- to determine α_j's for given z_j's, plug $\gamma(h)$'s expressible as
$$\gamma(h) = \alpha_1 z_1^{-h} + \alpha_2 z_2^{-h} + \cdots + \alpha_p z_p^{-h}$$
into
$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}, \quad 0 \leq k < m,$$
yielding a system of m linear equations to be solved for m unknowns, where the unknowns are either
  - $\alpha_1, \ldots, \alpha_p$ when $m = p$ or
  - $\alpha_1, \ldots, \alpha_p, \gamma(0), \ldots, \gamma(m-p-1)$ when $m = q+1 > p$

  (recall that this method is only of interest when $p \geq 1$)
- note: need to recall recursive scheme for computing ψ_j's given ϕ_j's and θ_j's (see overhead VIII–16)

Example – ARMA(1,1) Process: V

- for ARMA(1,1) process $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$,
$$\gamma(k) - \phi_1 \gamma(k-1) - \cdots - \phi_p \gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}$$
becomes $\gamma(k) - \phi\gamma(k-1) = \sigma^2 (\psi_{-k} + \theta\psi_{1-k})$, $k = 0, 1, \ldots$
- overhead VII–25: $\psi_0 = 1$ & $\psi_1 = \phi + \theta$; recalling that $\psi_j = 0$, $j < 0$, have
$$\begin{aligned} \gamma(0) - \phi\gamma(1) &= \sigma^2 (1 + \theta\psi_1) = \sigma^2 (1 + \theta[\phi + \theta]) && (1) \\ \gamma(1) - \phi\gamma(0) &= \sigma^2 \theta && (2) \\ \gamma(k) - \phi\gamma(k-1) &= 0, \quad k = 2, 3, \ldots && (3) \end{aligned}$$
- root z of $\phi(z) = 1 - \phi z$ is $1/\phi$, so $\gamma(h) = \alpha z^{-h} = \alpha\phi^h$, $h \geq 1$
- using $\gamma(1) = \alpha\phi$ in (1) and (2) yields two linear equations to be solved to get unknowns α and $\gamma(0)$

Example – ARMA(1,1) Process: VI

- two equations are thus

$$\begin{aligned} \gamma(0) - \alpha\phi^2 &= \sigma^2\left(1 + \theta[\phi + \theta]\right) \\ \alpha\phi - \phi\gamma(0) &= \sigma^2\theta \end{aligned}$$

- matrix formulation is

$$\begin{bmatrix} 1 & -\phi^2 \\ -\phi & \phi \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \alpha \end{bmatrix} = \begin{bmatrix} \sigma^2\left(1 + \theta[\phi + \theta]\right) \\ \sigma^2\theta \end{bmatrix}$$

- assuming $\phi \neq 0$ (i.e., ARMA(1,1) is not an MA(1)), have

$$\begin{aligned} \begin{bmatrix} \gamma(0) \\ \alpha \end{bmatrix} &= \frac{1}{\phi(1-\phi^2)} \begin{bmatrix} \phi & \phi^2 \\ \phi & 1 \end{bmatrix} \begin{bmatrix} \sigma^2\left(1 + \theta[\phi + \theta]\right) \\ \sigma^2\theta \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 1 + \frac{(\phi+\theta)^2}{1-\phi^2} \\ \frac{1+\theta(\phi+\theta)}{1-\phi^2} + \frac{\theta}{\phi(1-\phi^2)} \end{bmatrix} \end{aligned}$$

Example – ARMA(1,1) Process: VII

- thus
$$\frac{\gamma(0)}{\sigma^2} = 1 + \frac{(\phi + \theta)^2}{1 - \phi^2}$$
in agreement with expression obtained by 1st method (cf. overhead IX–3) – hurray!!!

- substituting value for α into $\gamma(h) = \alpha\phi^h$ yields
$$\frac{\gamma(h)}{\sigma^2} = \phi^h \left(\frac{1 + \theta(\phi + \theta)}{1 - \phi^2} + \frac{\theta}{\phi(1 - \phi^2)} \right),$$
which, after some algebra, becomes
$$\frac{\gamma(h)}{\sigma^2} = \phi^{h-1} \left(\phi + \theta + \frac{\phi(\phi + \theta)^2}{1 - \phi^2} \right),$$
the same expression we got using 1st method (see overhead IX–4) – we're really on a roll!!!

Example – AR(2) Process: I

- for causal AR(2) process $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-1} = Z_t$,

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}$$

  becomes $\gamma(k) - \phi_1\gamma(k-1) - \phi_2\gamma(k-2) = \sigma^2\psi_{-k}$, $k = 0, 1, \ldots$

- since $\psi_0 = 1$ while $\psi_j = 0$ when $j < 0$, get

$$\begin{aligned}\gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) &= \sigma^2 \\ \gamma(k) - \phi_1\gamma(k-1) - \phi_2\gamma(k-2) &= 0, \quad k = 1, 2, \ldots\end{aligned}$$

- assuming roots z_1 & z_2 of $\phi(z) = 0$ are such that $z_1 \neq z_2$, solution takes form

$$\gamma(h) = \alpha_1 z_1^{-h} + \alpha_2 z_2^{-h}, \quad h \geq 0$$

Example – AR(2) Process: II

- substituting $\gamma(h) = \alpha_1 z_1^{-h} + \alpha_2 z_2^{-h}$ into

$$\begin{aligned} \gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) &= \sigma^2 \\ \gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(1) &= 0 \end{aligned}$$

leads to

$$\begin{aligned} \alpha_1 + \alpha_2 - \phi_1\left(\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1}\right) - \phi_2\left(\alpha_1 z_1^{-2} + \alpha_2 z_2^{-2}\right) &= \sigma^2 \\ \alpha_1 z_1^{-1} + \alpha_2 z_2^{-1} - \phi_1(\alpha_1 + \alpha_2) - \phi_2\left(\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1}\right) &= 0 \end{aligned}$$

- collecting terms gives

$$\begin{aligned} \alpha_1\left(1 - \phi_1 z_1^{-1} - \phi_2 z_1^{-2}\right) + \alpha_2\left(1 - \phi_1 z_2^{-1} - \phi_2 z_2^{-2}\right) &= \sigma^2 \\ \alpha_1\left(z_1^{-1} - \phi_1 - \phi_2 z_1^{-1}\right) + \alpha_2\left(z_2^{-1} - \phi_1 - \phi_2 z_2^{-1}\right) &= 0 \end{aligned}$$

Example – AR(2) Process: III

- in matrix form, we have

$$\begin{bmatrix} 1-\phi_1 z_1^{-1}-\phi_2 z_1^{-2} & 1-\phi_1 z_2^{-1}-\phi_2 z_2^{-2} \\ z_1^{-1}-\phi_1-\phi_2 z_1^{-1} & z_2^{-1}-\phi_1-\phi_2 z_2^{-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$$

- now

$$\phi(z) = 1-\phi_1 z-\phi_2 z^2 = \left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right) = 1-\left(\frac{1}{z_1}+\frac{1}{z_2}\right)z+\frac{z^2}{z_1 z_2}$$

tells us that $\phi_1 = z_1^{-1}+z_2^{-1}$ and $\phi_2 = -z_1^{-1}z_2^{-1}$, which yields

$$\begin{bmatrix} 1-z_1^{-2}-z_1^{-1}z_2^{-1}+z_1^{-3}z_2^{-1} & 1-z_2^{-2}-z_1^{-1}z_2^{-1}+z_1^{-1}z_2^{-3} \\ -z_2^{-1}+z_1^{-2}z_2^{-1} & -z_1^{-1}+z_1^{-1}z_2^{-2} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$$

- solving above for α_1 and α_2 yields solutions in terms of σ^2 and roots z_1 and z_2

Example – AR(2) Process: IV

- plugging solutions for α_1 and α_2 into $\gamma(h) = \alpha_1 z_1^{-h} + \alpha_2 z_2^{-h}$ yields (after a considerable amount of reduction!)

$$\gamma(h) = \frac{\sigma^2 z_1^2 z_2^2}{(z_1 z_2 - 1)(z_2 - z_1)} \left[\frac{z_1^{1-h}}{z_1^2 - 1} - \frac{z_2^{1-h}}{z_2^2 - 1} \right]$$

- for complex conjugate roots $z_1 = re^{i\omega}$ and $z_2 = z_1^*$, have

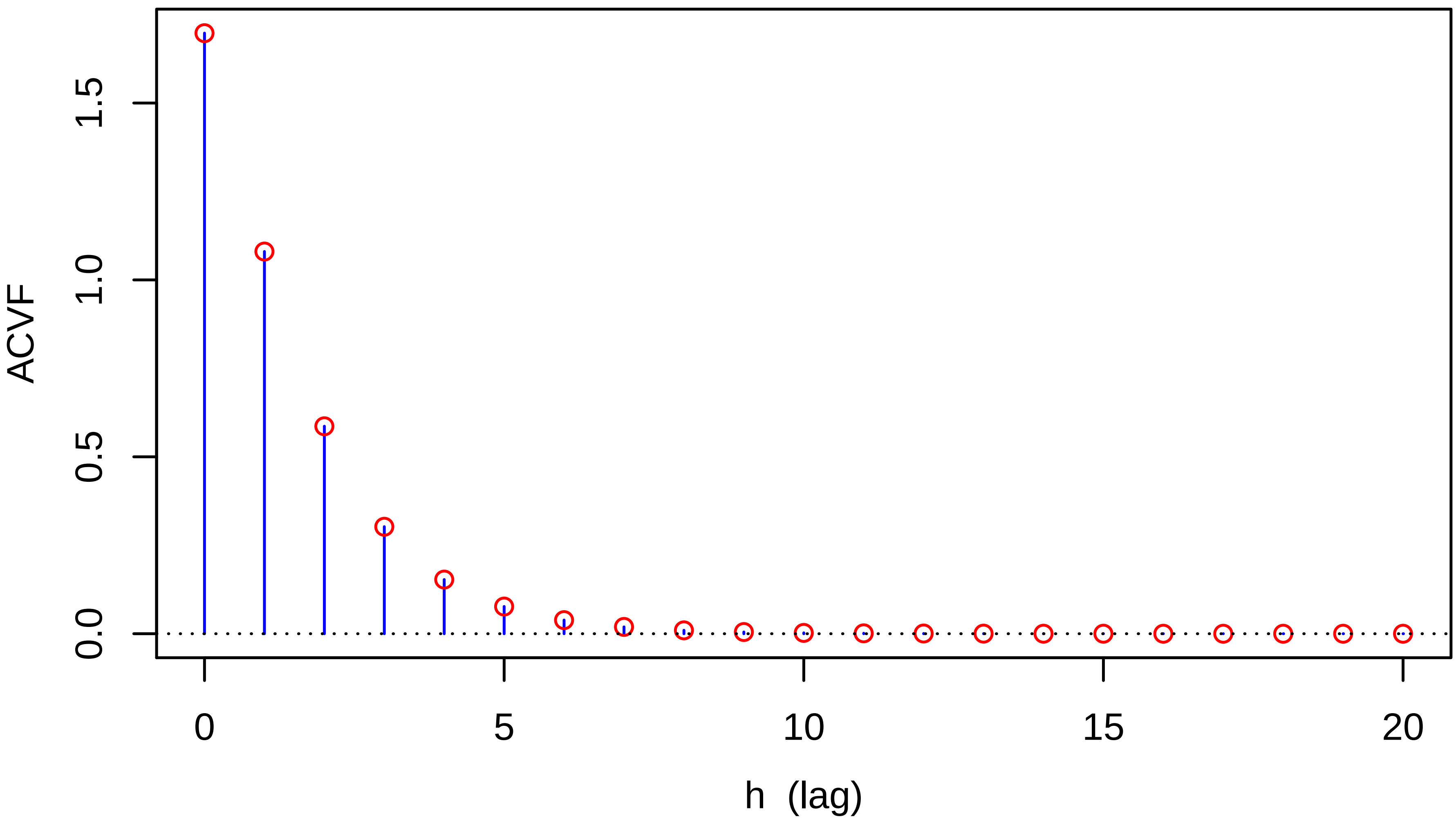
$$\gamma(h)/\gamma(0) = r^{-h} \sin(h\omega + \psi)/\sin(\psi), \qquad (*)$$

where

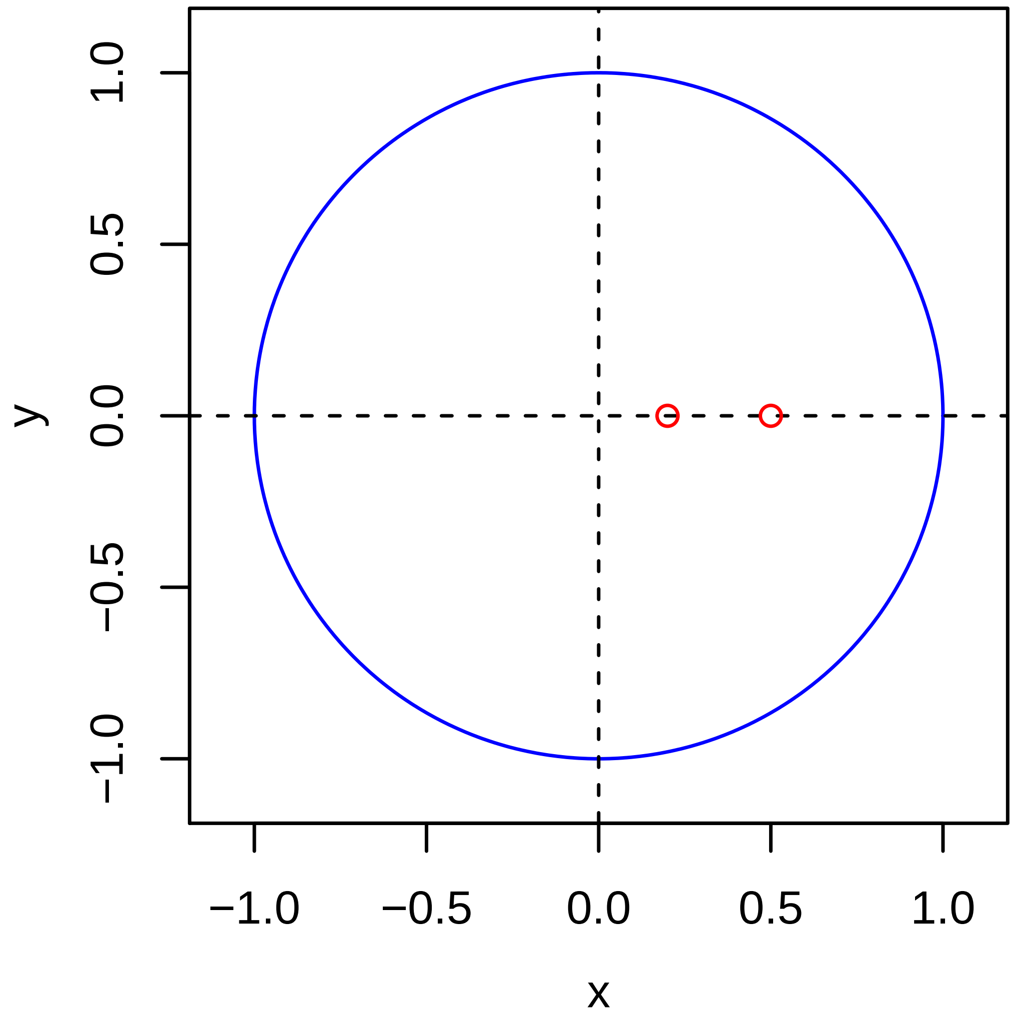
$$\gamma(0) = \frac{\sigma^2(r^6 + r^4)}{(r^2 - 1)(r^4 - 2r^2\cos(2\omega) + 1)} \text{ and } \tan(\psi) = \frac{r^2 + 1}{r^2 - 1}\tan(\omega)$$

  – note: $r > 1$ (roots are assumed to be *outside* unit circle)
  – $(*)$ is damped sinusoid with period $\frac{2\pi}{\omega}$, with damping slow when $r \approx 1$, i.e., when roots are close to unit circle

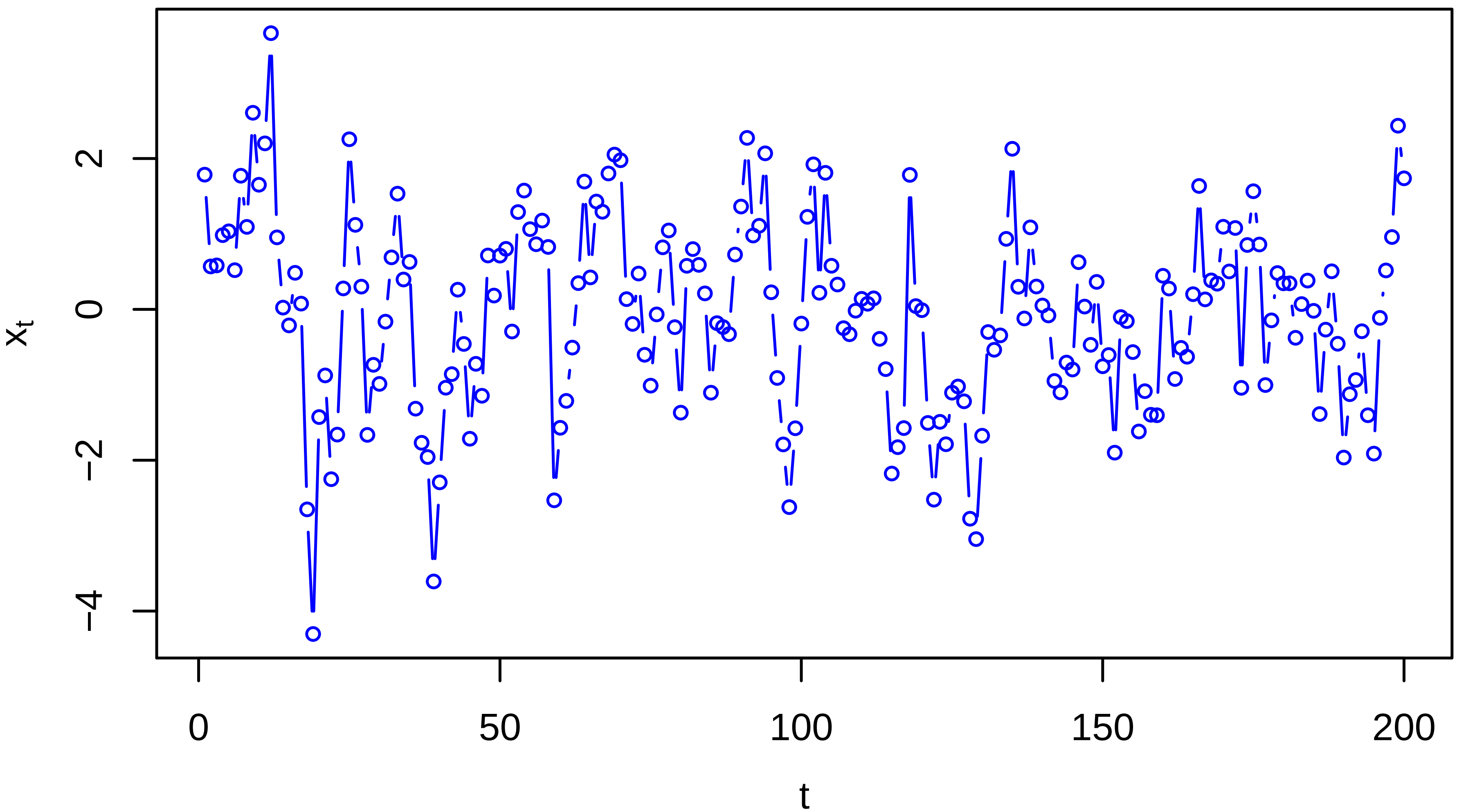
ACVF for AR(2) Process with $z_1 = 2$ & $z_2 = 5$



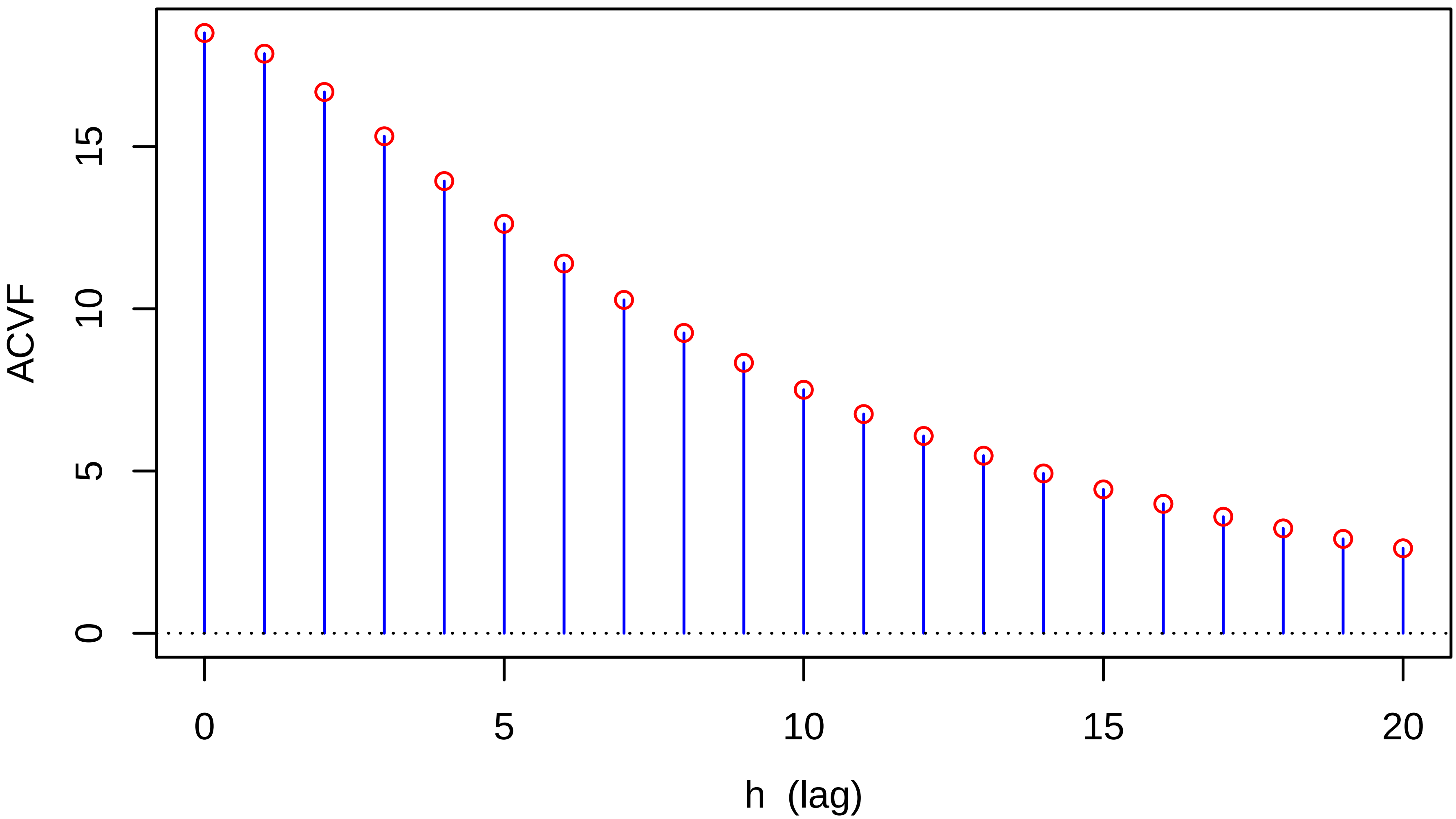
Reciprocal Roots Plot for $z_1 = 2$ & $z_2 = 5$



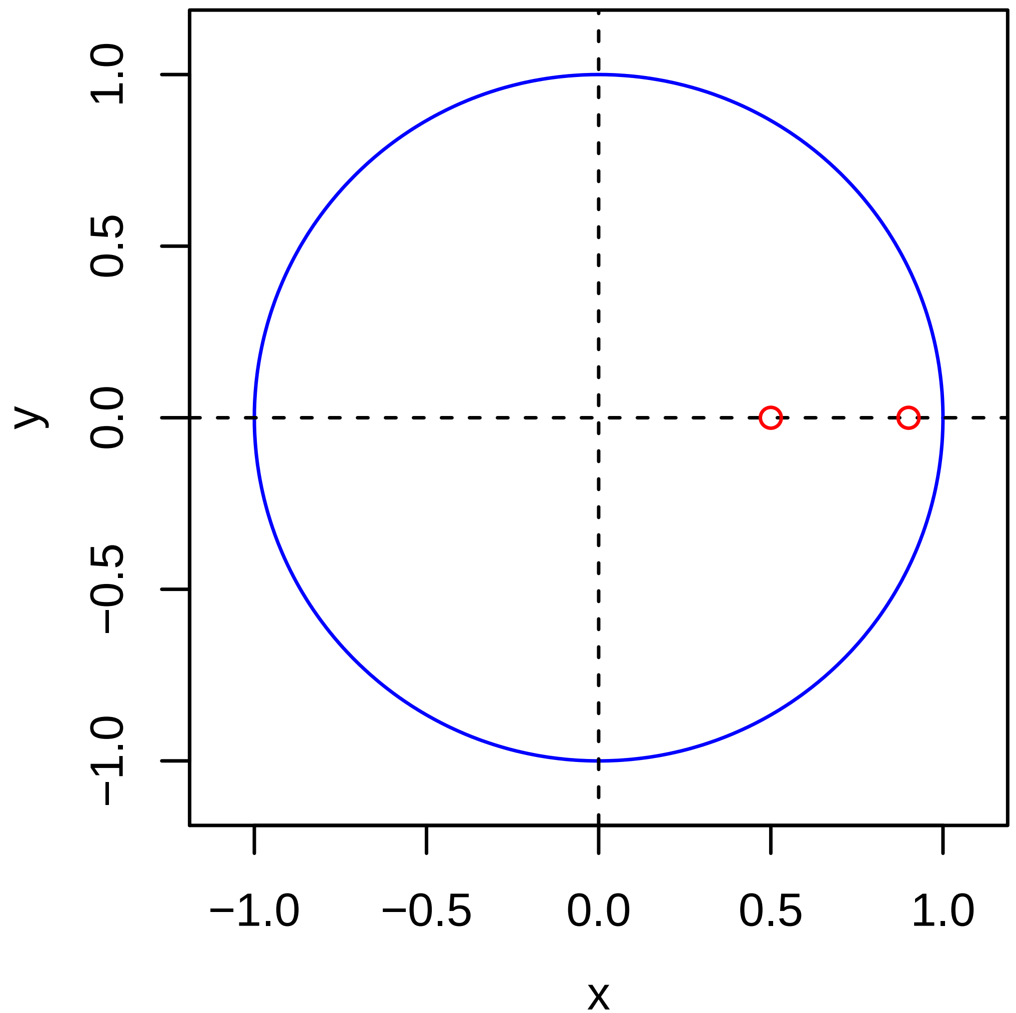
Realization of AR(2) Process



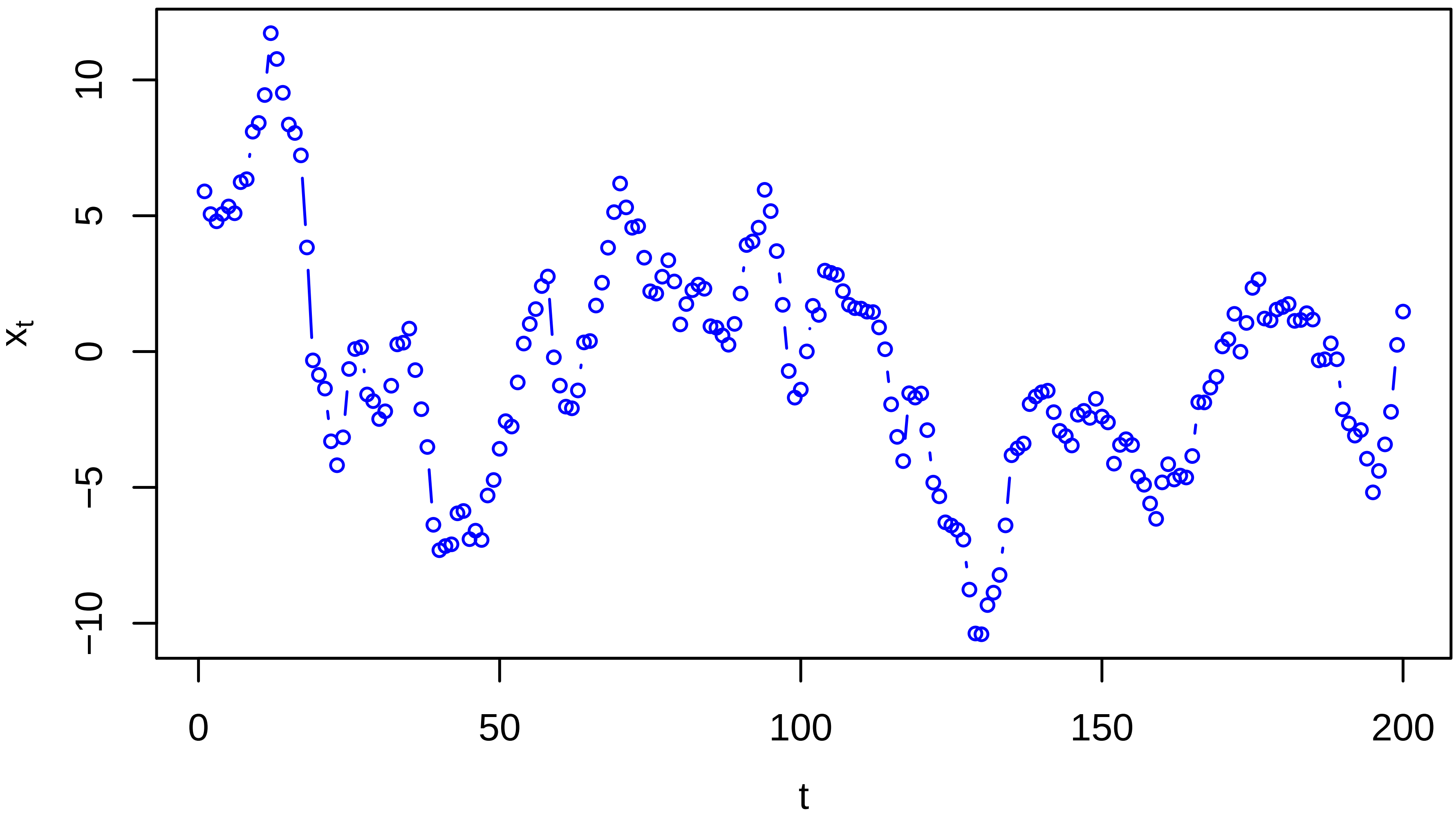
ACVF for AR(2) Process with $z_1 = \frac{10}{9}$ & $z_2 = 2$



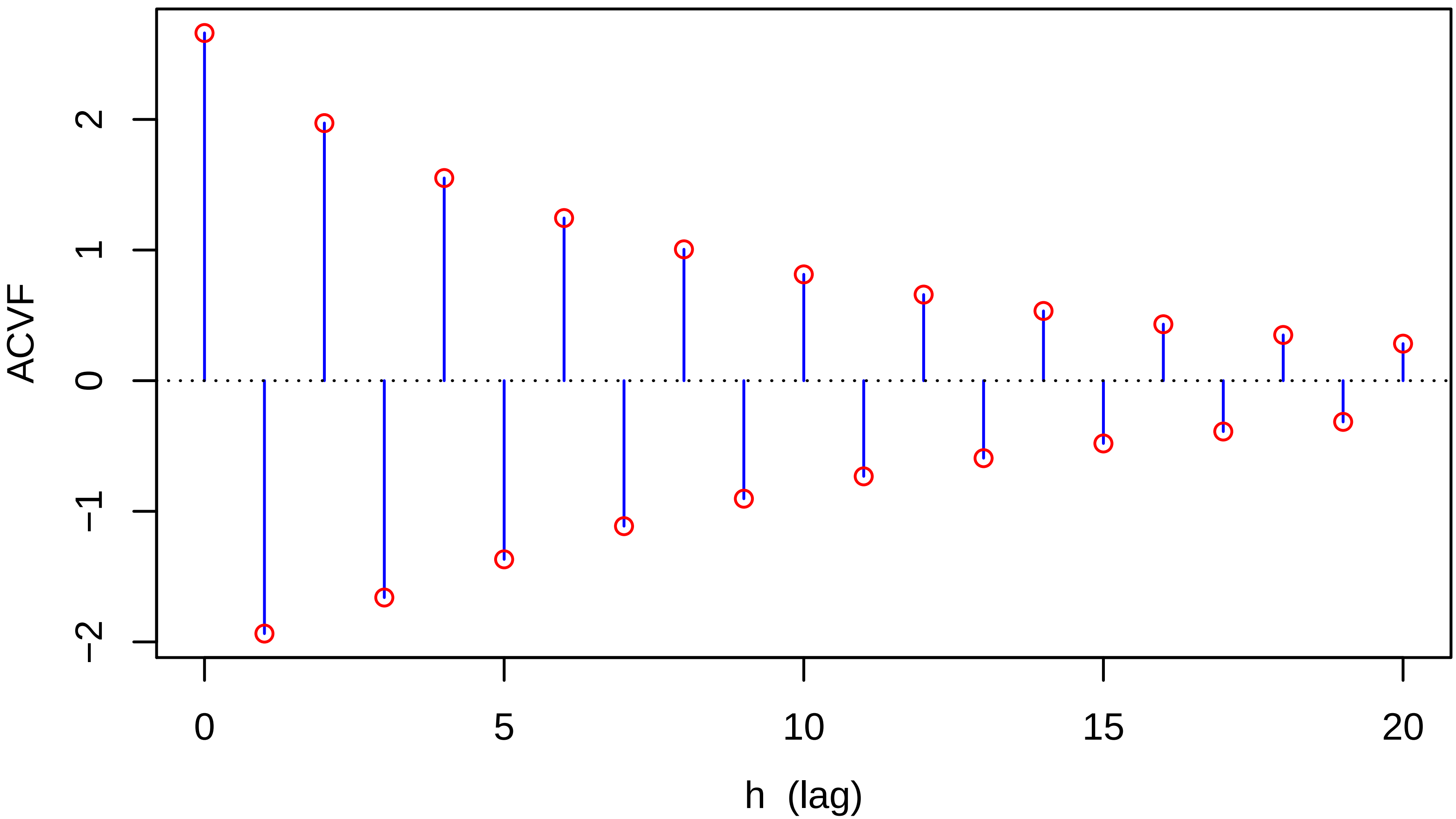
Reciprocal Roots Plot for $z_1 = \frac{10}{9}$ & $z_2 = 2$



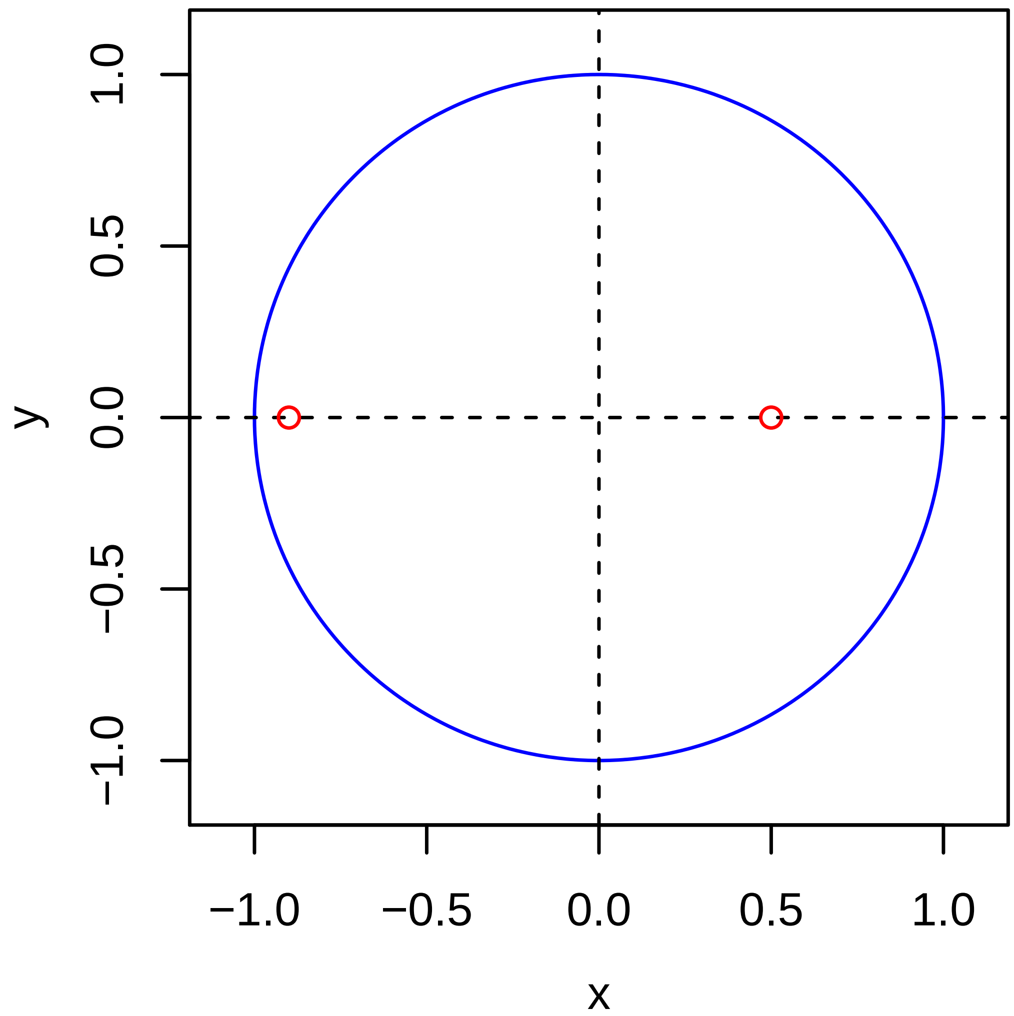
Realization of AR(2) Process



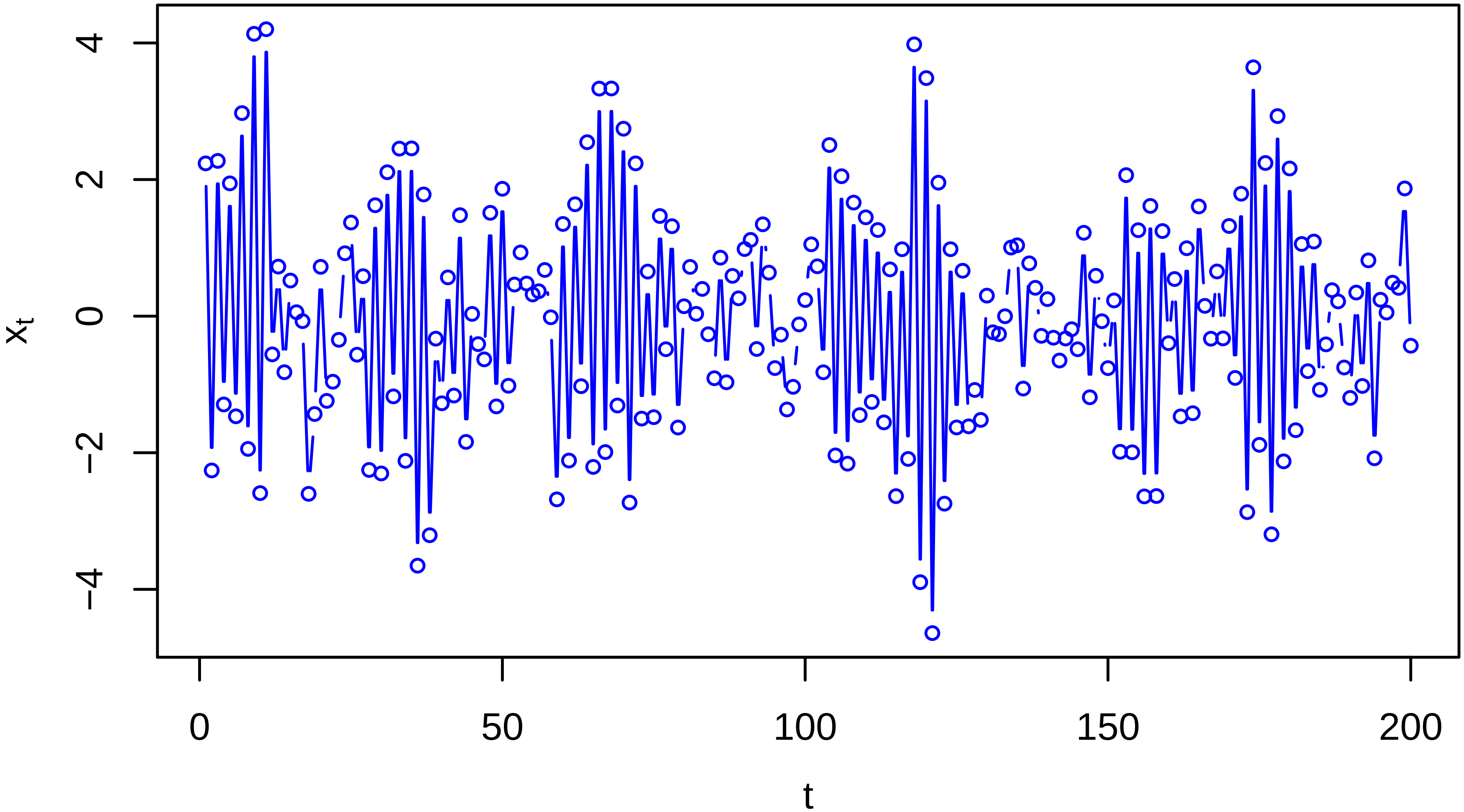
ACVF for AR(2) Process with $z_1 = -\frac{10}{9}$ & $z_2 = 2$



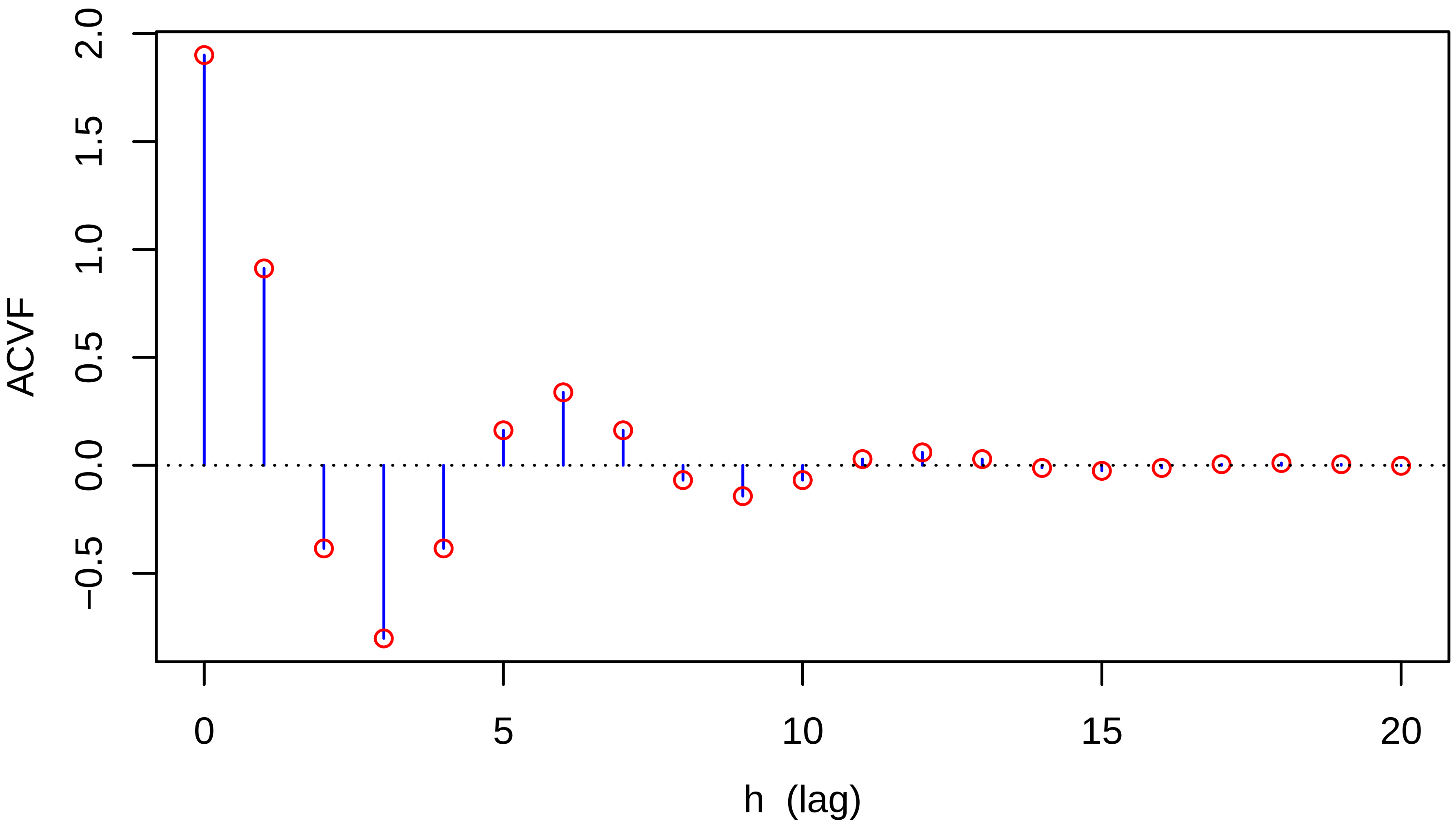
Reciprocal Roots Plot for $z_1 = -\frac{10}{9}$ & $z_2 = 2$



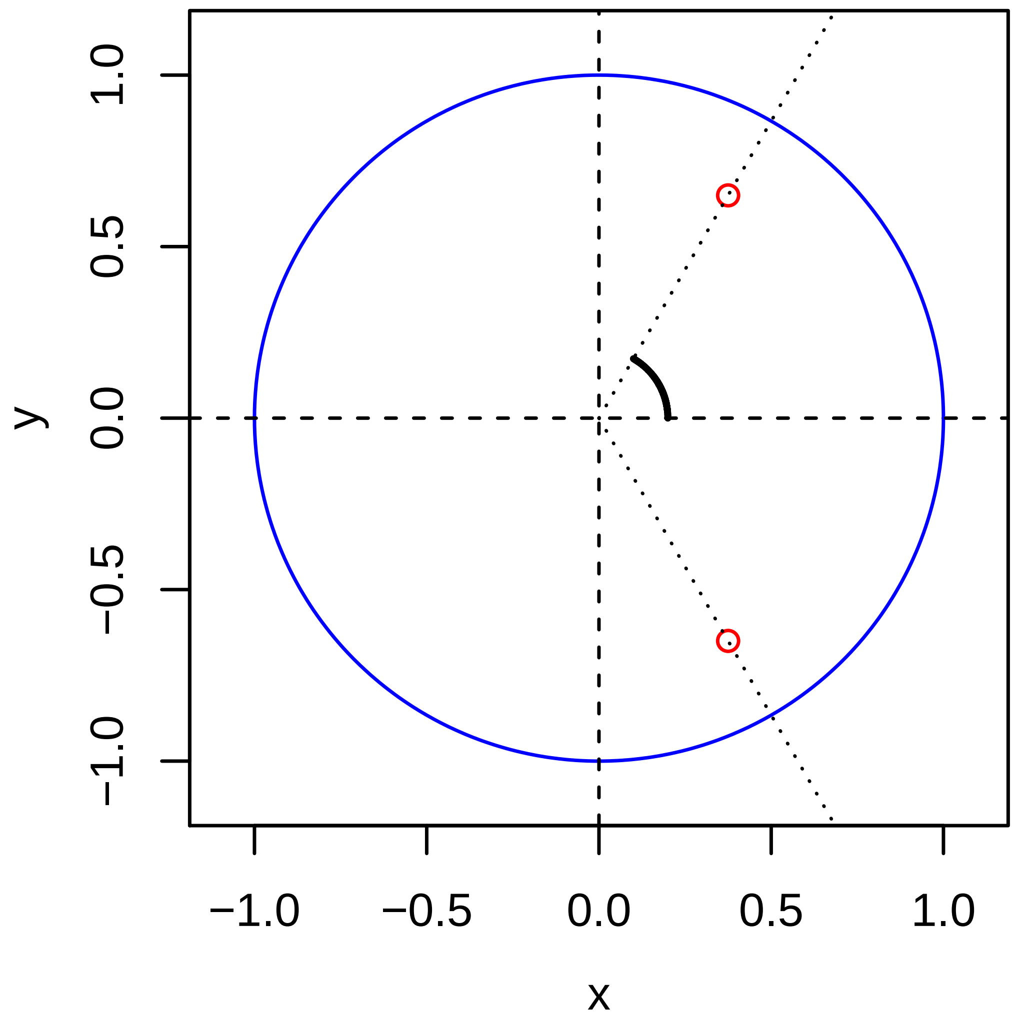
Realization of AR(2) Process



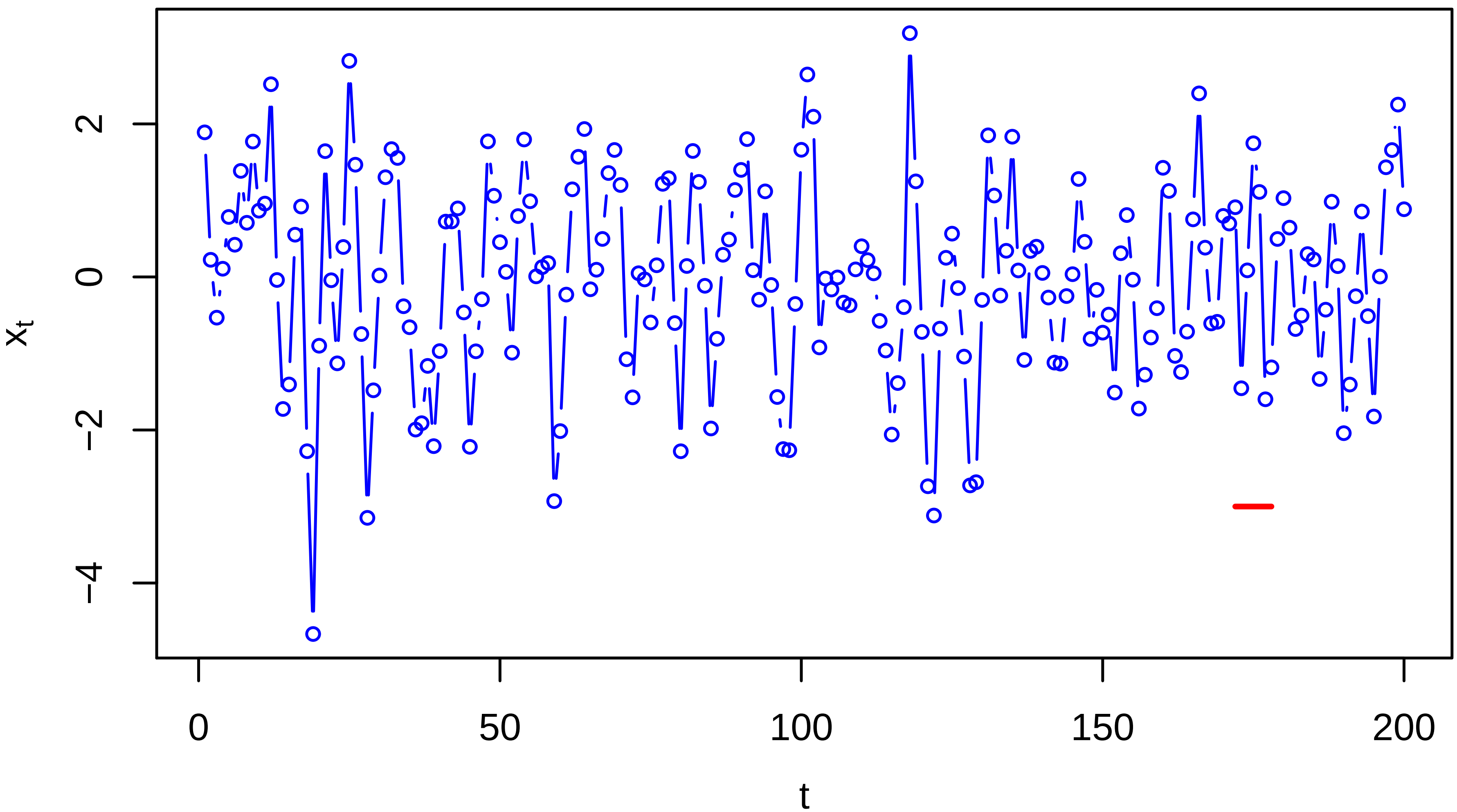
ACVF for AR(2) Process with $z_1 = \frac{2}{3} + i\frac{2}{\sqrt{3}}$ & $z_2 = z_1^*$



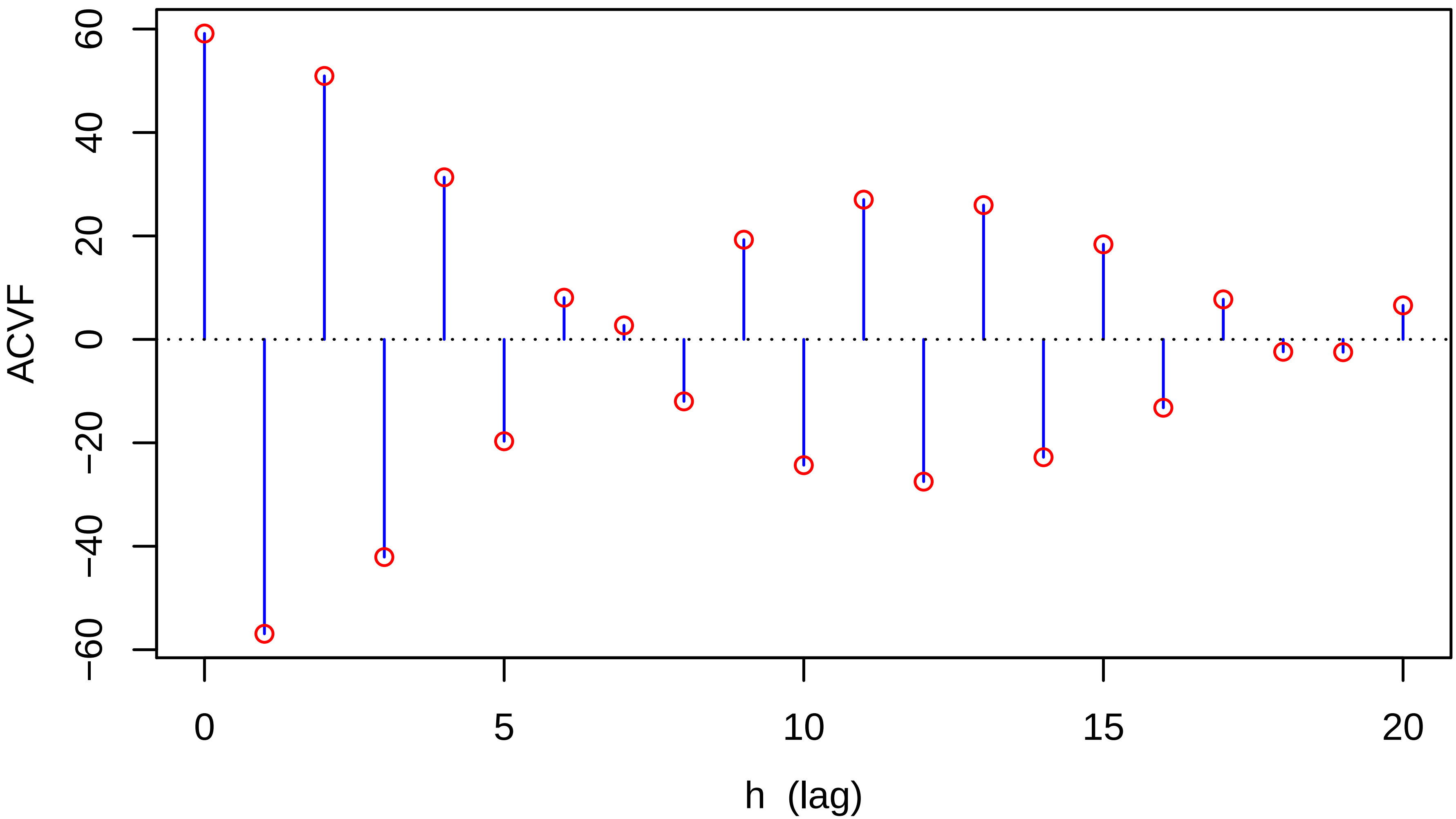
Reciprocal Roots Plot for $z_1 = \frac{2}{3} + i\frac{2}{\sqrt{3}}$ & $z_2 = z_1^*$



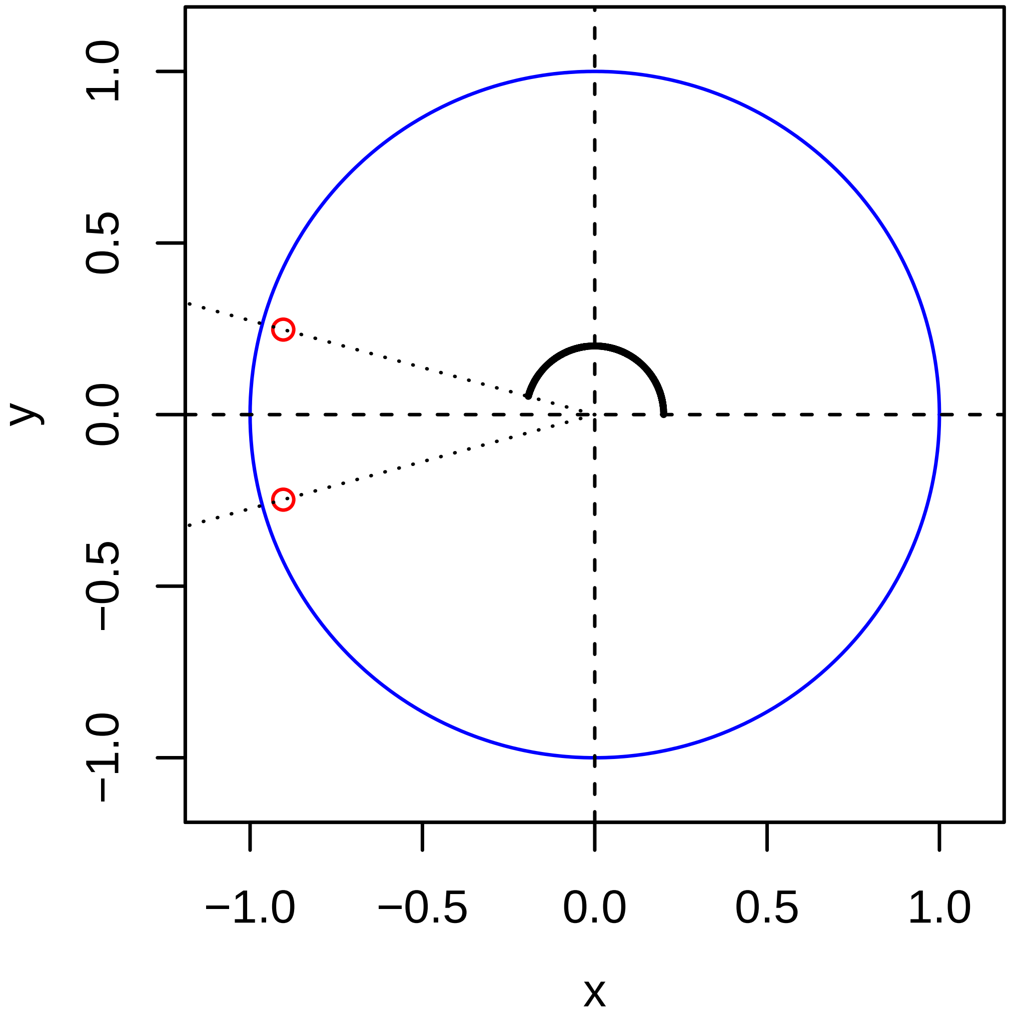
Realization of AR(2) Process



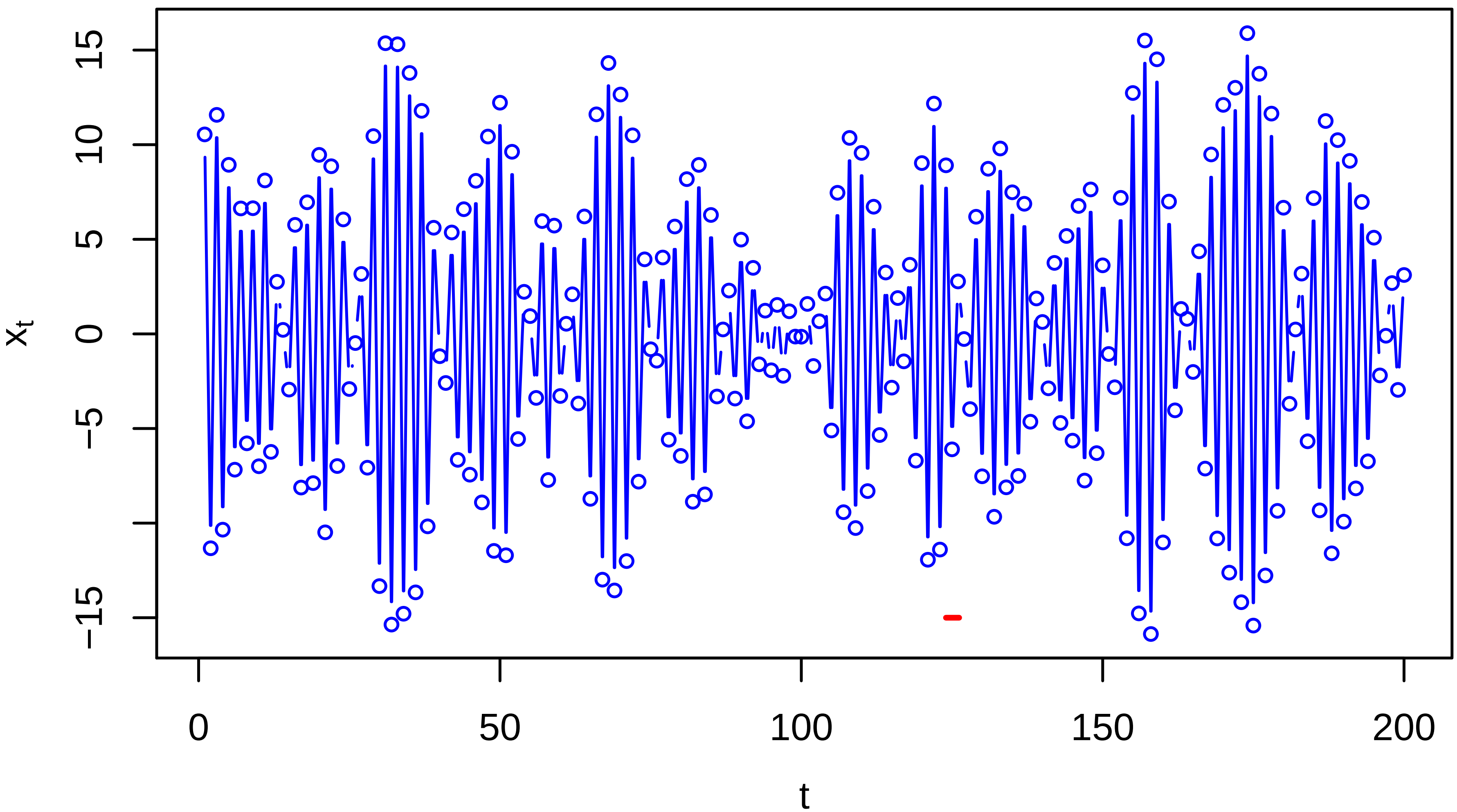
ACVF for AR(2) Process, $z_1 \doteq -1.03 + 0.28i$ & $z_2 = z_1^*$



Reciprocal Roots Plot for $z_1 \doteq -1.03 + 0.28i$ & $z_2 = z_1^*$



Realization of AR(2) Process



Calculation of ACVF for ARMA Process: VI

- 3rd method uses equations derived for 2nd method (IX–19):

$$\gamma(k)-\phi_1\gamma(k-1)-\cdots-\phi_p\gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}, \quad k \ge 0 \quad (*)$$

- letting $k = 0, 1, \ldots, p$ gives following system of equations:

$$\begin{aligned}
\gamma(0) - \phi_1\gamma(1) - \cdots - \phi_p\gamma(p) &= \sigma^2 \sum_{l=0}^{q} \theta_l \psi_l \stackrel{\text{def}}{=} c_0 \\
\gamma(1) - \phi_1\gamma(0) - \cdots - \phi_p\gamma(p-1) &= \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-1} \stackrel{\text{def}}{=} c_1 \\
&\vdots \\
\gamma(p) - \phi_1\gamma(p-1) - \cdots - \phi_p\gamma(0) &= \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-p} \stackrel{\text{def}}{=} c_p
\end{aligned}$$

Calculation of ACVF for ARMA Process: VII

- leads to following matrix equation:

$$\begin{bmatrix} 1 & -\phi_1 & -\phi_2 & \cdots & -\phi_{p-1} & -\phi_p \\ -\phi_1 & 1-\phi_2 & -\phi_3 & \cdots & -\phi_p & 0 \\ -\phi_2 & -\phi_1-\phi_3 & 1-\phi_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\phi_{p-1} & -\phi_{p-2}-\phi_p & -\phi_{p-3} & \cdots & 1 & 0 \\ -\phi_p & -\phi_{p-1} & -\phi_{p-2} & \cdots & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(p-1) \\ \gamma(p) \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{p-1} \\ c_p \end{bmatrix}$$

- solving above gives ACVF for lags 0 to p
- lags $k \geq p+1$ can be gotten recursively by rearranging $(*)$:

$$\gamma(k) = \phi_1\gamma(k-1) + \cdots + \phi_p\gamma(k-p) + \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}$$

- note: no need to find roots of $\phi(z)$ as required by 2nd method

Example – ARMA(1,1) Process: VIII

- as noted before, for ARMA(1,1) process,

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}$$

becomes $\gamma(k) - \phi_1\gamma(k-1) = \sigma^2\,(\psi_{-k} + \theta\psi_{1-k})$, yielding

$$\begin{aligned} \gamma(0) - \phi\gamma(1) &= \sigma^2\,(1 + \theta\psi_1) = \sigma^2\,(1 + \theta[\phi + \theta]) \\ \gamma(1) - \phi\gamma(0) &= \sigma^2\theta \\ \gamma(k) - \phi\gamma(k-1) &= 0, \quad k = 2, 3, \ldots \end{aligned}$$

- can get $\gamma(0)$ & $\gamma(1)$ by solving

$$\begin{bmatrix} 1 & -\phi \\ -\phi & 1 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \end{bmatrix} = \begin{bmatrix} \sigma^2\,(1 + \theta[\phi + \theta]) \\ \sigma^2\theta \end{bmatrix}$$

- remaining values gotten from $\gamma(k) = \phi\gamma(k-1)$, $k = 2, 3, \ldots$

Calculation of ACVF for ARMA Process: VIII

- 4th method uses the fact that an ARMA(p, q) process can be created by filtering an AR(p) process
- starting with the representation $\phi(B)X_t = \theta(B)Z_t$, note that causality allows us to write

$$X_t = \phi^{-1}(B)\theta(B)Z_t = \theta(B)\phi^{-1}(B)Z_t = \theta(B)Y_t, \text{ where } Y_t \stackrel{\text{def}}{=} \phi^{-1}(B)Z_t$$

- since we can also write

$$\phi(B)Y_t = Z_t,$$

  it follows that $\{Y_t\}$ is an AR(p) process, from which we can get $\{X_t\}$ by subjecting $\{Y_t\}$ to the MA filter

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

Calculation of ACVF for ARMA Process: IX

- recall that, if $\{Y_t\}$ is a stationary process with mean 0 and ACVF $\{\gamma_Y(h)\}$, then

$$X_t = \sum_{j=0}^{q} \theta_j Y_{t-j}, \text{ where, as usual, } \theta_0 \stackrel{\text{def}}{=} 1,$$

  is stationary with mean 0 and ACVF

$$\gamma_X(h) = \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_j \theta_k \gamma_Y(h + k - j)$$

  (the above follows readily from overhead VII–4)

- hence, if we can get ACVF for AR(p) process, we can readily compute ACVF for ARMA(p, q) process

Calculation of ACVF for ARMA Process: X

- reconsider equations derived for 2nd method (IX–19), namely,

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-k}, \quad k \geq 0$$

  and specialize them for AR(p) case (i.e., $q = 0$):

$$\gamma(k) - \phi_1\gamma(k-1) - \cdots - \phi_p\gamma(k-p) = \sigma^2 \psi_{-k}, \quad k \geq 0,$$

  where $\psi_0 = 1$, while $\psi_{-k} = 0$ for all $k > 0$

Calculation of ACVF for ARMA Process: XI

- leads to following matrix equation (special case of 3rd method):

$$\begin{bmatrix} 1 & -\phi_1 & -\phi_2 & \cdots & -\phi_{p-1} & -\phi_p \\ -\phi_1 & 1-\phi_2 & -\phi_3 & \cdots & -\phi_p & 0 \\ -\phi_2 & -\phi_1-\phi_3 & 1-\phi_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\phi_{p-1} & -\phi_{p-2}-\phi_p & -\phi_{p-3} & \cdots & 1 & 0 \\ -\phi_p & -\phi_{p-1} & -\phi_{p-2} & \cdots & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_Y(0) \\ \gamma_Y(1) \\ \gamma_Y(2) \\ \vdots \\ \gamma_Y(p-1) \\ \gamma_Y(p) \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

- after solving above to get $\gamma_Y(k)$ for lags 0 to p, can get it for lags $k \geq p+1$ recursively from

$$\gamma_Y(k) = \phi_1\gamma_Y(k-1) + \cdots + \phi_p\gamma_Y(k-p)$$

- exercise: use this approach to get ARMA(1,1) $\gamma_X(h)$
- note: after discussion of Levinson–Durbin recursions, can formulate a 5th method that is a variation on the 4th

Calculation of ACVF for ARMA Process – Summary

- 1st method can lead to analytic expression based directly on

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

  (if not, gives easy way to calculate $\gamma(h)$ approximately)

- 2nd method based on

$$\gamma(h) - \phi_1 \gamma(h-1) - \cdots - \phi_p \gamma(h-p) = \sigma^2 \sum_{l=0}^{q} \theta_l \psi_{l-h}$$

  (gives analytic expression and/or exact calculation of $\gamma(h)$)

- 3rd method is variation on 2nd (starts with same equations, but does not require finding roots of $\phi(z)$)

- 4th method gets $\gamma(h)$ via two-stage procedure using idea that ARMA process is result of filtering AR process with MA filter