Constructing Processes from White Noise: I

- given $\{Z_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$, can form MA(1) process

$$X_t = Z_t + \theta Z_{t-1},$$

for which $\gamma(1) = \sigma^2\theta$ but $\gamma(h) = 0$ for $|h| \geq 2$ (II–32 & II–33)

- obvious extension is moving average process of order q (MA(q)):

$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q},$$

where θ_j's are arbitrary real-valued constants (we'll insist, however, that $\theta_q \neq 0$)

- will argue later that $\{X_t\}$ is stationary with ACVF such that $\gamma(q) = \sigma^2\theta_q$ but $\gamma(h) = 0$ for $|h| > q$

Constructing Processes from White Noise: II

- processes with ACVFs that are nonzero at lag $h = q \geq 0$ but zero for $h > q$ are called *q-correlated processes*
- MA(1) process is 1-correlated, while white noise is 0-correlated
- factoid: *every* q-correlated process is an MA(q) process
  - B&D give a proof from their 1991 book *Time Series: Theory and Methods* (Second Edition)

Constructing Processes from White Noise: III

- extension of idea of contructing new processes from white noise leads to concept of a *linear process*, which takes the form

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \ \ t \in \mathbb{Z}, \qquad (*)$$

  where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and $\{\psi_j\}$ is a real-valued sequence such that $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ (hence sum in $(*)$ makes sense)
- can regard $\{X_t\}$ as result of filtering $\{Z_t\}$ with filter $\{\psi_j\}$
- note: alternative definitions for linear process involve use of
  - stronger condition $\{Z_t\} \sim \mathrm{IID}(0, \sigma^2)$ (implies $\{Z_t\}$ is white noise if $\sigma^2 < \infty$, but converse is not true)
  - weaker condition $\sum_j \psi_j^2 < \infty$ (implied by $\sum_j |\psi_j| < \infty$, but converse is not true)

Constructing Processes from White Noise: IV

- theorem: if $\{Y_t\}$ is a stationary process with mean 0 and ACVF $\{\gamma_Y(h)\}$ and if $\sum_j |\psi_j| < \infty$, then filtering it with the filter $\{\psi_j\}$ to obtain
$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}$$
yields a stationary process with mean 0 and ACVF
$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j)$$

- sketch of proof: first note that
$$E\{X_t\} = \sum_{j=-\infty}^{\infty} \psi_j E\{Y_{t-j}\} = 0, \quad \text{as required}$$

Constructing Processes from White Noise: V

- next note that

$$
\begin{aligned}
\operatorname{cov}\{X_{t+h}, X_t\} &= \operatorname{cov}\left\{\sum_{j=-\infty}^{\infty} \psi_j Y_{t+h-j}, \sum_{k=-\infty}^{\infty} \psi_k Y_{t-k}\right\} \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \operatorname{cov}\{Y_{t+h-j}, Y_{t-k}\} \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h+k-j) \\
&\stackrel{\text{def}}{=} \gamma_X(h),
\end{aligned}
$$

as required

Constructing Processes from White Noise: VI

- letting $\{Y_t\}$ be the white noise process $\{Z_t\}$,

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} \text{ becomes the linear process } X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

which is thus stationary with an ACVF given by

$$\begin{aligned}\gamma_X(h) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Z(h+k-j) \\ &= \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j-h} = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|h|}\end{aligned}$$

because $\gamma_Z(h+k-j) \neq 0$ only when $h+k-j=0$ (i.e., $k=j-h$), for which $\gamma_Z(0)=\sigma^2$ (last two equalities above follow because we must have $\gamma_X(-h)=\gamma_X(h)$)

MA(q) Process as a Linear Process

- an MA(q) process is obviously an example of a linear process: with $\psi_j = 0$ for $j < 0$ or $j > q$, and with $\psi_0 = 1$ and $\psi_j = \theta_j$ for $j = 1, \ldots, q$, we have

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j}$$

- its ACVF is thus

$$\gamma_X(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+|h|} = \sigma^2 \sum_{j=0}^{q} \theta_j \theta_{j+|h|} = \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}$$

where, for convenience, we define $\theta_0 = 1$; $\theta_j = 0$ for $j > q$; and last sum to be 0 when $q - |h| < 0$ (note: $\gamma_X(q) = \sigma^2 \theta_q$ and $\gamma_X(h) = 0$ when $|h| > q$, as claimed earlier)

MA(∞) Process as a Linear Process

- leads to notion of MA(∞) process:

$$X_t = Z_t + \sum_{j=1}^{\infty} \theta_j Z_{t-j},$$

where θ_j's are real-valued constants such that $\sum_{j=1}^{\infty} |\theta_j| < \infty$

- defining $\theta_0 = 1$ as before, its ACVF is

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} \theta_j \theta_{j+|h|}$$

- Q: why don't we gain more generality by using

$$X_t = \theta_0 Z_t + \sum_{j=1}^{\infty} \theta_j Z_{t-j},$$

with θ_0 allowed to be different from unity?

AR(1) Process as a Linear Process: I

- previously (overhead II–48), assumed AR(1) process to be stationary process $\{X_t\}$ satisfying

$$X_t = \mu + \phi(X_{t-1} - \mu) + Z_t, \quad t \in \mathbb{Z}, \qquad (*)$$

  where μ and ϕ are real-valued constants with $|\phi| < 1$, and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $\mathrm{cov}\,\{X_s, Z_t\} = 0$ for all $s < t$

- to show existence of such a process when $\mu = 0$, consider the linear process

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j},$$

  where condition $|\phi| < 1$ ensures that $\sum_j |\phi^j| < \infty$ (note: above is actually an MA(∞) process)

AR(1) Process as a Linear Process: II

- now, since $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$ says $X_{t-1} = \sum_{j=0}^{\infty} \phi^j Z_{t-1-j}$, consider the following sum:

$$
\begin{aligned}
Z_t + \phi X_{t-1} &= Z_t + \phi \sum_{j=0}^{\infty} \phi^j Z_{t-1-j} \\
&= Z_t + \sum_{j=0}^{\infty} \phi^{j+1} Z_{t-(j+1)} \\
&= Z_t + \sum_{j=1}^{\infty} \phi^j Z_{t-j} = \sum_{j=0}^{\infty} \phi^j Z_{t-j} = X_t,
\end{aligned}
$$

so this process satisfies defining equation $X_t = \phi X_{t-1} + Z_t$

- in addition, it is clear that $\text{cov}\,\{X_s, Z_t\} = 0$ for all $s < t$
- to handle $\mu \neq 0$, just replace X_t by $X_t - \mu$ to get $(*)$

AR(1) Process as a Linear Process: III

- focusing on $h \geq 0$, ACVF for AR(1) can be gotten from MA(∞) result by letting $\theta_j = \phi^j$:

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} \theta_j \theta_{j+h} = \sigma^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+h} = \sigma^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma^2 \phi^h}{1 - \phi^2},$$

  where we have appealed to the basic result (valid for $|x| < 1$)

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1 - x}$$

- agrees with what we got previously (overheads II–50 and II–51)

AR(1) Process as a Linear Process: IV

- focusing on $\mu = 0$ case, could have used 'telescoping' argument on $X_t = \phi X_{t-1} + Z_t$ to deduce appropriate linear process:

$$
\begin{aligned}
X_t &= \phi X_{t-1} + Z_t \\
&= \phi(\phi X_{t-2} + Z_{t-1}) + Z_t \\
&= \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t \\
&= \phi^2(\phi X_{t-3} + Z_{t-2}) + \phi Z_{t-1} + Z_t \\
&= \phi^3 X_{t-3} + \phi^2 Z_{t-2} + \phi Z_{t-1} + Z_t \\
&\;\vdots \\
&= \phi^J X_{t-J} + \sum_{j=0}^{J-1} \phi^j Z_{t-j} \to \sum_{j=0}^{\infty} \phi^j Z_{t-j}
\end{aligned}
$$

since $|\phi| < 1$ (can argue that stated MA(∞) process is the *only* stationary solution to $X_t = \phi X_{t-1} + Z_t$ when $|\phi| < 1$)

AR(1) Process as a Linear Process: V

- are there stationary solutions to $X_t = \phi X_{t-1} + Z_t$ for $|\phi| \geq 1$?
- answer is sometimes said to be 'no,' but there actually *are* when $|\phi| > 1$ (but exercise says there are none for $\phi = \pm 1$)
- to motivate solutions when $|\phi| > 1$, manipulate

$$X_t = \phi X_{t-1} + Z_t$$

to get

$$\begin{aligned}
\phi X_{t-1} &= X_t - Z_t \\
X_{t-1} &= \frac{1}{\phi} X_t - \frac{1}{\phi} Z_t \\
X_{t-1} &= \alpha X_t - \alpha Z_t \qquad (\text{with } \alpha \stackrel{\text{def}}{=} 1/\phi) \\
X_t &= \alpha X_{t+1} - \alpha Z_{t+1}
\end{aligned}$$

note: $|\alpha| < 1$ since $|\phi| > 1$

AR(1) Process as a Linear Process: VI

- consider telescoping as follows:

$$\begin{aligned}
X_t &= \alpha X_{t+1} - \alpha Z_{t+1} \\
&= \alpha(\alpha X_{t+2} - \alpha Z_{t+2}) - \alpha Z_{t+1} \\
&= \alpha^2 X_{t+2} - \alpha^2 Z_{t+2} - \alpha Z_{t+1} \\
&= \alpha^2(\alpha X_{t+3} - \alpha Z_{t+3}) - \alpha^2 Z_{t+2} - \alpha Z_{t+1} \\
&= \alpha^3 X_{t+3} - \alpha^3 Z_{t+3} - \alpha^2 Z_{t+2} - \alpha Z_{t+1} \\
&\vdots \\
&= \alpha^J X_{t+J} - \sum_{j=1}^{J} \alpha^j Z_{t+j} \to -\sum_{j=1}^{\infty} \alpha^j Z_{t+j} = -\sum_{j=1}^{\infty} \frac{1}{\phi^j} Z_{t+j}
\end{aligned}$$

AR(1) Process as a Linear Process: VII

- AR(1) stationary processes for which $|\phi| > 1$ do not bring anything new to the table: exercise says that the linear process

$$X_t = -\sum_{j=1}^{\infty} \frac{1}{\phi^j} Z_{t+j}$$

  is not only the solution to $X_t = \phi X_{t-1} + Z_t$, but is also the solution to $X_t = \frac{1}{\phi} X_{t-1} + W_t$, where $\{W_t\} \sim \mathrm{WN}(0, \sigma_W^2)$, and σ_W^2 depends on ϕ and σ^2 (in fact both solutions are unique)
- customary to dispose of $|\phi| > 1$ case by insisting upon *causality*, by which we mean that X_t only depends on Z_s's such that $s \leq t$ (amounts to expressing X_t as an MA(∞) process, i.e., a one-sided linear process)

AR(p) Process as a Linear Process

- with p being a positive integer, an AR(p) process $\{X_t\}$ with zero mean takes the form
$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + Z_t, \ \text{ where } \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$
- setting $p = 1$ in the above yields an AR(1) process
- by placing certain conditions on $\phi_1, \ldots, \phi_p$ (to be discussed later on), can write $\{X_t\}$ as an MA(∞) process, which is one form of a linear process:
$$X_t = Z_t + \sum_{j=1}^{\infty} \psi_j Z_{t-j},$$
where $\{\psi_j\}$ depends on $\phi_1, \ldots, \phi_p$, and $\sum_{j=1}^{\infty} |\psi_j| < \infty$

Invertibility of MA(1) Process: I

- AR(1) process $X_t = \phi X_{t-1} + Z_t$ has unique representation
$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j} \text{ as long as } |\phi| < 1$$
- consider $X_t = Z_t + \theta Z_{t-1} = Z_t - \varphi Z_{t-1}$, where $\varphi \stackrel{\text{def}}{=} -\theta$
- rewriting MA(1) equation as $Z_t = \varphi Z_{t-1} + X_t$ suggests
$$Z_t = \sum_{j=0}^{\infty} \varphi^j X_{t-j} \text{ as long as } |\varphi| < 1$$
- linear process $\{X_t\}$ such that Z_t can be expressed as a linear combination of just $X_t, X_{t-1}, X_{t-2}, \ldots$ is called *invertible*
- coefficients of linear combination must be absolutely summable, so invertibility condition for MA(1) process is $|\theta| < 1$

BD: 49, CC: 79–80, SS: 83

Invertibility of MA(1) Process: II

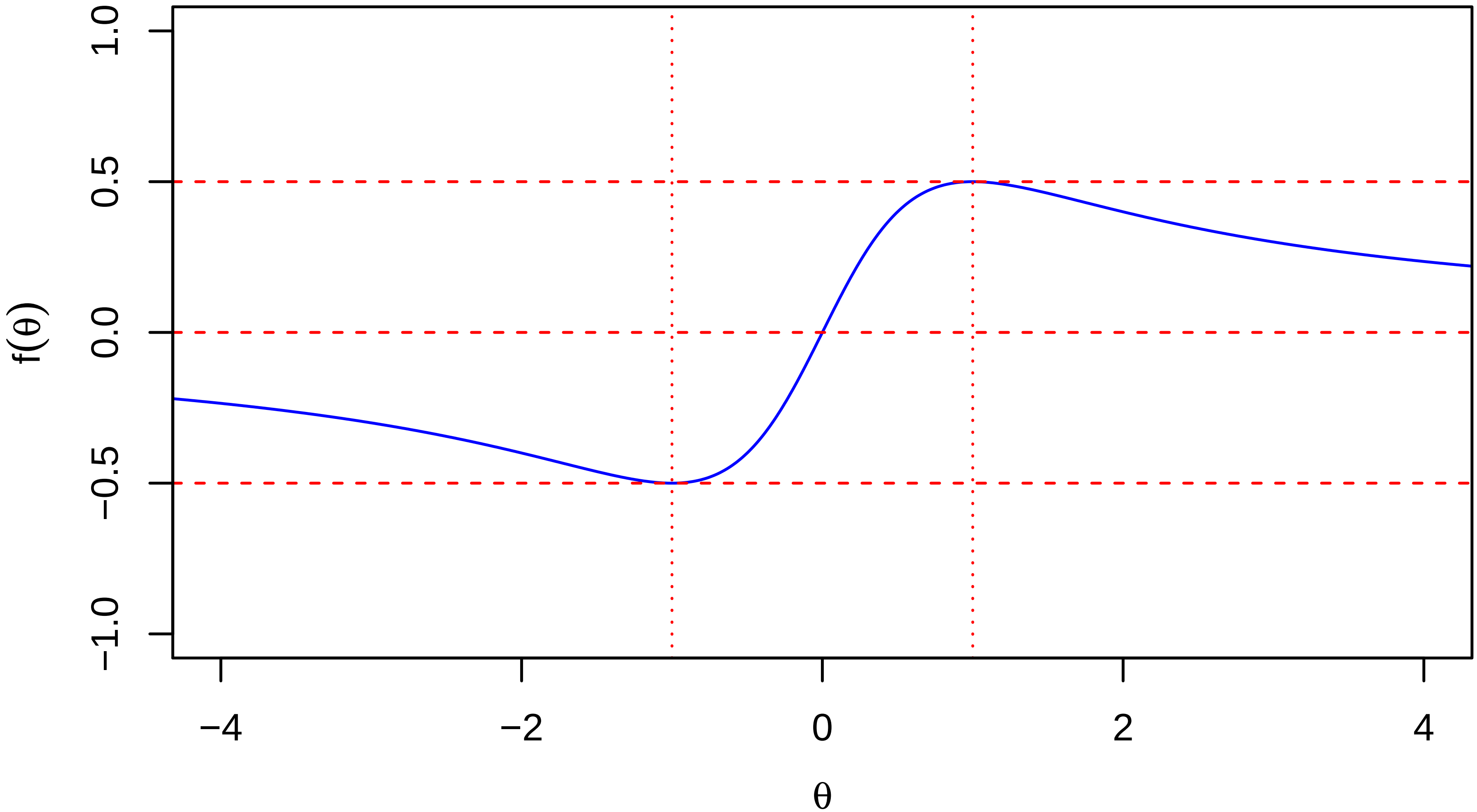
- Q: what condition do we need on ϕ so that the AR(1) process
$$X_t = \phi X_{t-1} + Z_t$$
is invertible?

- reconsider ACF for MA(1) process:
$$\rho(h) = \begin{cases} 1, & h = 0, \\ \theta/(1+\theta^2), & h = \pm 1, \\ 0, & \text{otherwise} \end{cases}$$

- as was done in solution to Problem 3, consider $f(\theta) = \theta/(1+\theta^2)$ for $\theta \in \mathbb{R}$ (see next overhead)

- conclusion: don't really lose much by imposing invertibility restriction $|\theta| < 1$ (main loss is cases $\theta = \pm 1$, which are actually of interest; e.g., $X_t = \nabla Z_t = Z_t - Z_{t-1}$)

Unit-Lag Value of ACF as a Function of θ



Successive Applications of Filters: I

- recall that filtering a stationary process $\{Y_t\}$ yields another such process $\{X_t\}$:

$$X_t = \sum_{j=-\infty}^{\infty} \alpha_j Y_{t-j},$$

where we need $\sum_j |\alpha_j| < \infty$ for $\{X_t\}$ to be well-defined

- can write above compactly as

$$X_t = \alpha(B)Y_t, \text{ where } \alpha(B) \stackrel{\text{def}}{=} \sum_{j=-\infty}^{\infty} \alpha_j B^j$$

- now consider subjecting $\{X_t\}$ to another filter, say $\beta(B) = \sum_k \beta_k B^k$, where $\sum_k |\beta_k| < \infty$, to obtain

$$W_t = \beta(B)X_t = \beta(B)\alpha(B)Y_t \stackrel{\text{def}}{=} \psi(B)Y_t$$

Successive Applications of Filters: II

- now

$$\begin{aligned}\psi(B) = \alpha(B)\beta(B) &= \Big(\sum_j \alpha_j B^j\Big)\Big(\sum_k \beta_k B^k\Big) \\ &= \sum_j \sum_k \alpha_j \beta_k B^{j+k} \\ &= \sum_j \sum_l \alpha_j \beta_{l-j} B^l \quad \text{(letting } l = j+k\text{)} \\ &= \sum_l \Big(\sum_j \alpha_j \beta_{l-j}\Big) B^l \stackrel{\text{def}}{=} \sum_l \psi_l B^l\end{aligned}$$

  with $\psi_l = \sum_j \alpha_j \beta_{l-j}$, but $\psi_l = \sum_j \beta_j \alpha_{l-j}$ also (why?)

- conclusion: sequence of filtering operations involving filters $\alpha(B)$, $\beta(B)$, ..., $\omega(B)$ yields same output if done in *any* order or with a single composite filter $\varphi(B) = \alpha(B)\beta(B)\cdots\omega(B)$

ARMA(1,1) Process as a Linear Process: I

- time series is called a zero-mean ARMA(1,1) process if it is stationary and satisfies

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}, \quad t \in \mathbb{Z},$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $\phi + \theta \neq 0$ (to generalize to non-zero mean case, just replace X_t with $X_t - \mu$ everywhere)

- equivalently we can write the above as

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

and then more compactly as

$$\phi(B) X_t = \theta(B) Z_t,$$

where $\phi(B)$ and $\theta(B)$ are linear filters given by

$$\phi(B) = 1 - \phi B \ \text{ and } \ \theta(B) = 1 + \theta B$$

ARMA(1,1) Process as a Linear Process: II

- assuming $|\phi| < 1$, consider filter

$$\chi(B) = \sum_{j=0}^{\infty} \phi^j B^j$$

- note that $\sum_j |\phi^j| < \infty$ and that

$$\begin{aligned}
\chi(B)\phi(B) &= \left(\sum_{j=0}^{\infty} \phi^j B^j\right)(1 - \phi B) \\
&= \sum_{j=0}^{\infty} \phi^j B^j - \sum_{j=0}^{\infty} \phi^{j+1} B^{j+1} \\
&= \sum_{j=0}^{\infty} \phi^j B^j - \sum_{j=1}^{\infty} \phi^j B^j = 1
\end{aligned}$$

ARMA(1,1) Process as a Linear Process: III

- since $\chi(B)\phi(B) = 1$, can consider $\chi(B)$ as inverse of $\phi(B)$
- form that $\chi(B)$ takes is in keeping with power series expansion of $1/(1-x)$ with $|x| < 1$; i.e., using

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

  with $x = \phi B$ says that

$$\frac{1}{\phi(B)} = \frac{1}{1-\phi B} = 1 + \phi B + \phi^2 B^2 + \cdots = \chi(B)$$

- note: henceforth, rather than denoting inverse of $\phi(B)$ by $\chi(B)$, will denote it as $\phi^{-1}(B)$, with the understanding that this should be interpreted as per the last line above

ARMA(1,1) Process as a Linear Process: IV

- applying $\phi^{-1}(B)$ to both sides of $\phi(B)X_t = \theta(B)Z_t$ yields

$$X_t = \phi^{-1}(B)\theta(B)Z_t = \psi(B)Z_t,$$

  where

$$\begin{aligned}\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j &= (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)(1 + \theta B) \\ &= 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots \\ &\quad + \theta B + \theta\phi B^2 + \theta\phi^2 B^3 + \cdots\end{aligned}$$

  so $\psi_0 = 1$, $\psi_1 = \phi + \theta$, $\psi_2 = \phi^2 + \theta\phi$, $\psi_3 = \phi^3 + \theta\phi^2$, ...

- hence $\psi_0 = 1$ and $\psi_j = (\phi + \theta)\phi^{j-1}$ for $j \geq 1$, and we have $\sum_j |\psi_j| < \infty$ (can also see why $\phi + \theta \neq 0$ is assumed)

ARMA(1,1) Process as a Linear Process: V

- exercise: use argument similar to that used in AR(1) case to show that

$$X_t = \psi(B)Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

  is stationary solution to $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$ when $|\phi| < 1$ (in fact, it is a unique solution)

- can also argue that

$$X_t = -\frac{\theta}{\phi} Z_t - (\phi + \theta) \sum_{j=1}^{\infty} \frac{1}{\phi^{j+1}} Z_{t+j}$$

  is unique stationary solution to $X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}$ when $|\phi| > 1$ and that no solutions exist when $|\phi| = 1$

ARMA(1,1) Process as a Linear Process: V

- now let's consider the filter

$$\xi(B) = \sum_{j=0}^{\infty} (-\theta)^j B_j \text{ assuming } |\theta| < 1 \text{ so that } \sum_{j=0}^{\infty} |(-\theta)^j| < \infty$$

- note that $\theta(B) = 1 + \theta B = 1 - (-\theta)B$ and that

$$\begin{aligned}
\xi(B)\theta(B) &= \left(\sum_{j=0}^{\infty} (-\theta)^j B^j \right) (1 - (-\theta)B) \\
&= \sum_{j=0}^{\infty} (-\theta)^j B^j - \sum_{j=0}^{\infty} (-\theta)^{j+1} B^{j+1} \\
&= \sum_{j=0}^{\infty} (-\theta)^j B^j - \sum_{j=1}^{\infty} (-\theta)^j B^j = 1
\end{aligned}$$

ARMA(1,1) Process as a Linear Process: VI

- rather than denoting inverse of $\theta(B)$ by $\xi(B)$, henceforth will denote it by $\theta^{-1}(B)$:

$$\theta^{-1}(B) = \frac{1}{\theta(B)} = \frac{1}{1+\theta B} = \sum_{j=0}^{\infty} (-\theta)^j B^j,$$

which is in keeping with power series expansion of $1/(1+x)$ with $|x| < 1$:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

ARMA(1,1) Process as a Linear Process: VII

- applying $\theta^{-1}(B)$ to both sides of $\phi(B)X_t = \theta(B)Z_t$ yields

$$\theta^{-1}(B)\phi(B)X_t = \theta^{-1}(B)\theta(B)Z_t = Z_t,$$

i.e.,

$$Z_t = \theta^{-1}(B)\phi(B)X_t \stackrel{\text{def}}{=} \pi(B)X_t,$$

where

$$\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j = (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \cdots)(1 - \phi B),$$

with $\pi_0 = 1$ and $\pi_j = -(\phi + \theta)(-\theta)^{j-1}$ for $j \geq 1$

- thus, when $|\theta| < 1$, the ARMA(1,1) process is invertible since we can write

$$Z_t = X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}$$

ARMA(1,1) Process as a Linear Process: VIII

- when $|\theta| > 1$, can express Z_t in terms of X_{t+1}, X_{t+2}, $\ldots$, which goes *against* the notion of invertibility (remaining case $|\theta| = 1$ raises subtle issues!)
- ignoring subtleties arising when $|\theta| = 1$, if we restrict our attention to ARMA(1,1) models that are causal ($|\phi| < 1$) and invertible ($|\theta| < 1$), there is
  - nothing lost from a second-order point of view
  - much to be gained from the point of view of identifiability, i.e., ACVFs mapping to a single ARMA(1,1) model

Wold Decomposition: I

- if we insist upon causality, can express stationary zero-mean AR(1) and ARMA(1,1) processes as infinite moving averages:

$$X_t = \sum_{j=0}^{\infty} \theta_j Z_{t-j}, \quad t \in \mathbb{Z},$$

  where θ_j's are real-valued constants such that $\sum_j |\theta_j| < \infty$, $\theta_0 \stackrel{\text{def}}{=} 1$ and $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, $0 \leq \sigma^2 < \infty$
  - note: MA(q) processes can obviously be expressed as such
- noncausal stationary models 'uninteresting': for any such model, there always exists a causal stationary model with same ACVF
- Q: how general are infinite MA processes?
- answered is provided by *Wold Decomposition*

Wold Decomposition: II

- *any* stationary process $\{X_t\}$ can be expressed as

$$X_t = \sum_{j=0}^{\infty} \theta_j Z_{t-j} + V_t,$$

where

1. $\theta_0 = 1$ and $\sum_{j=0}^{\infty} \theta_j^2 < \infty$ (note: $\sum_j |\theta_j| < \infty$ implies $\sum_j \theta_j^2 < \infty$, but the converse does not hold);
2. $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, $0 < \sigma^2 < \infty$;
3. $\mathrm{cov}\,(Z_s, V_t) = 0$ for all s and t; and
4. $\{V_t\}$ is *deterministic* in the sense that, given V_{t-1}, V_{t-2}, $\ldots$, it is possible to predict V_t *perfectly*

Wold Decomposition: III

- two examples of perfectly predictable stationary processes $\{V_t\}$
  - $V_t = \mu$, a trivial stationary process
  - $V_t = W_1 \cos(\omega t) + W_2 \sin(\omega t)$, with W_1 and W_2 being RVs with zero mean and a common variance σ_W^2 that are uncorrelated with each other and all RVs in $\{Z_t\}$ (solution to Problem 2(b) says that $\{V_t\}$ is stationary with ACVF $\gamma_V(h) = \sigma_W^2 \cos(\omega h)$ – this does *not* decay to 0 as $h \to \infty$)
- processes that can be written as

$$X_t = \sum_{j=0}^{\infty} \theta_j Z_{t-j}$$

are said to be *purely nondeterministic* – these have an ACVF such that $\gamma_X(h) \to 0$ as $h \to \infty$