ARMA Models: I

- autoregressive moving-average (ARMA) processes play a key role in time series analysis
- for any positive integer p & any purely nondeterministic process $\{X_t\}$ with ACVF $\{\gamma_X(h)\}$, there is an AR(p) process $\{Y_t\}$ with ACVF $\{\gamma_Y(h)\}$ such that $\gamma_Y(h) = \gamma_X(h)$ for $|h| \leq p$
- corresponding statement does *not* hold for MA(q) processes (cf. AR(1) and MA(1) processes), but adding MA component to form ARMA processes increases flexibility by defining potentially useful models with small number of parameters
- will now extend notions introduced for ARMA(1,1) model to higher order ARMA models

ARMA Models: II

- $\{X_t\}$ is said to be an ARMA(p, q) process if it is stationary and if, for $t \in \mathbb{Z}$,

  $X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$,

  where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and the polynomials

  $$1 - \phi_1 z - \cdots - \phi_p z^p \text{ and } 1 + \theta_1 z + \cdots + \theta_q z^q$$

  have no common roots (factors)

  – in above z is a complex-valued variable

  – above assumes that $\phi_p \neq 0$ if $p > 0$ and $\theta_q \neq 0$ if $q > 0$

- note: ARMA model sometimes written in 3 other ways:

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t - \theta_1 Z_{t-1} - \cdots - \theta_q Z_{t-q}$$
$$X_t + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$
$$X_t + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} = Z_t - \theta_1 Z_{t-1} - \cdots - \theta_q Z_{t-q}$$

ARMA Models: III

- polynomial condition is sometimes stated in terms of
$$1 - \phi_1 z^{-1} - \cdots - \phi_p z^{-p} \text{ and } 1 + \theta_1 z^{-1} + \cdots + \theta_q z^{-q}$$
having no common roots (as will be noted later, this equivalent formulation has one distinct advantage)
- to see why no common root is stipulated, recall ARMA(1,1) process $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$, for which $\phi + \theta \neq 0$ was stipulated
- reason for this stipulation became clear when we considered causal (and hence) stationary solution
$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j};$$
note that $\{X_t\}$ degenerates into WN model when $\phi + \theta = 0$

ARMA Models: IV

- ARMA(1,1) polynomial condition says $1 - \phi z$ & $1 + \theta z$ should not have a common root
- $1 - \phi z = 0$ & $1 + \theta z = 0$ yield roots of $1/\phi$ & $-1/\theta$, and $1/\phi \neq -1/\theta$ is equivalent to $\phi \neq -\theta$ and to stipulation $\phi + \theta \neq 0$
- can write ARMA(p, q) model more compactly as
$$\phi(B)X_t = \theta(B)Z_t,$$
with $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ & $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ (as before, B is the backward shift operator)
- needed conditions $|\phi| < 1$ & $|\theta| < 1$ on ARMA(1,1) parameters for process to be causal (and hence stationary) & invertible
- similarly, need conditions on ϕ_j's and θ_k's for ARMA(p,q) process to be stationary, causal and invertible – these can be stated as conditions on polynomials $\phi(z)$ and $\theta(z)$

ARMA Models: V

1. there is a (unique) *stationary* solution to $\phi(B)X_t = \theta(B)Z_t$ if and only if $\phi(z) \neq 0$ for all $|z| = 1$
2. ARMA(p, q) process is *causal*, meaning that, for $t \in \mathbb{Z}$,

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \psi(B)Z_t \text{ with } \psi(B) = \sum_{j=0}^{\infty} \psi_j B^j \;\&\; \sum_{j=0}^{\infty} |\psi_j| < \infty,$$

   if $\phi(z) \neq 0$ for all $|z| \leq 1$

3. ARMA(p, q) process is *invertible*, meaning that, for $t \in \mathbb{Z}$,

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = \pi(B)X_t \text{ with } \pi(B) = \sum_{j=0}^{\infty} \pi_j B^j \;\&\; \sum_{j=0}^{\infty} |\pi_j| < \infty,$$

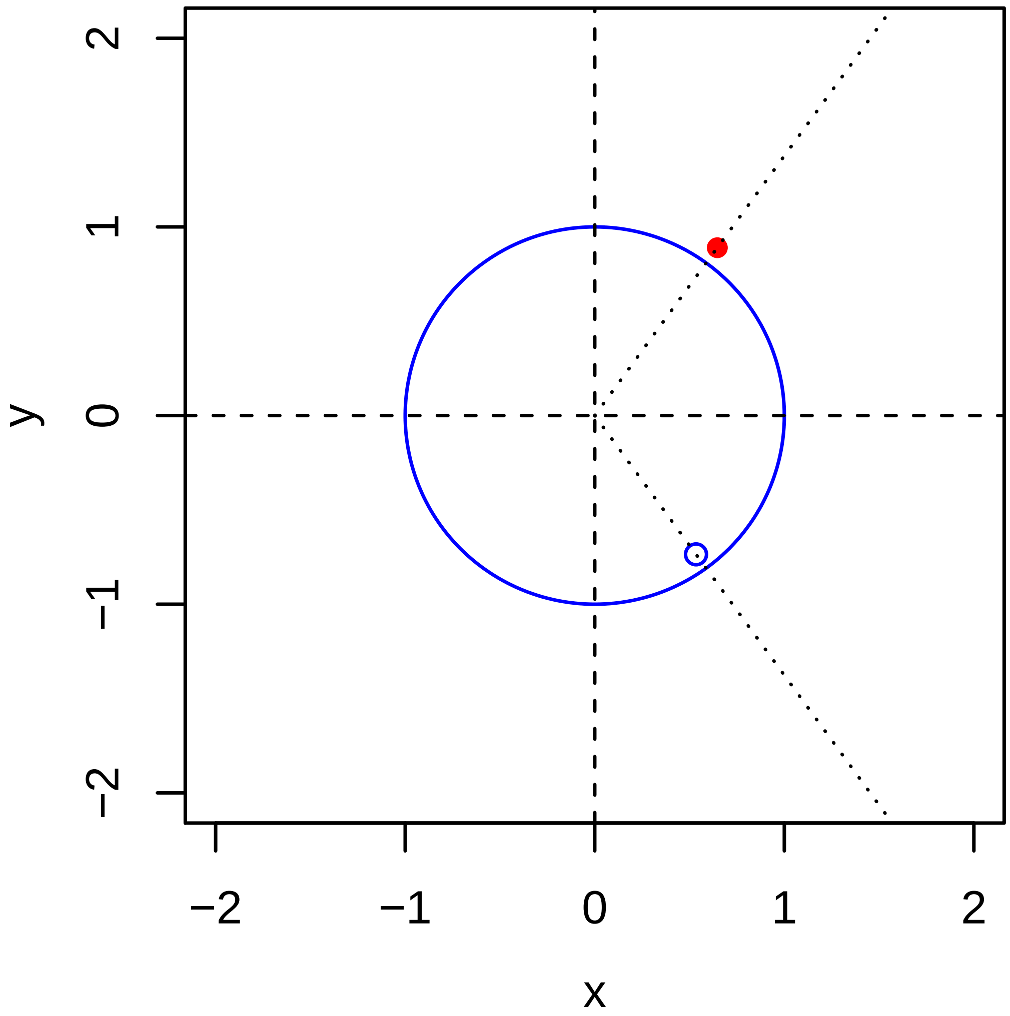
   if $\theta(z) \neq 0$ for all $|z| \leq 1$

ARMA Models: VI

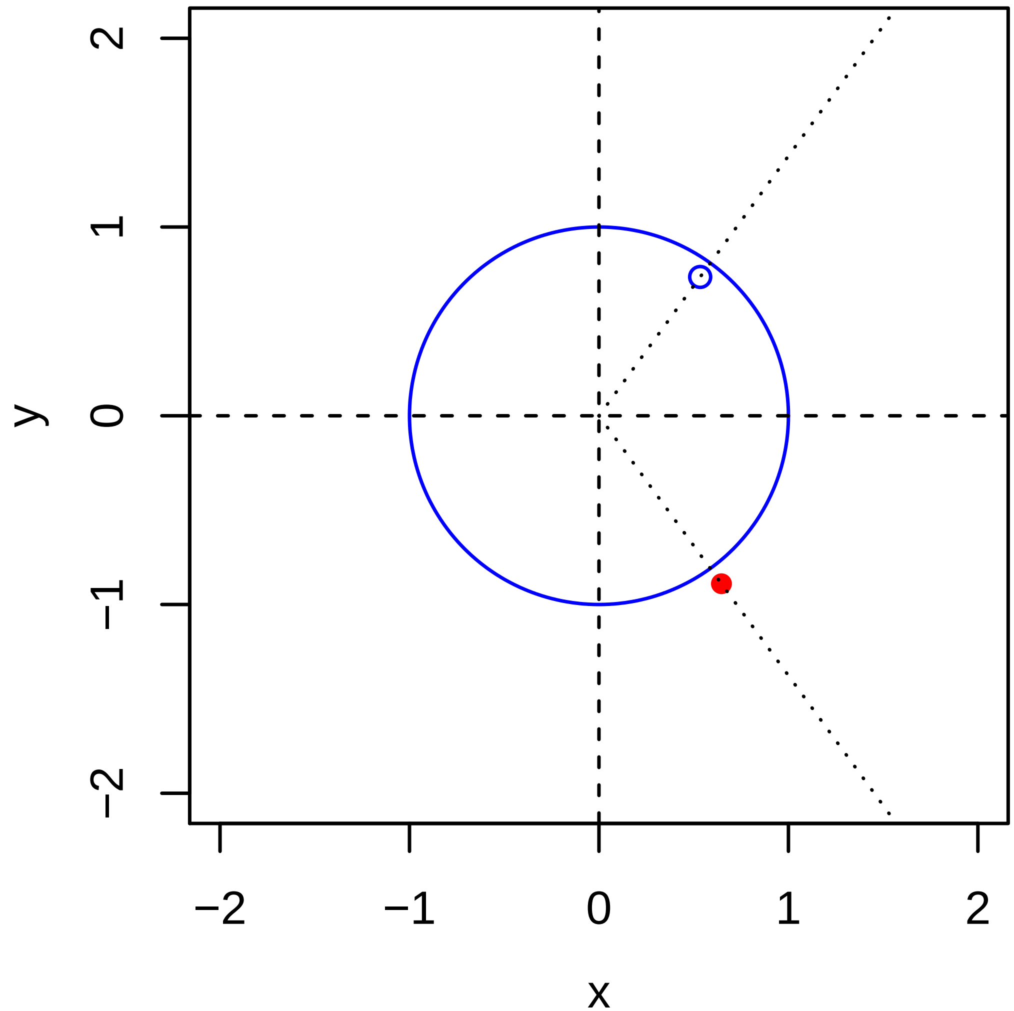
- for complex variable $z = x + iy$, where $i \stackrel{\text{def}}{=} \sqrt{-1}$, *unit circle* defined to be set of all z's such that $|z|^2 = x^2 + y^2 = 1$
- unit circle handily described by $e^{i\omega} \stackrel{\text{def}}{=} \cos(\omega) + i\sin(\omega)$ as ω varies from 0 to 2π (note that $|e^{i\omega}|^2 = \cos^2(\omega) + \sin^2(\omega) = 1$)
- conditions can be restated in terms of roots of $\phi(z)$ and $\theta(z)$, i.e., values z_l and z_m such that $\phi(z_l) = 0$ and $\theta(z_m) = 0$

1. *stationarity*: requires all roots z_l of $\phi(z)$ be *off* the unit circle; i.e., must have $|z_l| \neq 1$
2. *causality*: requires all roots z_l of $\phi(z)$ to be *outside* the unit circle; i.e., must have $|z_l| > 1$
3. *invertibility*: requires all roots z_m of $\theta(z)$ to be *outside* the unit circle; i.e., must have $|z_m| > 1$

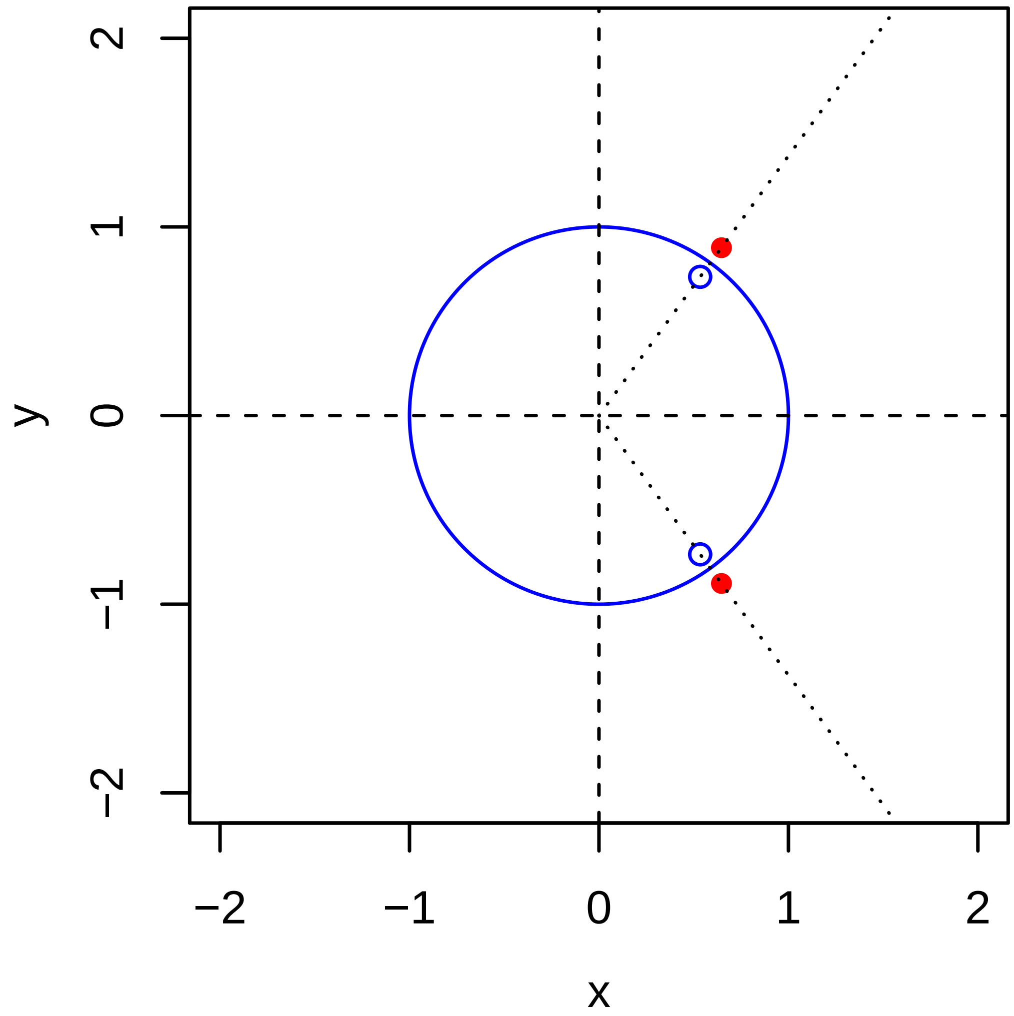
Unit Circle, Root z and Its Reciprocal $1/z$



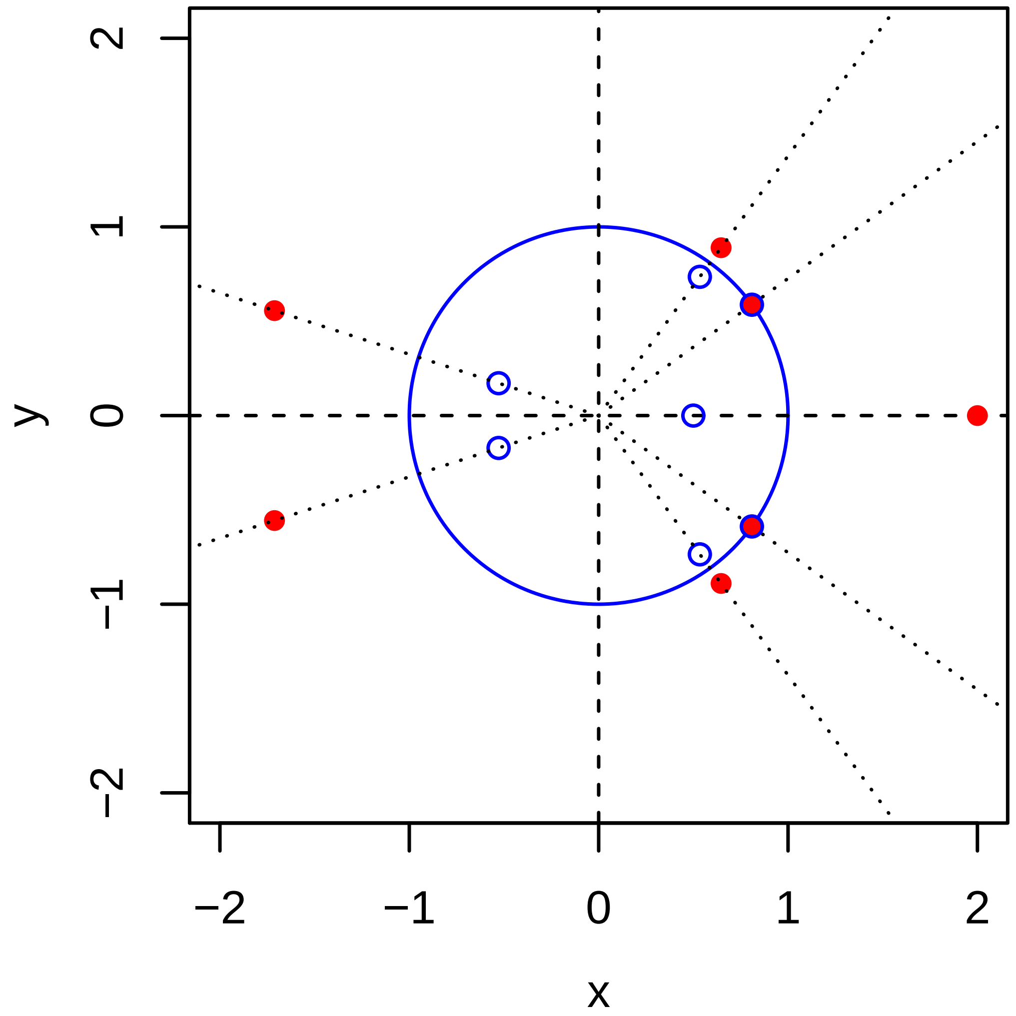
Unit Circle, Root z and Its Reciprocal $1/z$



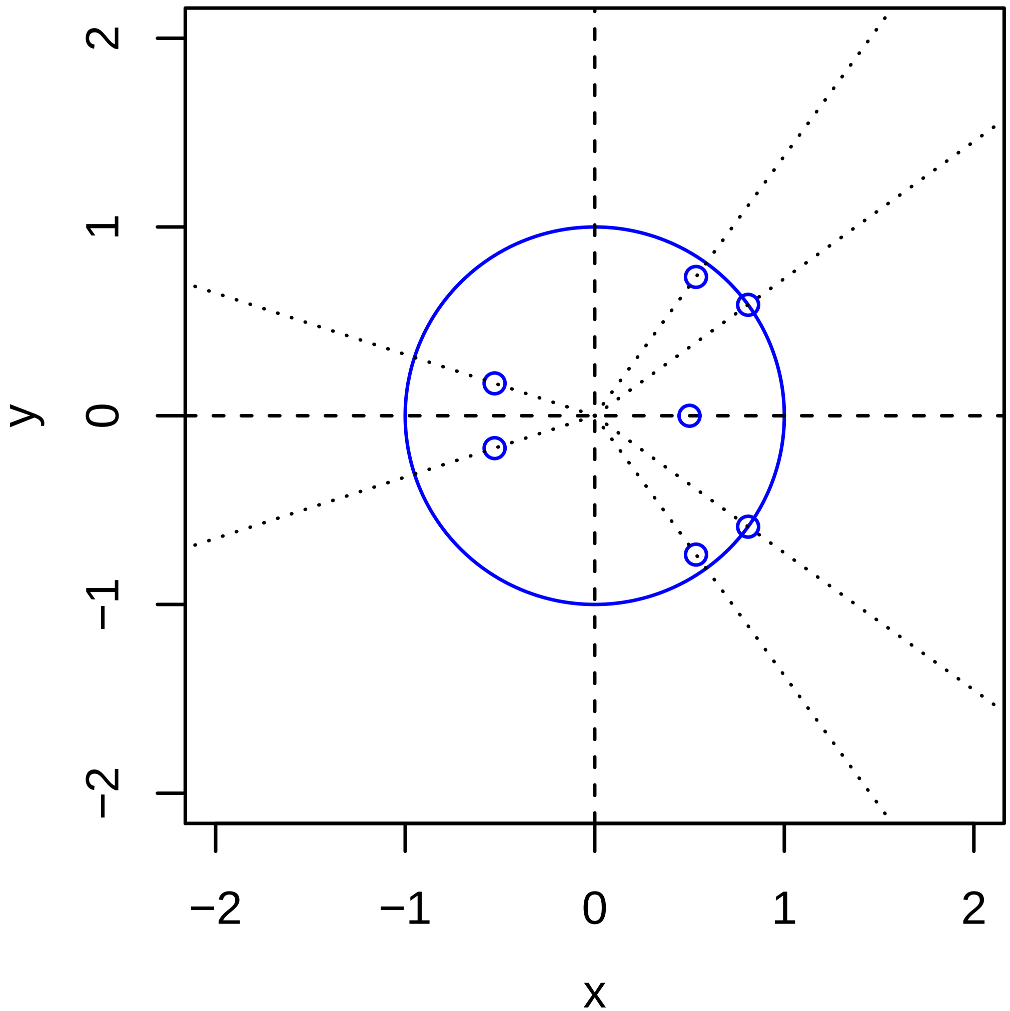
Unit Circle, 2 Conjugate Roots z and Reciprocals $1/z$



Unit Circle, 7 Roots z and Reciprocals $1/z$



7 Reciprocal Roots



ARMA Models: VII

- causality condition on $\phi(z)$ implies filter $\phi(B)$ has an inverse $\phi^{-1}(B)$ such that
$$\phi^{-1}(B)\phi(B) = \phi(B)\phi^{-1}(B) = 1,$$
where the coefficients for $\phi^{-1}(B)$ are absolutely summable
- likewise, invertibility condition on $\theta(z)$ implies filter $\theta(B)$ has an inverse $\theta^{-1}(B)$ such that
$$\theta^{-1}(B)\theta(B) = \theta(B)\theta^{-1}(B) = 1,$$
where the coefficients for $\theta^{-1}(B)$ are absolutely summable
- since $\phi(B)X_t = \theta(B)Z_t$ says that $X_t = \phi^{-1}(B)\theta(B)Z_t$ and since $X_t = \psi(B)Z_t$ also indicates that $\psi(B) = \phi^{-1}(B)\theta(B)$, might seem we would need to know coefficients for $\phi^{-1}(B)$ to figure out those for $\psi(B)$; however, this is not the case, as the following overheads indicate

ARMA Models: VIII

1. definition of ARMA process says $\phi(B)X_t = \theta(B)Z_t$
2. causality of ARMA process says $X_t = \psi(B)Z_t$
3. multiplication of above by $\phi(B)$ says $\phi(B)X_t = \phi(B)\psi(B)Z_t$

- comparison of 3 & 1 says $\phi(B)\psi(B) = \theta(B)$ and hence

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(\psi_0 + \psi_1 B + \cdots) = 1 + \theta_1 B + \cdots + \theta_q B^q \quad (*)$$

- expanding out left-hand side (LHS) of $(*)$ yields

$$\begin{array}{rrrrl} \psi_0 + & \psi_1 B + & \psi_2 B^2 + & \psi_3 B^3 & + \cdots \\ & - \phi_1\psi_0 B & - \phi_1\psi_1 B^2 & - \phi_1\psi_2 B^3 & - \cdots \\ & & - \phi_2\psi_0 B^2 & - \phi_2\psi_1 B^3 & - \cdots \\ & & & - \phi_3\psi_0 B^3 & - \cdots \end{array}$$

ARMA Models: IX

- now take

$$
\begin{aligned}
\psi_0 + \quad \psi_1 B + \quad & \psi_2 B^2 + \quad \psi_3 B^3 + \cdots \\
- \phi_1\psi_0 B - & \phi_1\psi_1 B^2 - \phi_1\psi_2 B^3 - \cdots \\
- & \phi_2\psi_0 B^2 - \phi_2\psi_1 B^3 - \cdots \\
& \qquad\qquad - \phi_3\psi_0 B^3 - \cdots
\end{aligned}
$$

collect together coefficients for $B, B^2, B^3, \ldots$ to get

$$\psi_0 + (\psi_1 - \phi_1\psi_0)B + (\psi_2 - \phi_1\psi_1 - \phi_2\psi_0)B^2 + (\psi_3 - \phi_1\psi_2 - \phi_2\psi_1 - \phi_3\psi_0)B^3 + \cdots$$

and equate with $1 + \theta_1 B + \theta_2 B^2 + \theta_3 B^3 + \cdots$ (RHS of $(*)$):

$$
\begin{aligned}
1 &= \psi_0 \\
\theta_1 &= \psi_1 - \phi_1\psi_0 \\
\theta_2 &= \psi_2 - \phi_1\psi_1 - \phi_2\psi_0 \\
\theta_3 &= \psi_3 - \phi_1\psi_2 - \phi_2\psi_1 - \phi_3\psi_0
\end{aligned}
$$

ARMA Models: X

- rewrite

$$\begin{aligned} 1 &= \psi_0 \\ \theta_1 &= \psi_1 - \phi_1\psi_0 \\ \theta_2 &= \psi_2 - \phi_1\psi_1 - \phi_2\psi_0 \\ \theta_3 &= \psi_3 - \phi_1\psi_2 - \phi_2\psi_1 - \phi_3\psi_0 \\ &\vdots \end{aligned}$$

as

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 &= \phi_1\psi_0 + \theta_1 \\ \psi_2 &= \phi_1\psi_1 + \phi_2\psi_0 + \theta_2 \\ \psi_3 &= \phi_1\psi_2 + \phi_2\psi_1 + \phi_3\psi_0 + \theta_3 \\ &\vdots \end{aligned}$$

ARMA Models: XI

- stare at

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \phi_1\psi_0 + \theta_1 \\
\psi_2 &= \phi_1\psi_1 + \phi_2\psi_0 + \theta_2 \\
\psi_3 &= \phi_1\psi_2 + \phi_2\psi_1 + \phi_3\psi_0 + \theta_3 \\
&\vdots
\end{aligned}
$$

to see recursive scheme for computing ψ_j's:

$$\psi_j = \sum_{k=1}^{p} \phi_k \psi_{j-k} + \theta_j, \quad j = 0, 1, 2, \ldots,$$

for which we need to define $\theta_0 = 1$, $\theta_j = 0$ for $j > q$ and $\psi_j = 0$ for $j < 0$ (also take $\sum_{k=1}^{p} \phi_k\psi_{j-k}$ to be 0 if $p = 0$)

ARMA Models: XII

- now start with

1. definition of ARMA process: $\theta(B)Z_t = \phi(B)X_t$
2. invertibility of ARMA process: $Z_t = \pi(B)X_t$
3. multiplication of above by $\theta(B)$: $\theta(B)Z_t = \theta(B)\pi(B)X_t$

- comparison of 3 & 1 says $\theta(B)\pi(B) = \phi(B)$ and hence

$$(1 + \theta_1 B + \cdots + \theta_q B^q)(\pi_0 + \pi_1 B + \cdots) = 1 - \phi_1 B - \cdots - \phi_p B^p$$

- same argument as before (with ϕ_k replaced by $-\theta_k$ and with θ_j replaced by $-\phi_j$) leads to scheme for computing π_j's:

$$\pi_j = -\sum_{k=1}^{q} \theta_k \pi_{j-k} - \phi_j, \quad j = 0, 1, 2, \ldots,$$

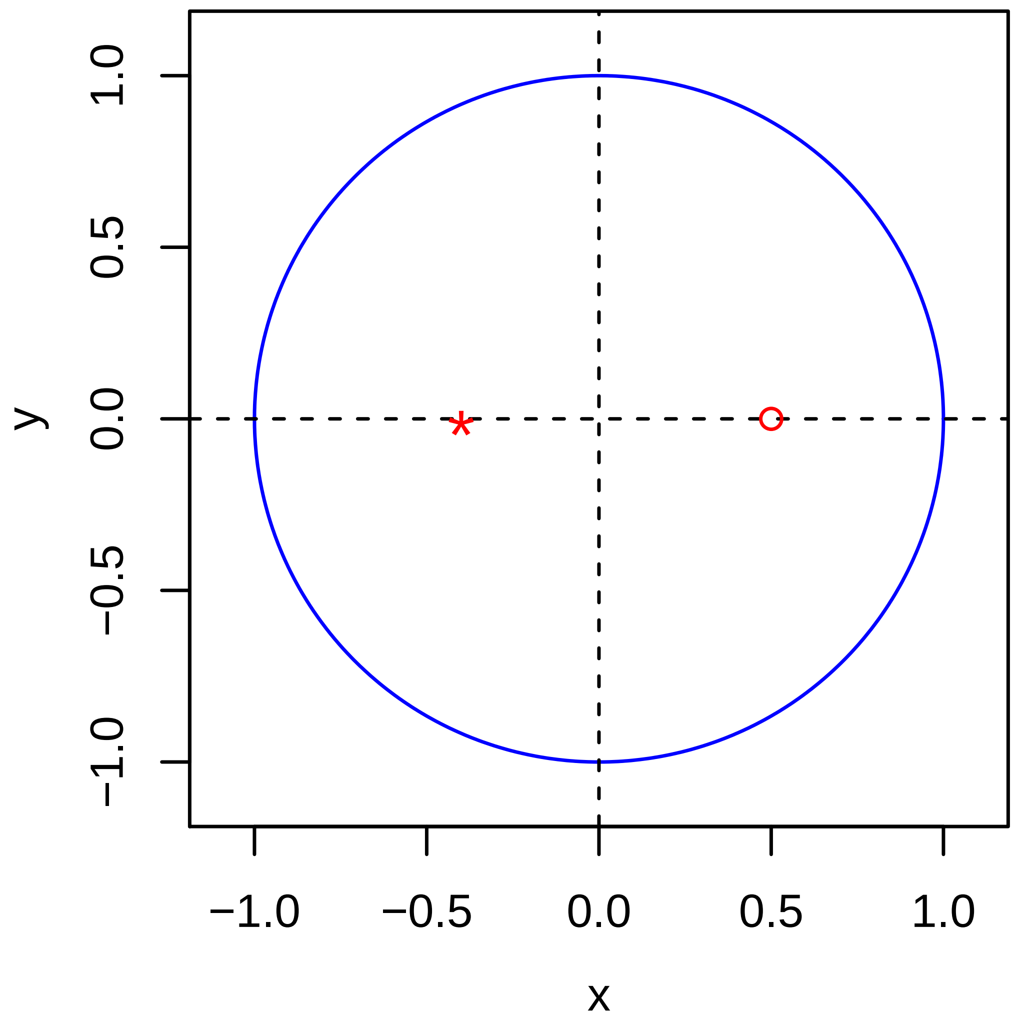
where $\phi_0 \stackrel{\text{def}}{=} -1$, $\phi_j \stackrel{\text{def}}{=} 0$ for $j > p$ and $\pi_j \stackrel{\text{def}}{=} 0$ for $j < 0$

Example – ARMA(1,1) Process: I

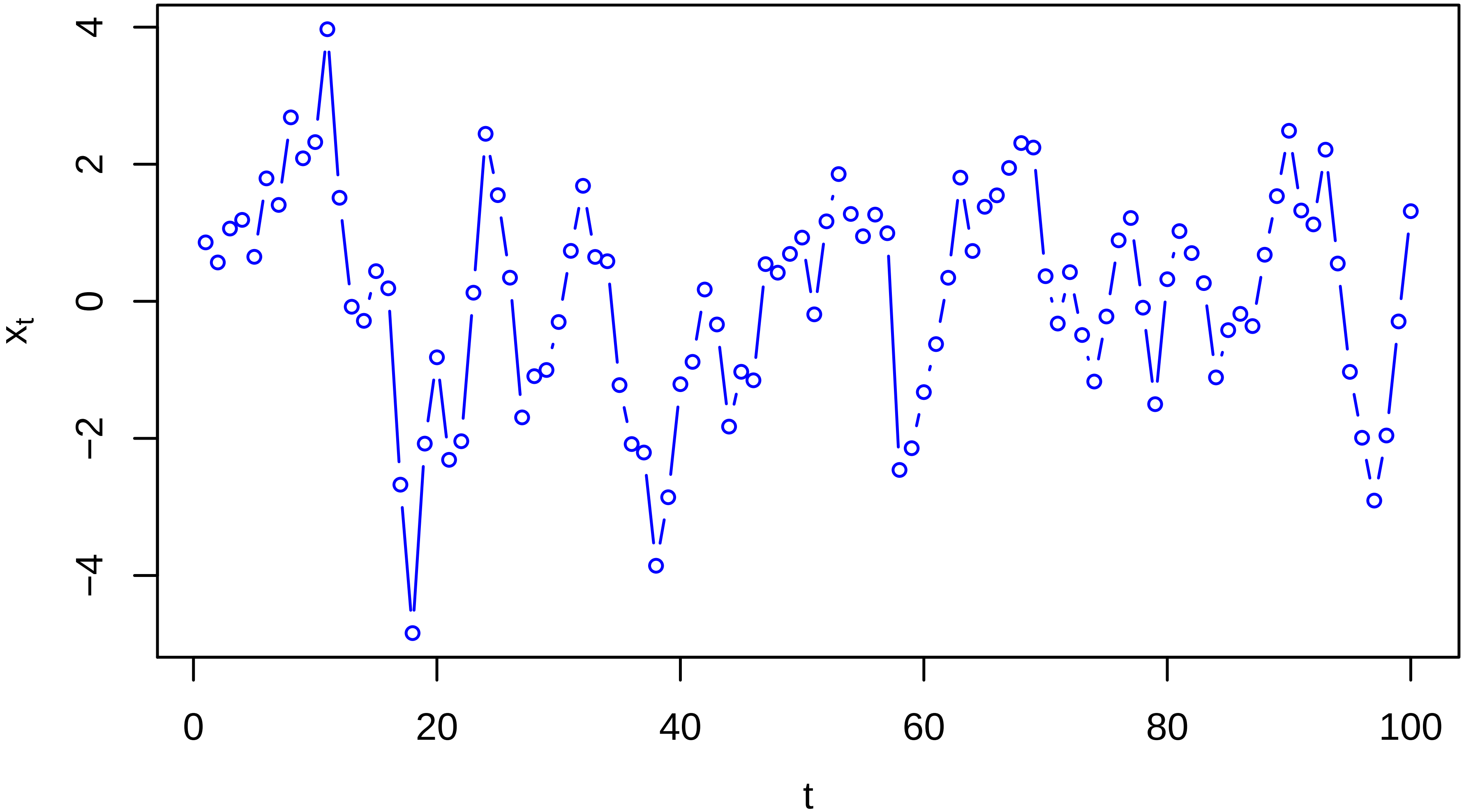
- note: already considered in overheads VII–21 to VII–29
- process takes the form $X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$
- here $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$
- roots of $\phi(z) = 0$ and $\theta(z) = 0$ are $1/\phi$ and $-1/\theta$
- causal (and hence stationary) and invertible if $|1/\phi| > 1$ and $|-1/\theta| > 1$, i.e., $|\phi| < 1$ and $|\theta| < 1$ (easily checked!)
- have already noted $\psi_0 = 1$ and $\psi_j = (\phi + \theta)\phi^{j-1}$ for $j \geq 1$
- also have $\pi_0 = 1$ and $\pi_j = -(\phi + \theta)(-\theta)^{j-1}$ for $j \geq 1$
- next overheads show (1) plot of reciprocal roots and (2) one realization for specific ARMA(1,1) model

$$X_t - 0.5X_{t-1} = Z_t + 0.4Z_{t-1}, \quad \{Z_t\} \sim \text{Gaussian WN}(0, 1)$$

Reciprocal Roots Plot (○ for AR and ∗ for MA)



Realization of ARMA(1,1) Process



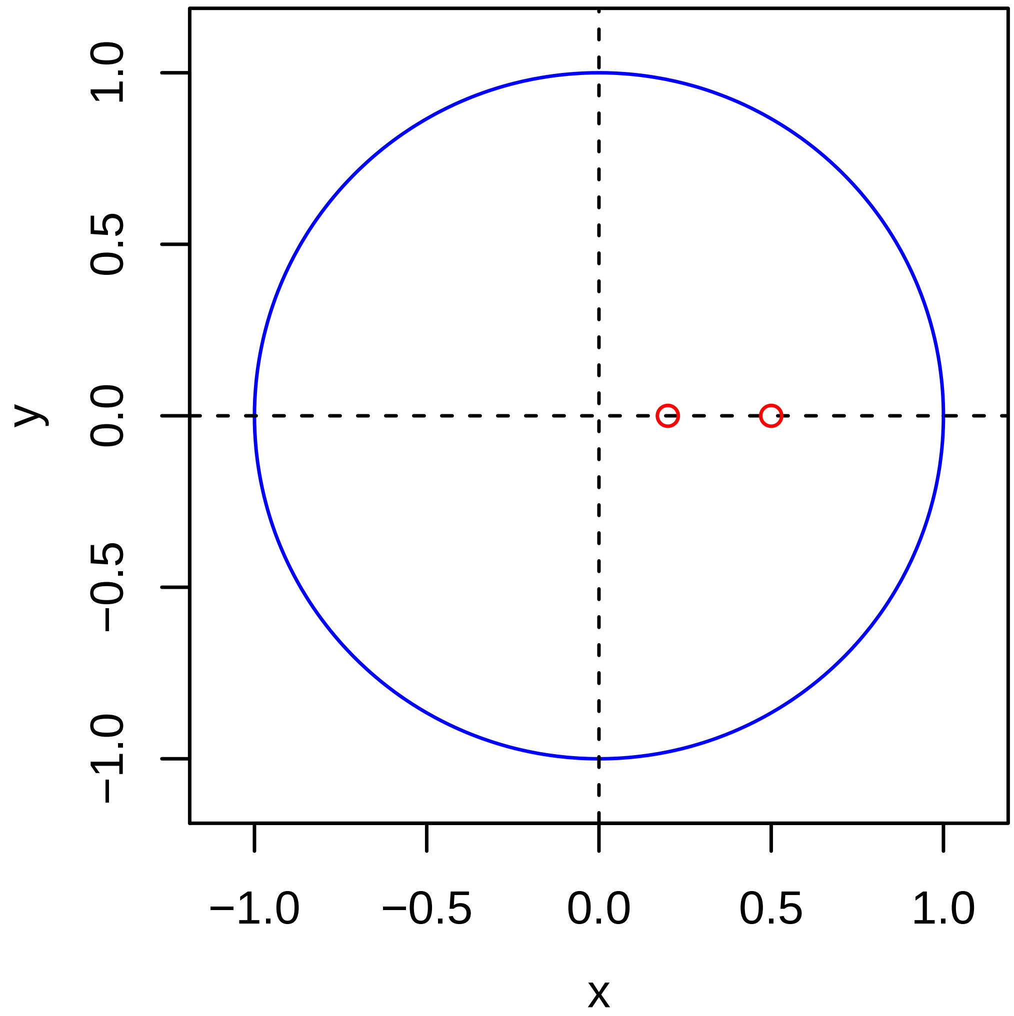
Example – B&D's AR(2) Process: I

- AR(2) process takes form $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$
- invertibility trivially true: $Z_t = X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}$
- here $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ (note: $\pi_1 = -\phi_1$ and $\pi_2 = -\phi_2$)
- need to find roots z_1 and z_2 to see if $\{X_t\}$ is causal
- B&D consider $X_t = 0.7X_{t-1} - 0.1X_{t-2} + Z_t$, for which

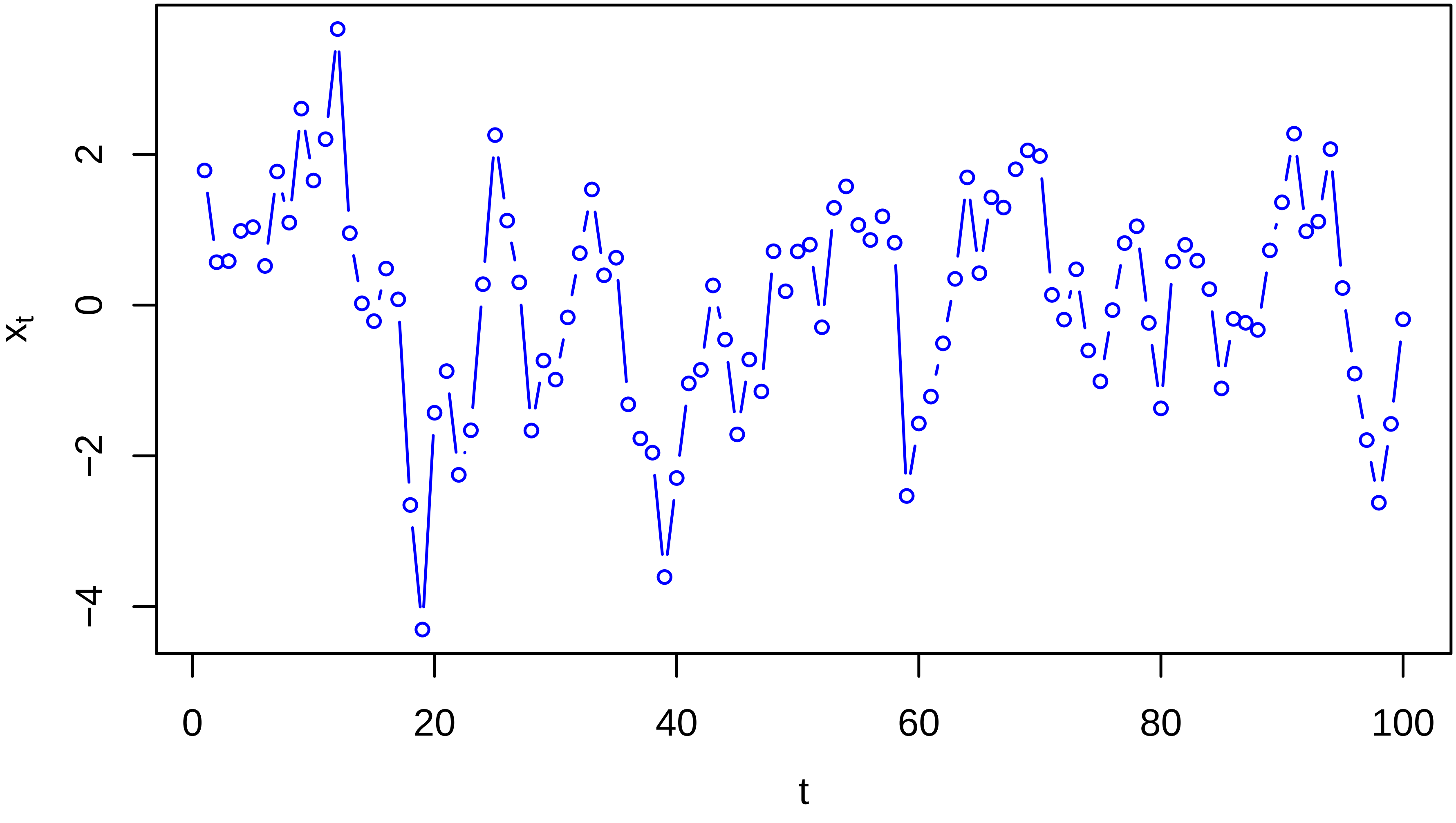
$$\phi(z) = 1 - 0.7z + 0.1z^2 = (1 - 0.5z)(1 - 0.2z)$$

- roots are thus $z_1 = 2$ and $z_2 = 5$
- both $|z_1|$ and $|z_2|$ are outside the unit circle
- process is thus causal (and hence stationary)
- next overheads show plots of reciprocal roots and one realization, for which $\{Z_t\} \sim$ Gaussian WN$(0, 1)$

Reciprocal Roots Plot



Realization of B&D's AR(2) Process



Example – B&D's AR(2) Process: II

- for AR(2) processes, recursive scheme for computing ψ_j's, namely,

$$\psi_j = \sum_{k=1}^{2} \phi_k \psi_{j-k} + \theta_j, \quad j = 0, 1, 2, \ldots,$$

  leads to $\psi_0 = 1$, $\psi_1 = \phi_1$ & $(*)$ $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}$, $j \geq 2$
- theory of homogeneous linear difference equations says that, if roots z_1 and z_2 are distinct, have

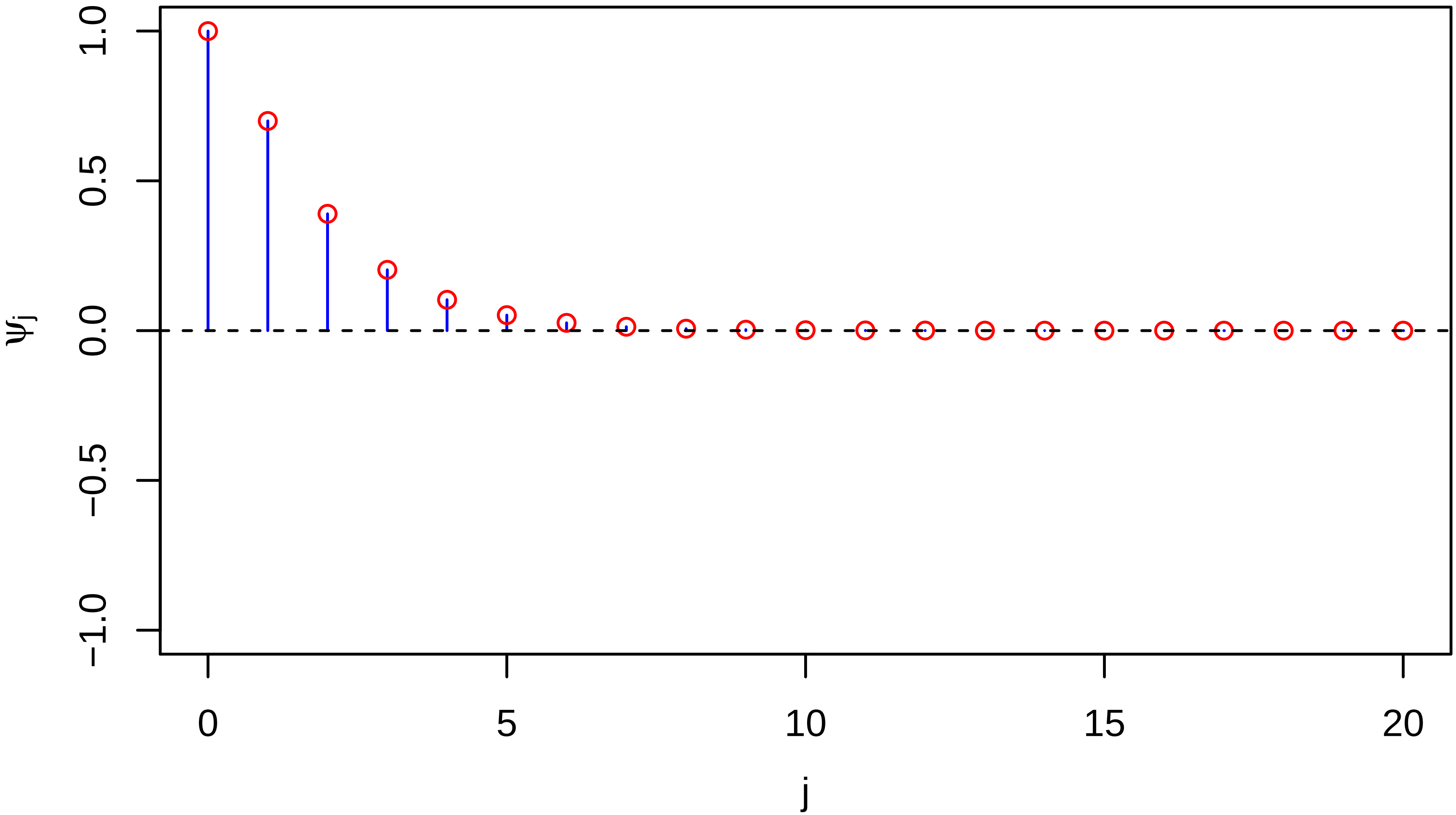
$$\psi_j = \alpha_1 z_1^{-j} + \alpha_2 z_2^{-j}, \quad j \geq 2$$

- since $(*)$ says $\psi_2 = \phi_1^2 + \phi_2$ and $\psi_3 = \phi_1^3 + 2\phi_1\phi_2$, can solve for α_l's using

$$\psi_2 = \alpha_1 z_1^{-2} + \alpha_2 z_2^{-2} \text{ and } \psi_3 = \alpha_1 z_1^{-3} + \alpha_2 z_2^{-3}$$

- for B&D AR(2) process, get $\psi_j = \frac{5}{3} \cdot 2^{-j} - \frac{2}{3} \cdot 5^{-j}$, $j \geq 2$

ψ_j's for B&D's AR(2) Process



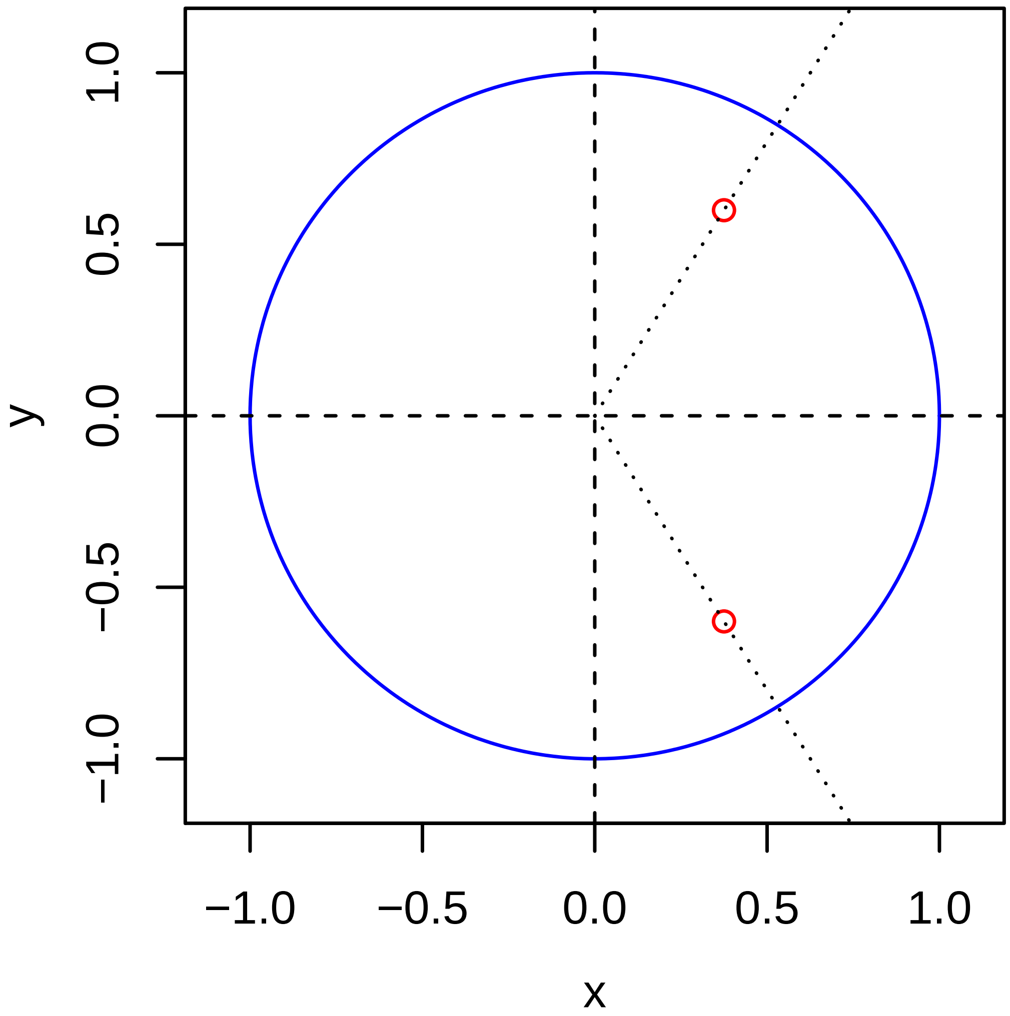
Example – Second AR(2) Process: I

- now consider $X_t = 0.75X_{t-1} - 0.5X_{t-2} + Z_t$, for which

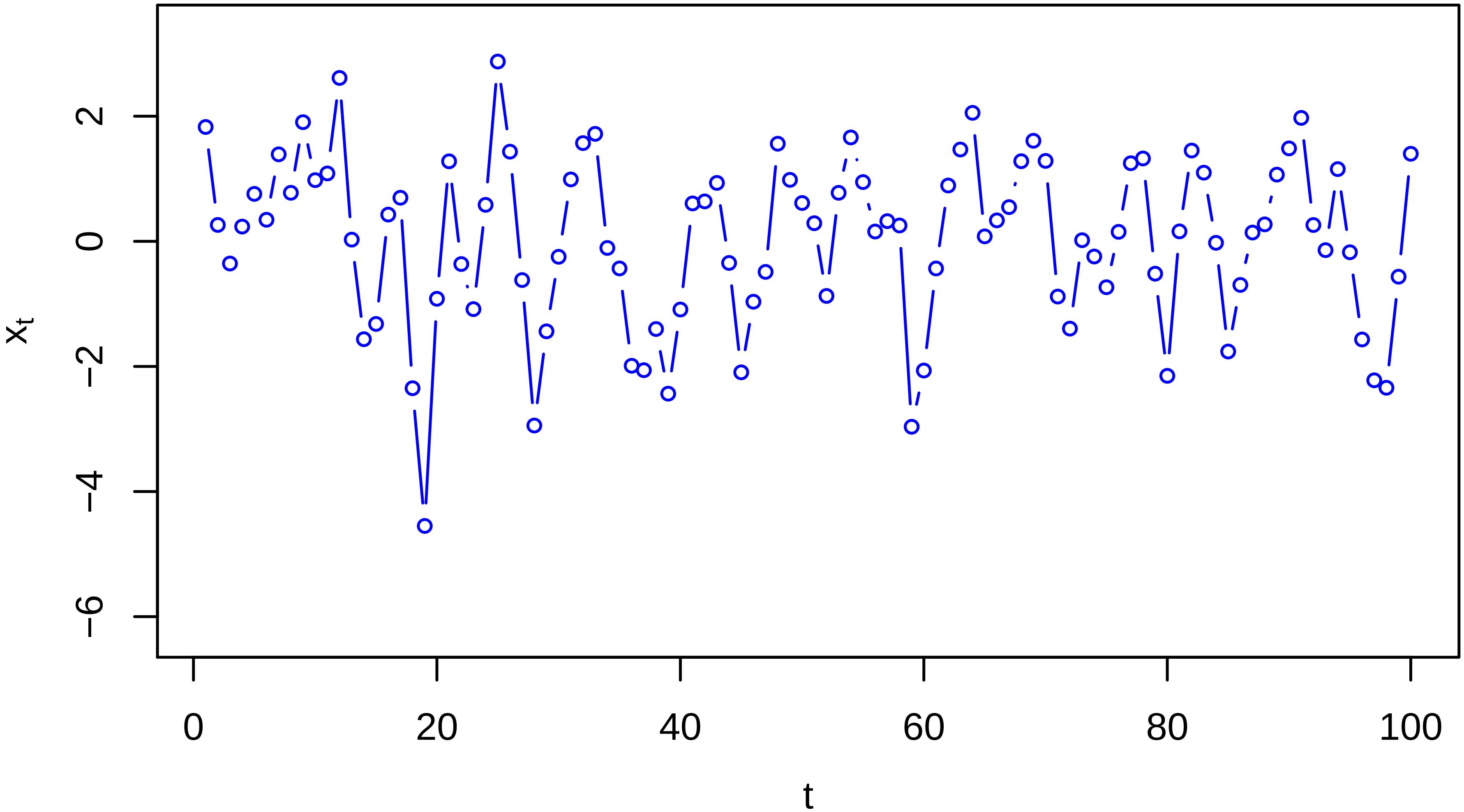
$$\phi(z) = 1 - 0.75z + 0.5z^2 = \left(1 - \frac{z}{\frac{3}{4} + \frac{\sqrt{23}}{4}i}\right)\left(1 - \frac{z}{\frac{3}{4} - \frac{\sqrt{23}}{4}i}\right)$$

- roots are $\frac{3}{4} \pm \frac{\sqrt{23}}{4}i$ (complex conjugates) – denote as z_1 & z_1^*
- here $|z_1| = |z_1^*| = \sqrt{2}$, so roots are outside the unit circle
- process is thus causal (and hence stationary)
- next overheads show plots of reciprocal roots and one realization, for which $\{Z_t\} \sim$ Gaussian WN$(0,1)$

Reciprocal Roots Plot



Realization of Second AR(2) Process



Example – Second AR(2) Process: II

- as before, $\psi_0 = 1$ & $\psi_1 = \phi_1$, but now ψ_j's for $j \geq 2$ satisfy

$$\begin{aligned}\psi_j &= \alpha z_1^{-j} + \alpha^*(z_1^*)^{-j} \quad \text{for some yet-to-be-determined } \alpha \\ &= \alpha|z_1|^{-j}e^{-i\omega j} + \alpha^*|z_1|^{-j}e^{i\omega j} \quad \text{taking } z_1 = |z_1|e^{i\omega} \\ &= \alpha|z_1|^{-j}e^{-i\omega j} + \left(\alpha|z_1|^{-j}e^{-i\omega j}\right)^* \\ &= 2\Re\left\{\alpha|z_1|^{-j}e^{-i\omega j}\right\}, \quad \text{where } \Re\{z\} \text{ is real part of } z\end{aligned}$$

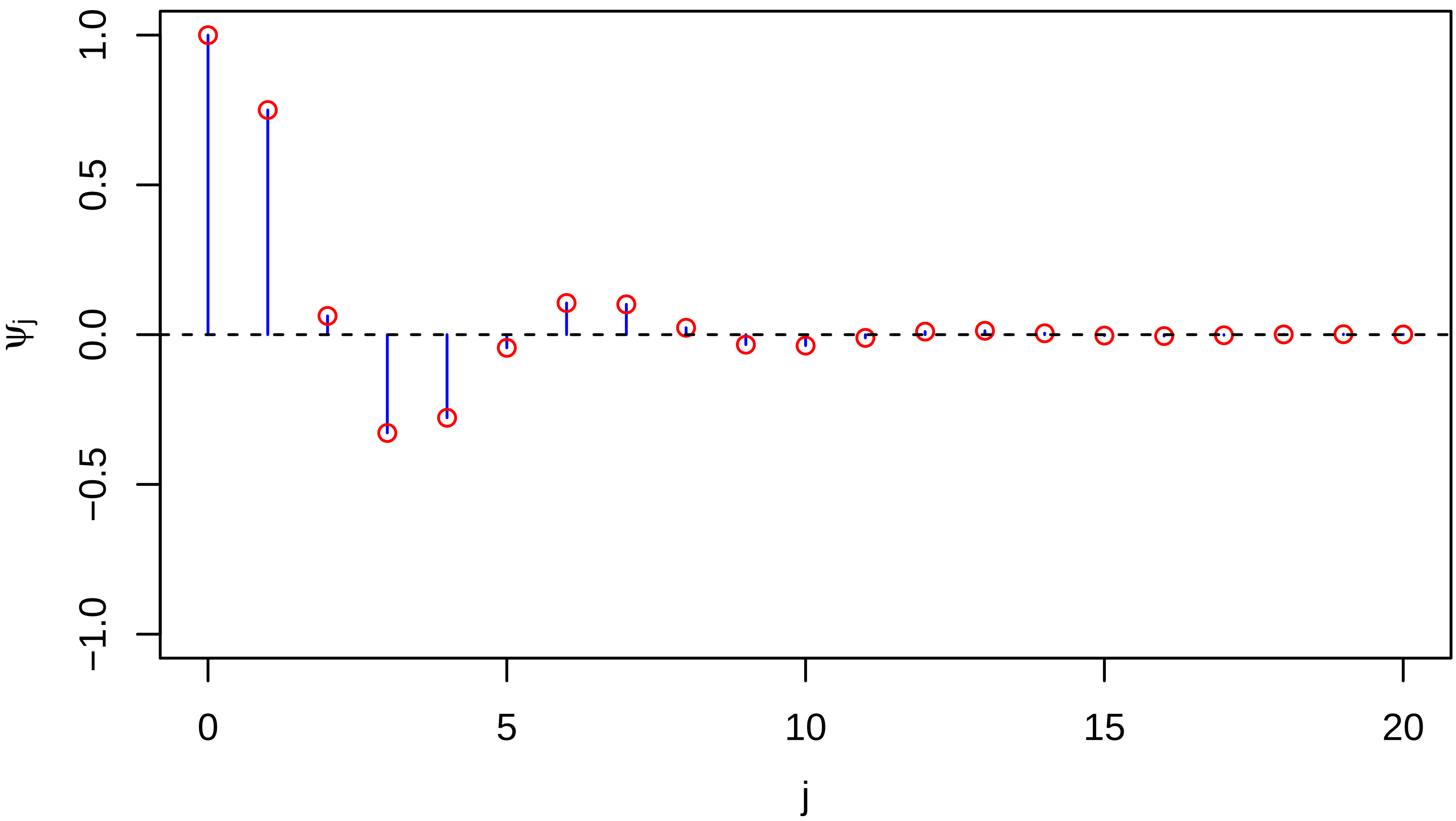
- writing $\alpha = x + iy$ and recalling $e^{-iu} = \cos(u) - i\sin(u)$, have

$$\psi_j = 2\Re\{(x+iy)|z_1|^{-j}e^{-i\omega j}\} = 2[x\cos(\omega j)+y\sin(\omega j)]|z_1|^{-j}, \quad \text{yielding}$$

$$\psi_2 = 2[x\cos(2\omega)+y\sin(2\omega)]|z_1|^{-2} \;\; \& \;\; \psi_3 = 2[x\cos(3\omega)+y\sin(3\omega)]|z_1|^{-3}$$

- as before, can use $\psi_2 = \phi_1^2 + \phi_2$ and $\psi_3 = \phi_1^3 + 2\phi_1\phi_2$, yielding two equations to solve to get $x = 0.5$ & $y \doteq 0.313$

ψ_j's for Second AR(2) Process



Example – AR(4) Process

- now consider AR(4) process

$$X_t = 2.7607X_{t-1} - 3.8106X_{t-2} + 2.6535X_{t-3} - 0.9238X_{t-4} + Z_t,$$

where $\{Z_t\} \sim$ Gaussian WN$(0, 1)$

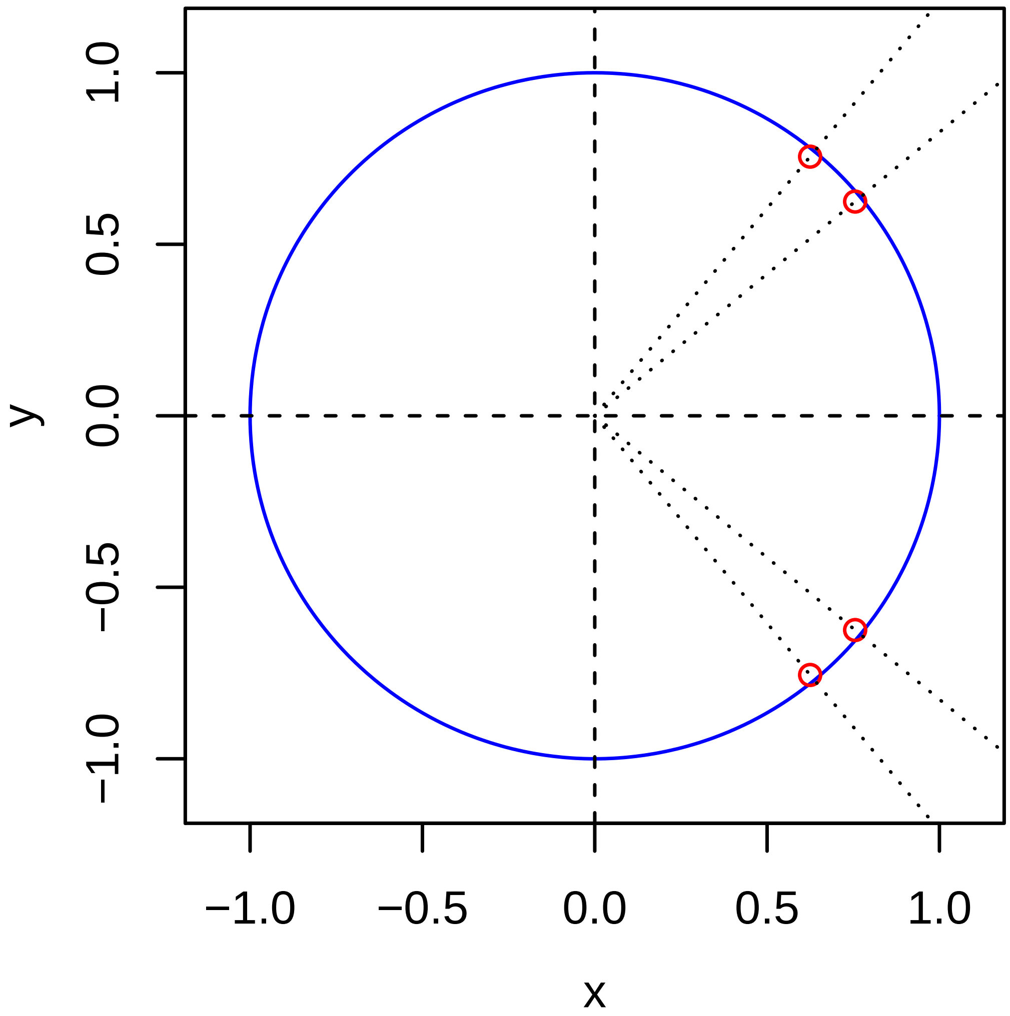
- thus $\phi(z) = 1 - 2.7607z + 3.8106z^2 - 2.6535z^3 + 0.9238z^4$
- `polyroot` function in `R` calculates roots as

$$0.650 \pm 0.786i \text{ and } 0.786 \pm 0.650i,$$

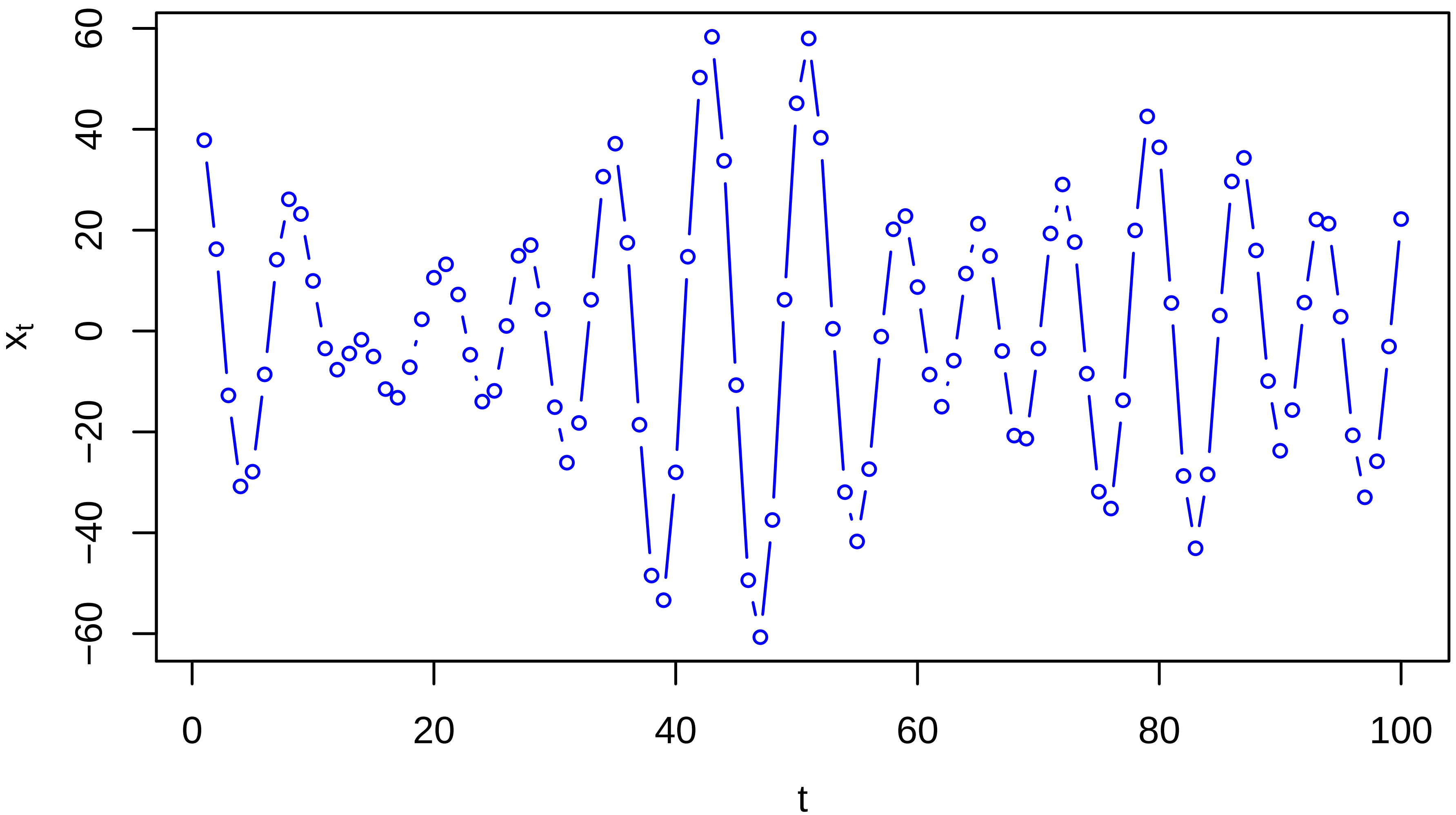
with corresponding magnitudes 1.0199 and 1.0201

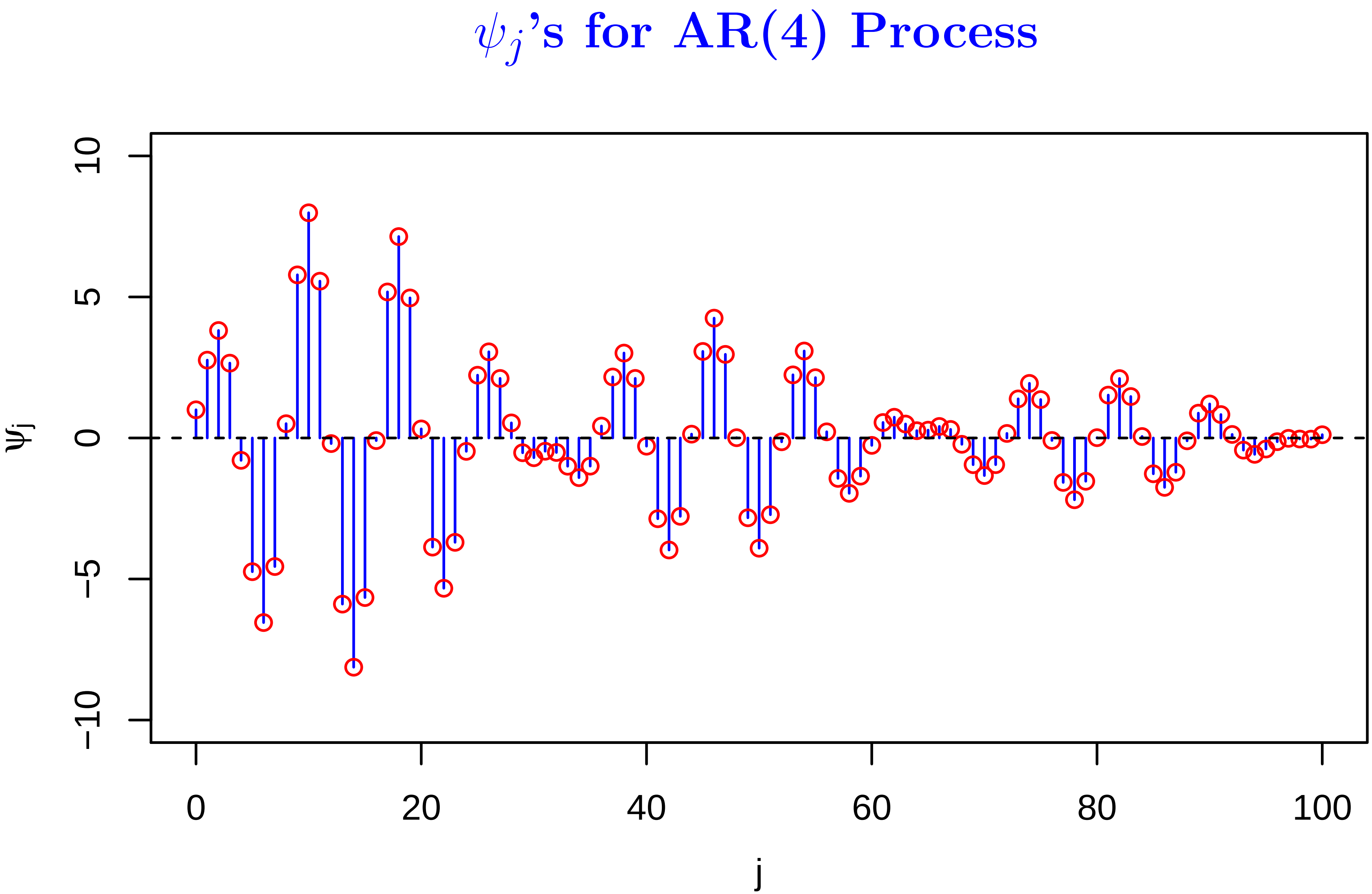
- thus $\{X_t\}$ is causal (and hence stationary)
- getting closed form expression for ψ_j's is tedious, so opt to just compute them using recursive scheme

Reciprocal Roots Plot



Realization of AR(4) Process

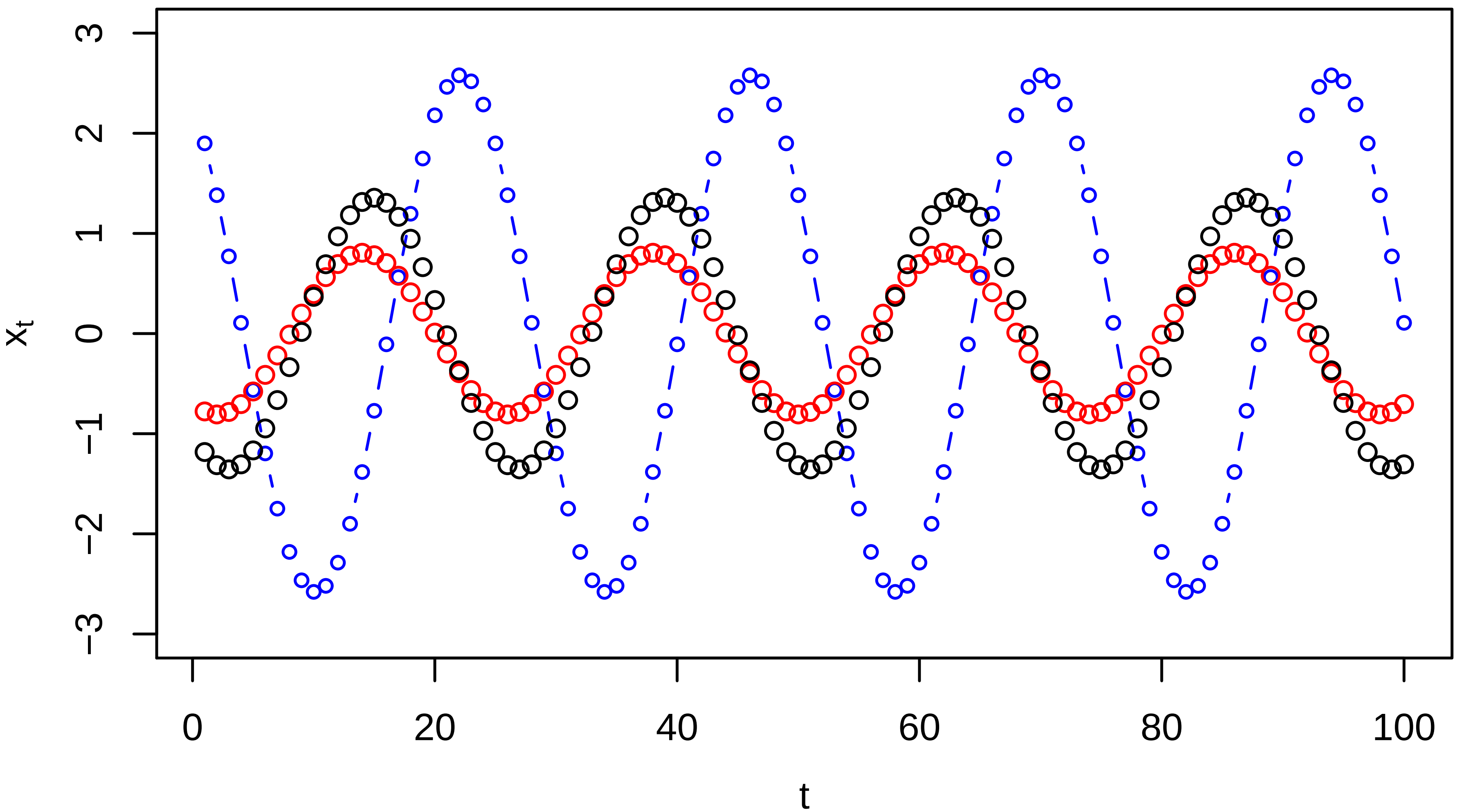



ψ_j's for AR(4) Process
ψ_j
10
5
0
−5
−10
0
20
40
60
80
100
j

Aside – Harmonic Processes: I

- reconsider stationary process of Problem 2(b):
$$X_t = Z_2 \cos(\omega t) + Z_1 \sin(\omega t),$$
where Z_2 and Z_1 are independent $\mathcal{N}(0, 1)$ RVs
- above is an example of a *harmonic process*
- realizations of harmonic processes are qualitatively very different from those for ARMA processes (see next overhead)
- exercise: given X_1 and X_2, can write
$$X_t = 2\,\cos(\omega) X_{t-1} - X_{t-2}, \quad t \in \mathbb{Z}$$
- above resembles AR(2) process
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t$$
if we set $\phi_1 = 2\cos(\omega)$, $\phi_2 = -1$ and $Z_t = 0$ (can achieve by stipulating $\{Z_t\} \sim \mathrm{WN}(0, 0)$)

Three Realizations of Harmonic Process ($\omega = \pi/12$)



Aside – Harmonic Processes: II

- since $X_t = 2\cos(\omega)X_{t-1} - X_{t-2}$, can *perfectly* predict X_t given X_{t-1} & X_{t-2}
  - note: $\{X_t\}$ is example of a *deterministic* stationary process
- regarding $\{X_t\}$ as an AR(2) process with $\{Z_t\} \sim \mathrm{WN}(0,0)$, have

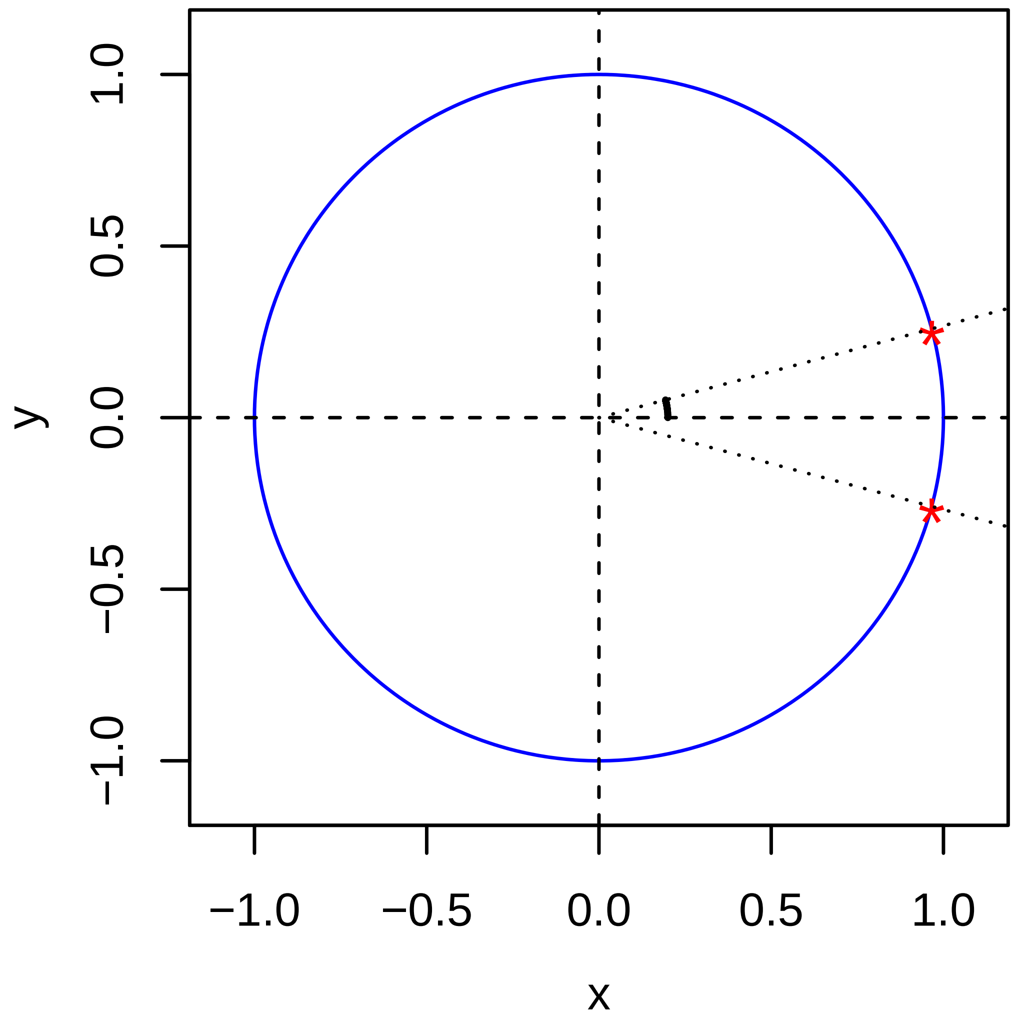
$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 1 - 2\cos(\omega)z + z^2,$$

  which has roots $e^{\pm i\omega}$ since

$$\begin{aligned}\phi(e^{i\omega}) &= 1 - 2\cos(\omega)e^{i\omega} + e^{i2\omega} \\ &= 1 - (e^{i\omega} + e^{-i\omega})e^{i\omega} + e^{i2\omega} = 0,\end{aligned}$$

  where we have made use of $2\cos(\omega) = e^{i\omega} + e^{-i\omega}$
- since $|e^{\pm i\omega}|^2 = \cos^2(\omega) + \sin^2(\omega) = 1$, roots are *on* unit circle
- reconsider example $\omega = \pi/12$, which has period $\frac{2\pi}{\omega} = 24$

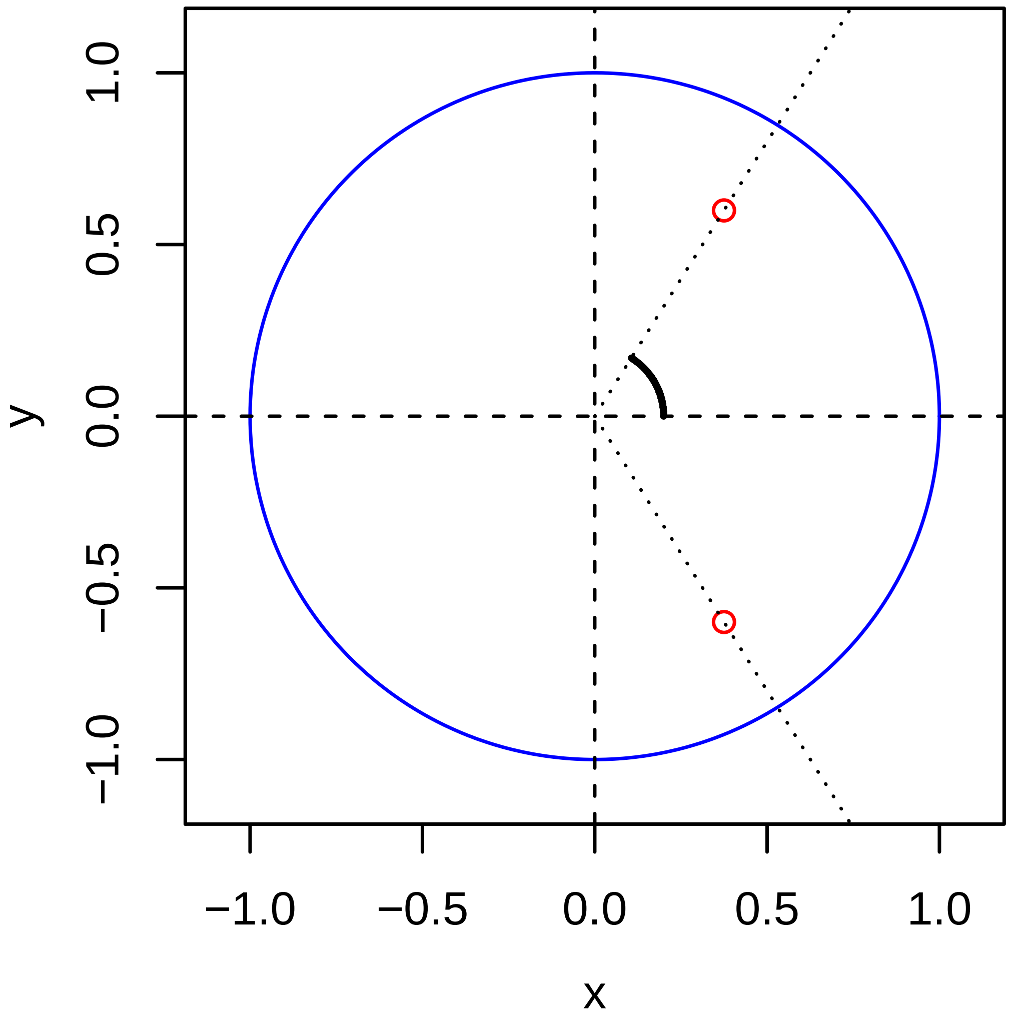
Roots Plot for Harmonic Process



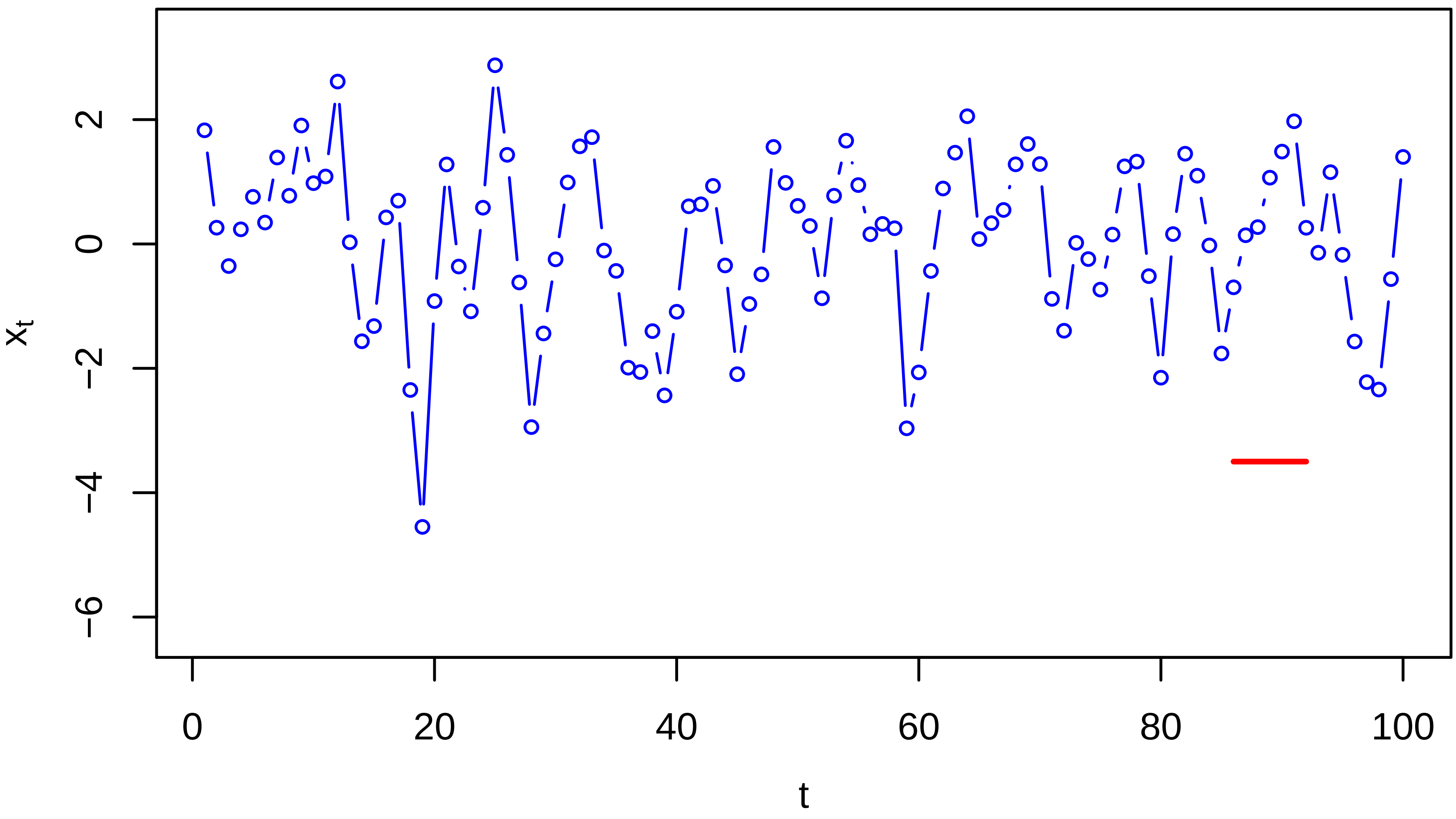
Second AR(2) Process Reconsidered

- reconsider $X_t = 0.75X_{t-1} - 0.5X_{t-2} + Z_t$, for which roots of
$$\phi(z) = 1 - 0.75z + 0.5z^2$$
are complex conjugates $\frac{3}{4} \pm \frac{\sqrt{23}}{4}i$
- denoting these roots as z_1 & z_1^*, have $|z_1| = |z_1^*| = \sqrt{2}$
- can reexpress roots as $\sqrt{2}e^{\pm i\omega}$, where $\omega \doteq 1.01$ radians $(58.0°)$
- realizations will tend to fluctuate roughly with period $\frac{2\pi}{\omega} \doteq 6.2$
- next overheads revisit plots of reciprocal roots and realization

Reciprocal Roots Plot



Realization of Second AR(2) Process



AR(4) Process Reconsidered: I

- reconsider AR(4) process

$$X_t = 2.7607X_{t-1} - 3.8106X_{t-2} + 2.6535X_{t-3} - 0.9238X_{t-4} + Z_t,$$

  for which roots of

$$\phi(z) = 1 - 2.7607z + 3.8106z^2 - 2.6535z^3 + 0.9238z^4$$

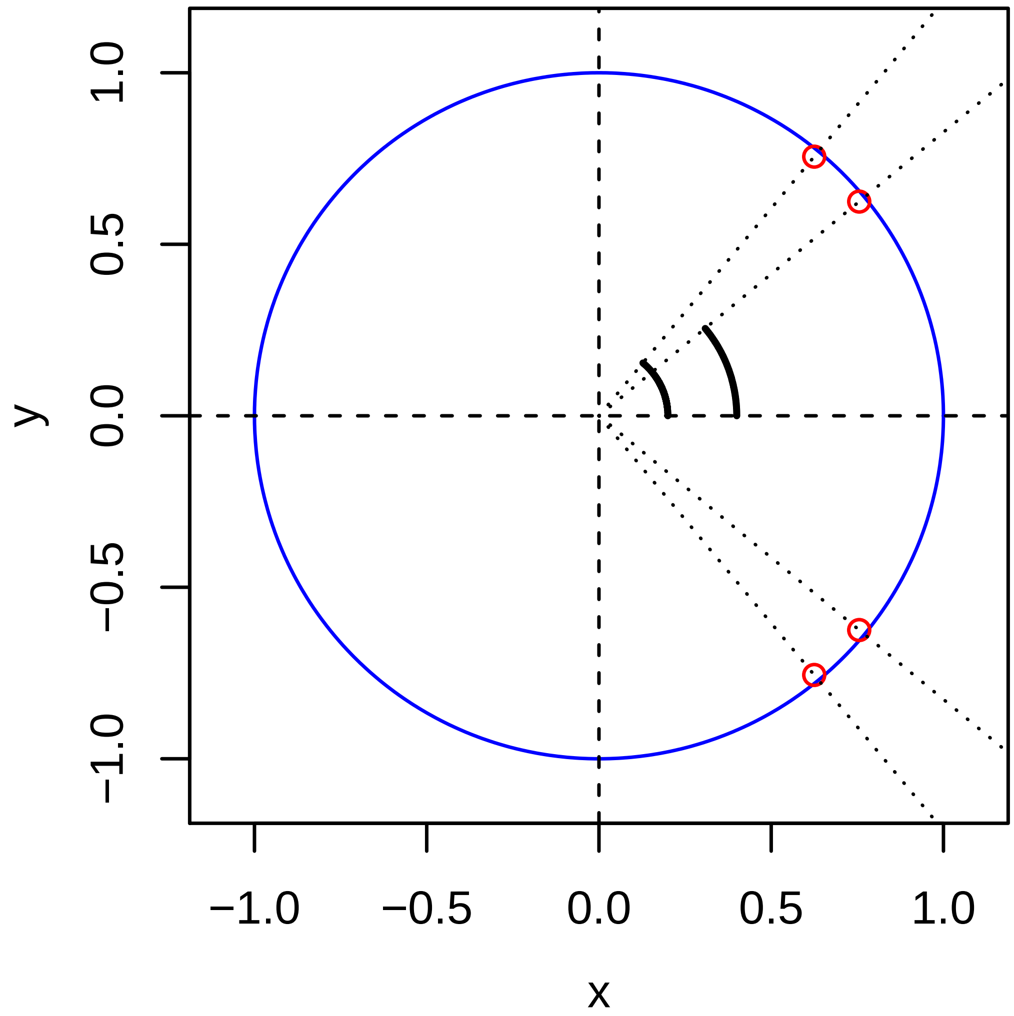
  are $z_1 \doteq 0.650 + 0.786i$, $z_2 \doteq 0.786 + 0.650i$ and their complex conjugates z_1^* and z_2^*

- can reexpress z_1 and z_2 as $|z_1|e^{i\omega_1}$ and $|z_2|e^{i\omega_2}$, where $\omega_1 \doteq 0.88$ radians (50.4°) and $\omega_2 \doteq 0.69$ radians (39.6°)
- realizations will tend to fluctuate roughly as a linear combination of sinusoids with periods of $\frac{2\pi}{\omega_1} \doteq 7.1$ and $\frac{2\pi}{\omega_2} \doteq 9.1$

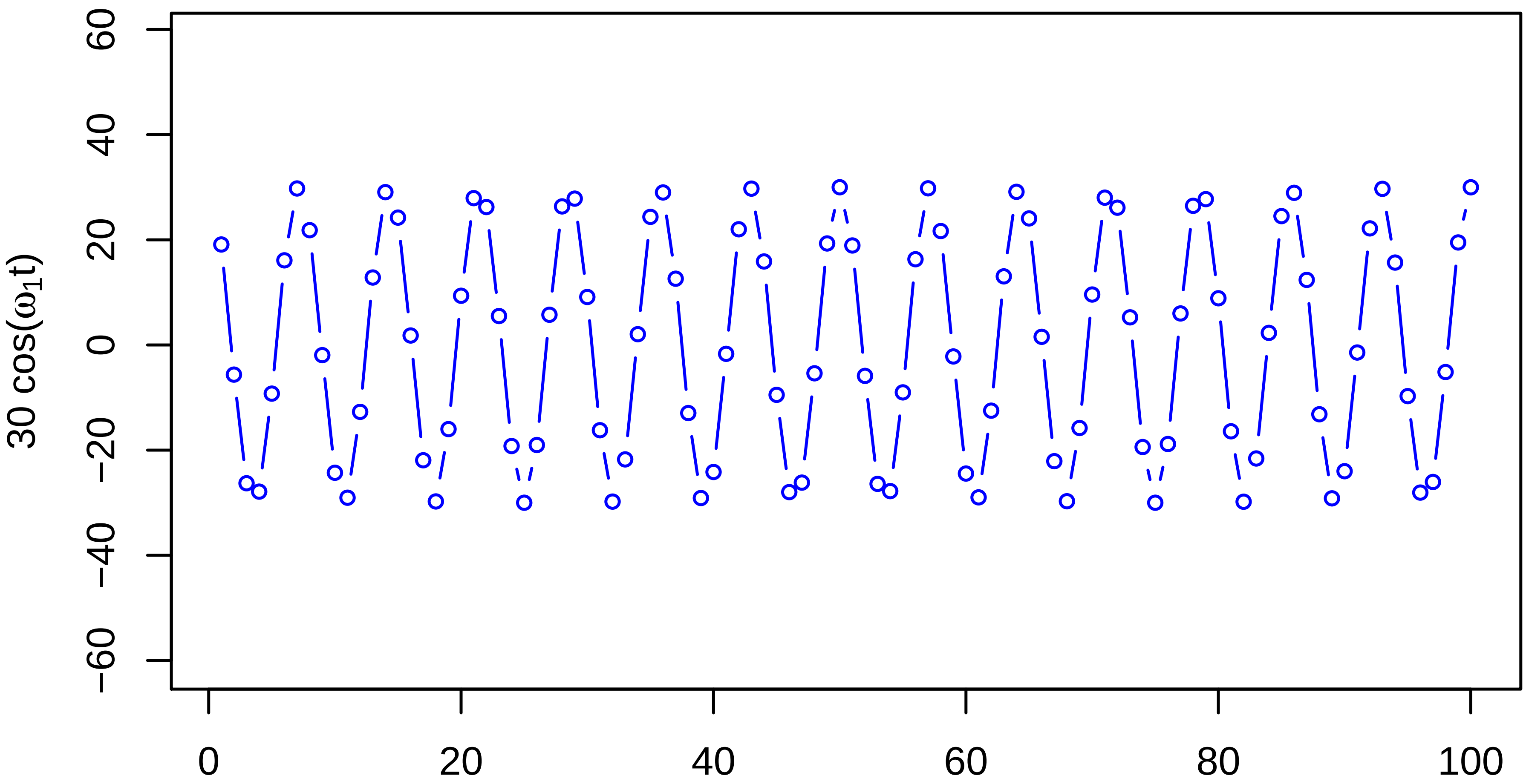
AR(4) Process Reconsidered: II

- next five overheads show
  - reciprocal roots plot with ω_1 and ω_2 indicated by arcs
  - $30\cos(\omega_1 t)$ versus $t = 1, 2, \ldots, 100$
  - $30\cos(\omega_2 t)$ versus $t = 1, 2, \ldots, 100$
  - $30\cos(\omega_1 t) + 30\cos(\omega_2 t)$ versus $t = 1, 2, \ldots, 100$
  - realization of AR(4) process

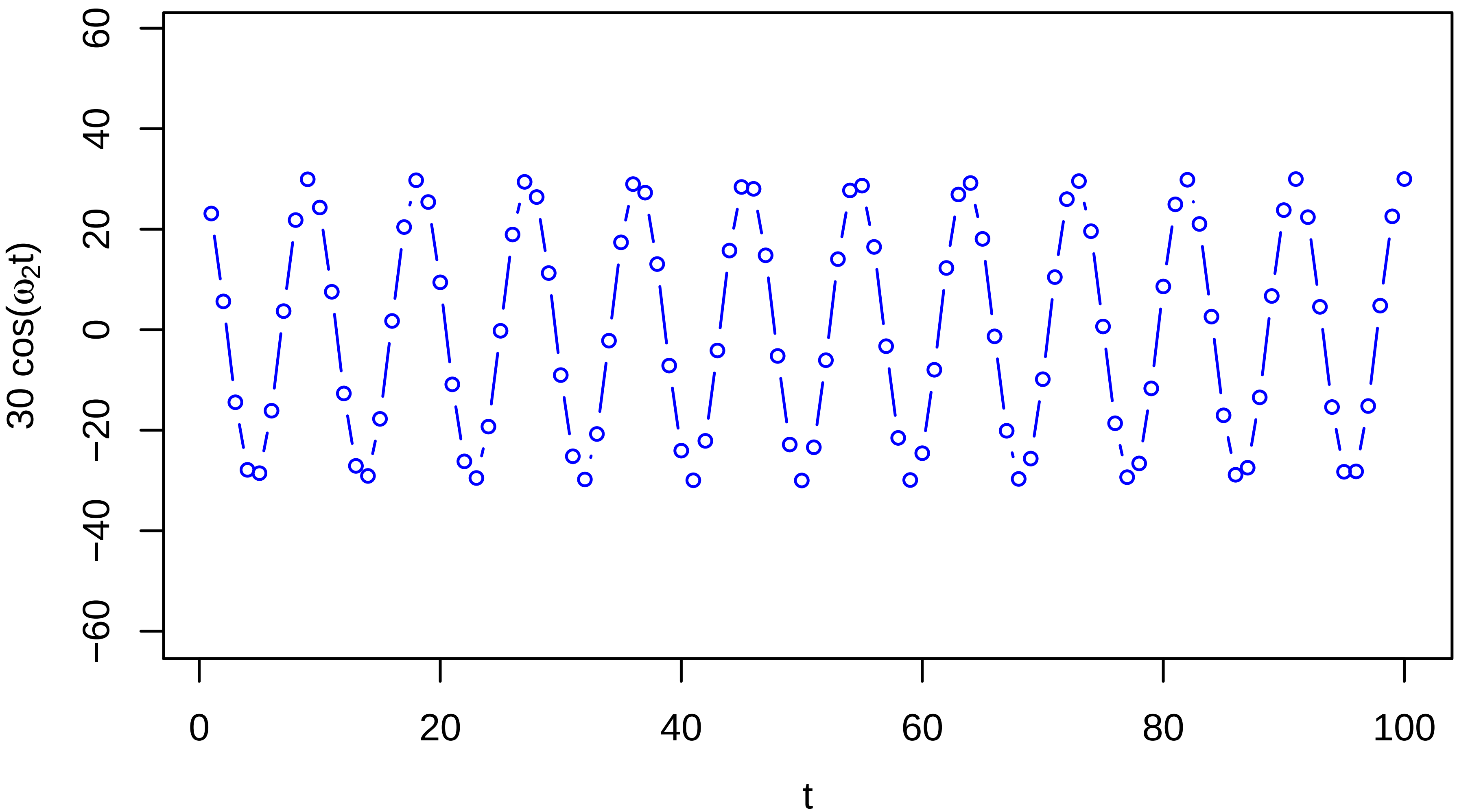
Reciprocal Roots Plot



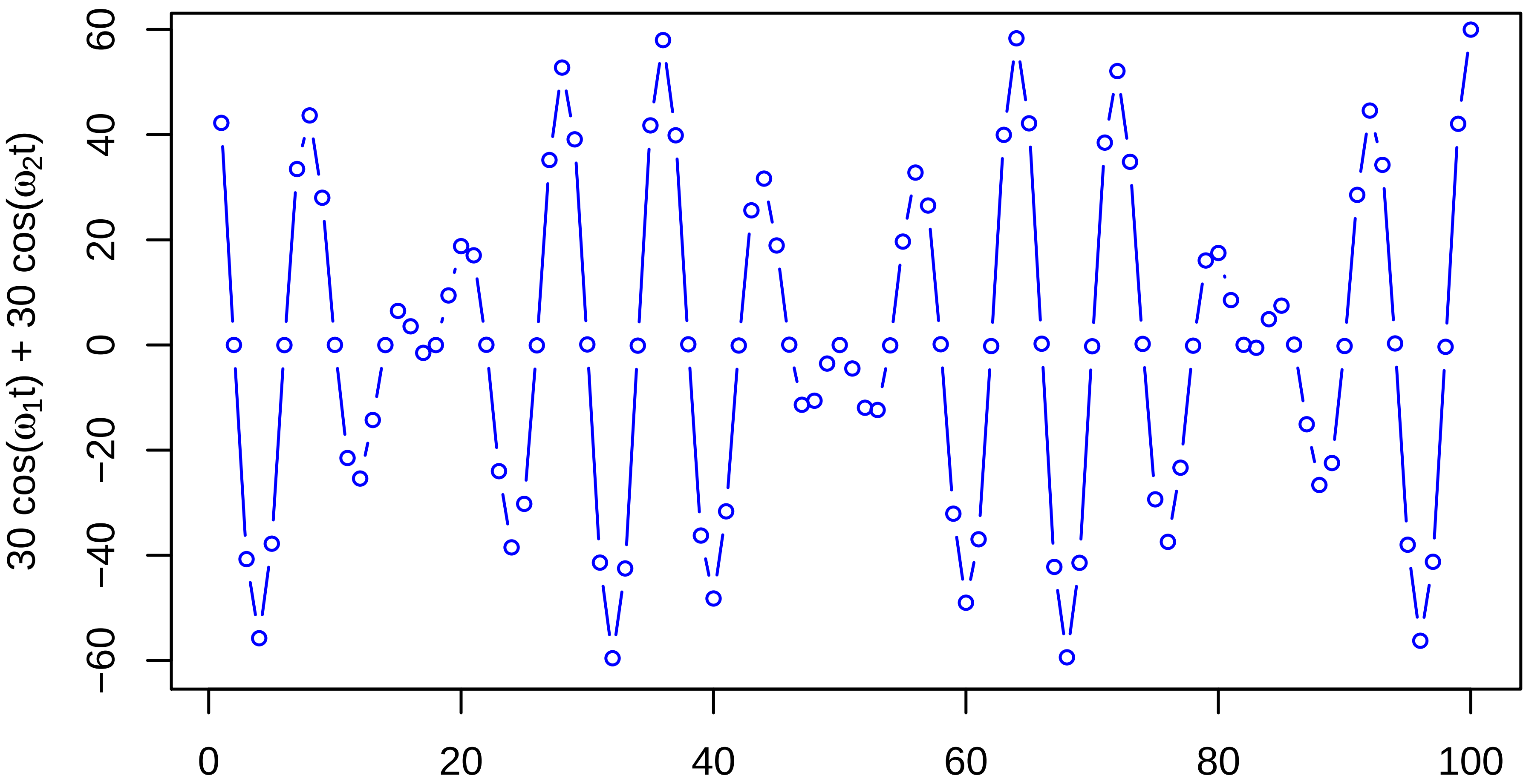
$30\cos(\omega_1 t)$ **versus** t



$30\cos(\omega_2 t)$ **versus** t



$30\cos(\omega_1 t) + 30\cos(\omega_2 t)$ **versus** t



Realization of AR(4) Process