Simple Time Series Models: I

- definition: time series model for $\{x_t\}$ is specification of joint distributions (or summaries thereof) of a sequence of random variables (RVs) $\{X_t\}$, one of whose realizations is assumed to be $\{x_t\}$
  - sequence of RVs $\{X_t\}$ called a stochastic process
  - realizations of $\{X_t\}$ are coupled together as dictated by their joint distribution
- will use term 'time series' to refer to both
  - $\{x_t\}$ (actual data or realization of stochastic process) and
  - $\{X_t\}$ (stochastic process itself)

Simple Time Series Models: II

- complete model for $\{x_t : t \in T\}$, $T = \{1, 2, \ldots, n\}$, requires specification of joint distributions of RVs X_1, X_2, $\ldots$, X_n, which is equivalent to specification of all probabilities

$$P[X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n],$$

where $-\infty < a_t < \infty$ for all $t \in T$

- while fully general, too complicated, so will concentrate on summary of joint distributions afforded by first- and second-order moments, i.e.,

$$E\{X_t\} \text{ and } E\{X_{t+h}X_t\}, \quad 1 \leq t + h \leq n$$

or, equivalently, $E\{X_t\}$ and $\operatorname{cov}\{X_{t+h}, X_t\}$, where

$$\operatorname{cov}\{U, V\} \stackrel{\text{def}}{=} E\{(U - E\{U\})(V - E\{V\})\} = E\{UV\} - E\{U\}E\{V\}$$

Simplest Time Series Model – IID Noise: I

- assume that X_1, X_2, …, X_n are independent and identically distributed (IID) RVs so that

$$\begin{aligned} P[X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n] \\ &= P[X_1 \leq a_1] \times P[X_2 \leq a_2] \times \cdots \times P[X_n \leq a_n] \\ &= F(a_1) \times F(a_2) \times \cdots \times F(a_n), \end{aligned}$$

  where $F(\cdot)$ is cumulative probability distribution function of each of the IID RVs

- model says that there is no dependence between observations
- extend model to include future observations by requiring RVs X_1, X_2, …, X_n, X_{n+1}, …, X_{n+h} to be IID for any $h \geq 1$

Simplest Time Series Model – IID Noise: II

- for extended model, conditioning on observed time series does not alter probabilistic description of X_{n+h} since

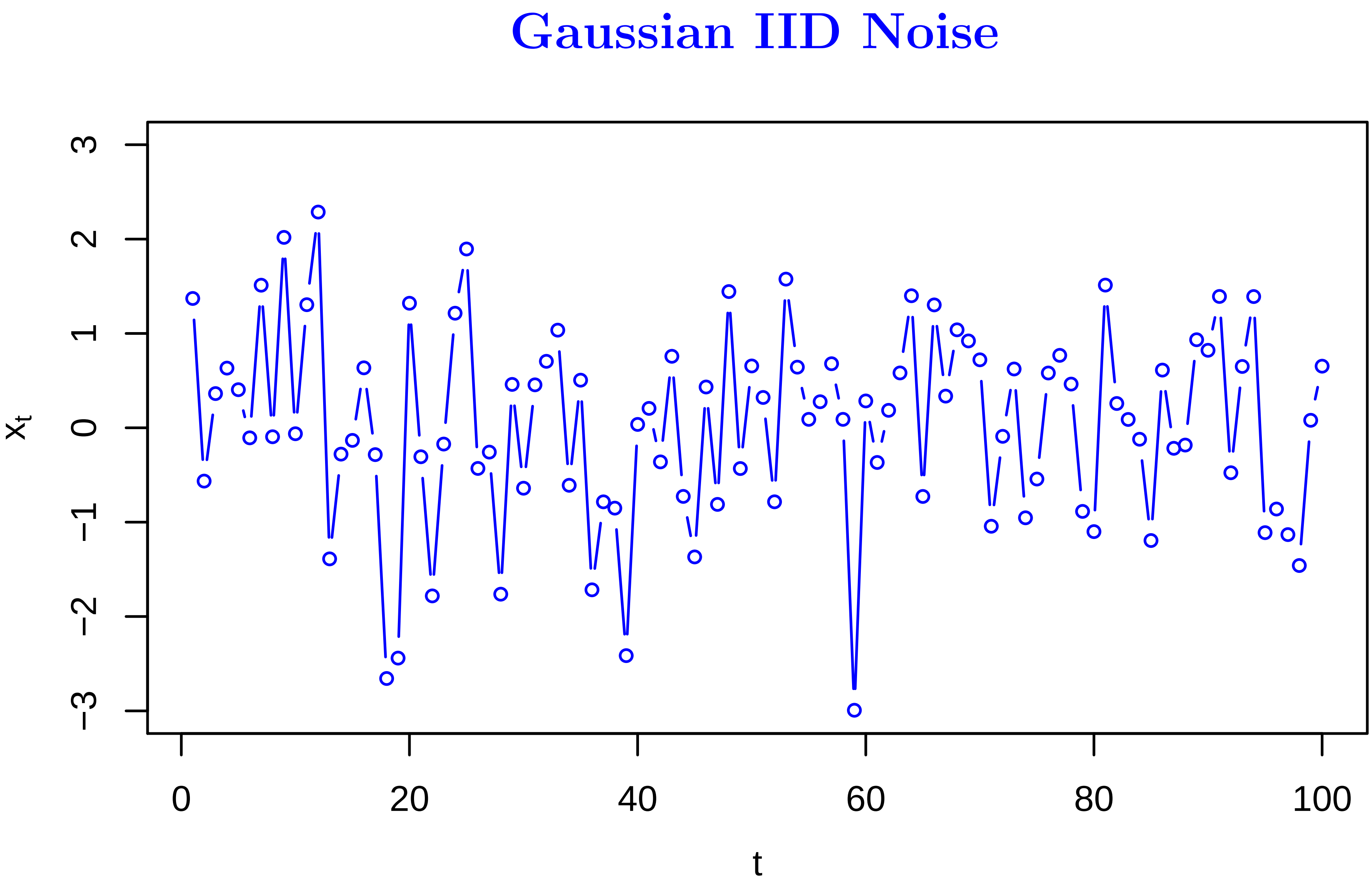
$$P[X_{n+h} \leq a \mid X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n] = P[X_{n+h} \leq a]$$

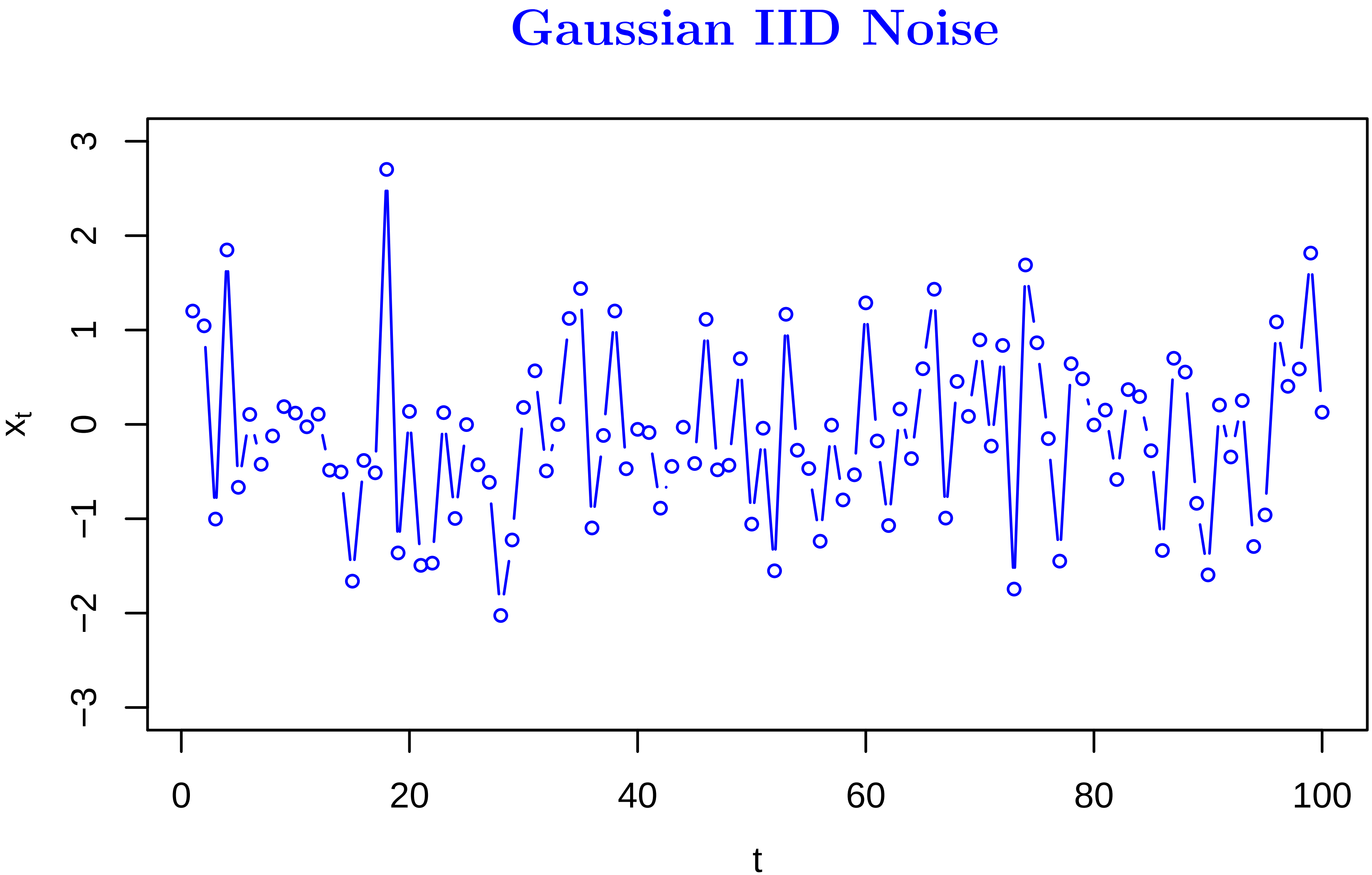
- implies that function f minimizing the mean squared error

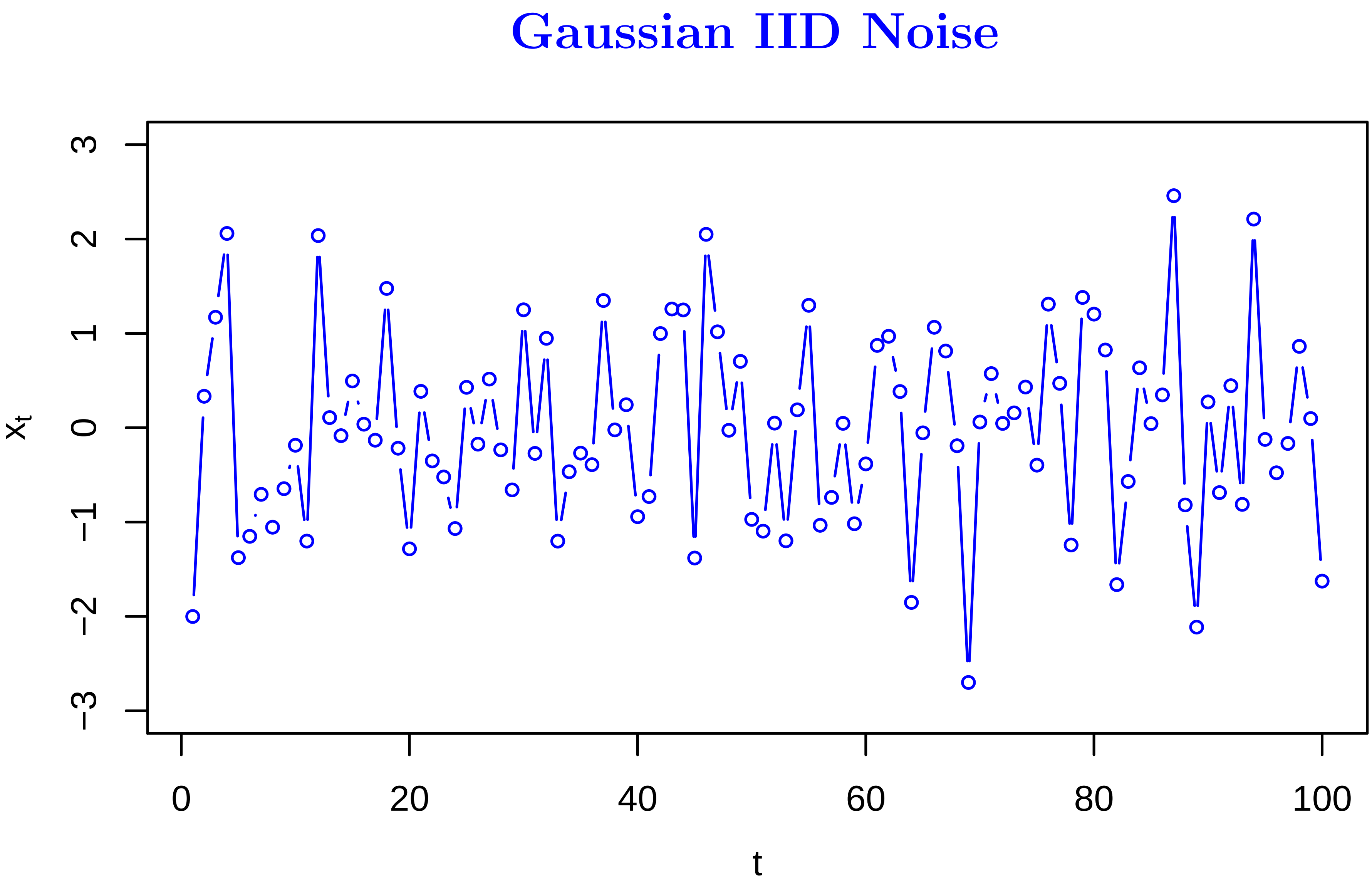
$$E\{[X_{n+h} - f(X_1, X_2, \ldots, X_n)]^2\}$$

  is $f(X_1, X_2, \ldots, X_n) = \mu$, where μ is the common mean of all the X_t's; i.e., best predictor of X_{n+h} in the mean square sense is $\mu = E\{X_{n+h}\}$, which does not depend on X_1, X_2, $\ldots$, X_n

- here are realizations of IID noise with $n = 100$ and with $F(\cdot)$ given by standard normal (Gaussian) distribution (i.e., zero mean and unit variance)


Gaussian IID Noise
3
2
1
0
−1
−2
−3
x_t
0
20
40
60
80
100
t


Gaussian IID Noise
x_t
3
2
1
0
−1
−2
−3
0
20
40
60
80
100
t


Gaussian IID Noise
3
2
1
0
−1
−2
−3
x_t
0
20
40
60
80
100
t

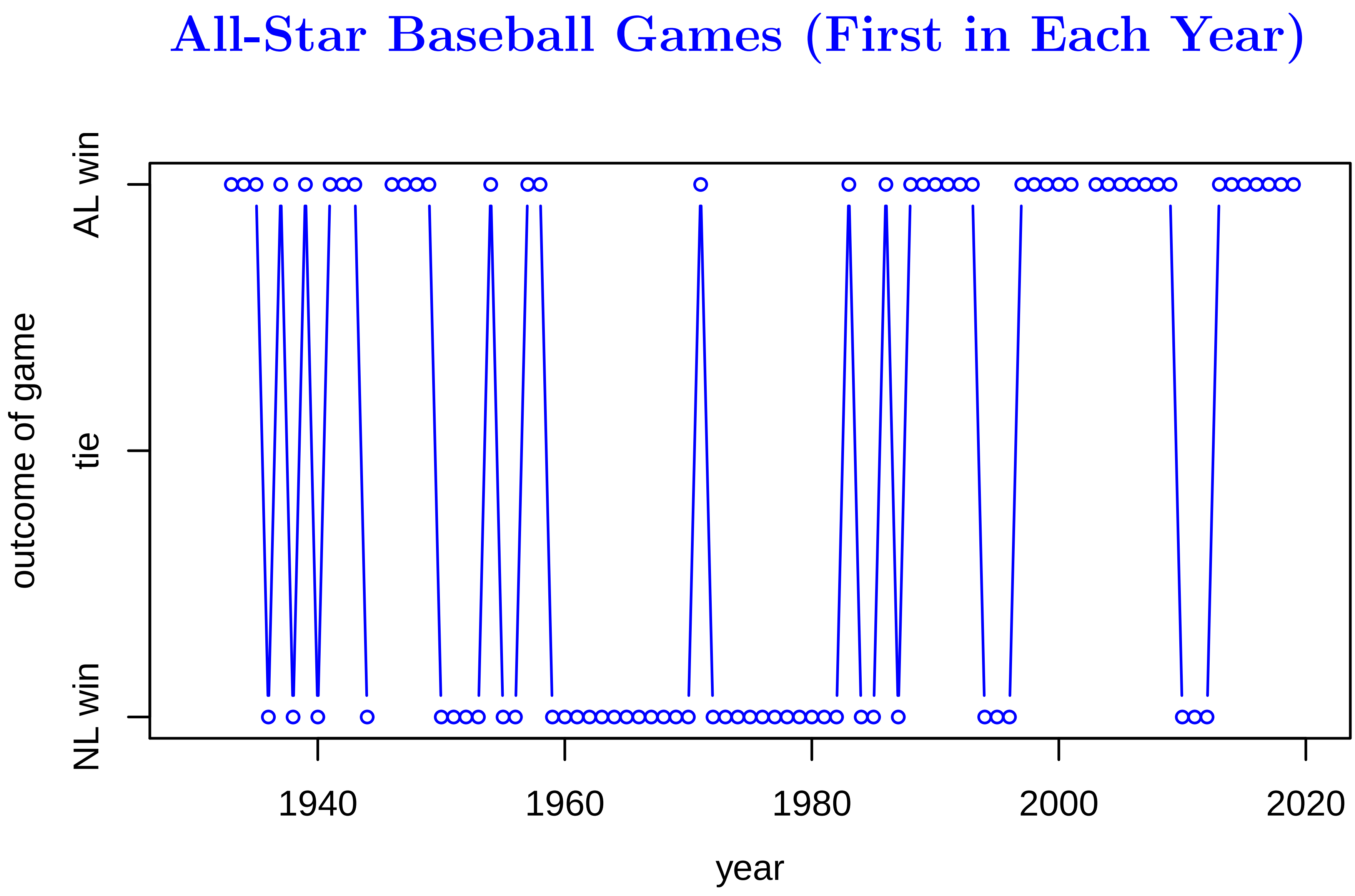
Binary-valued IID Noise

- as second example of IID noise, suppose X_t is binary-valued:

$$P[X_t = 1] = p \text{ and } P[X_t = -1] = 1 - p, \quad 0 \le p \le 1,$$

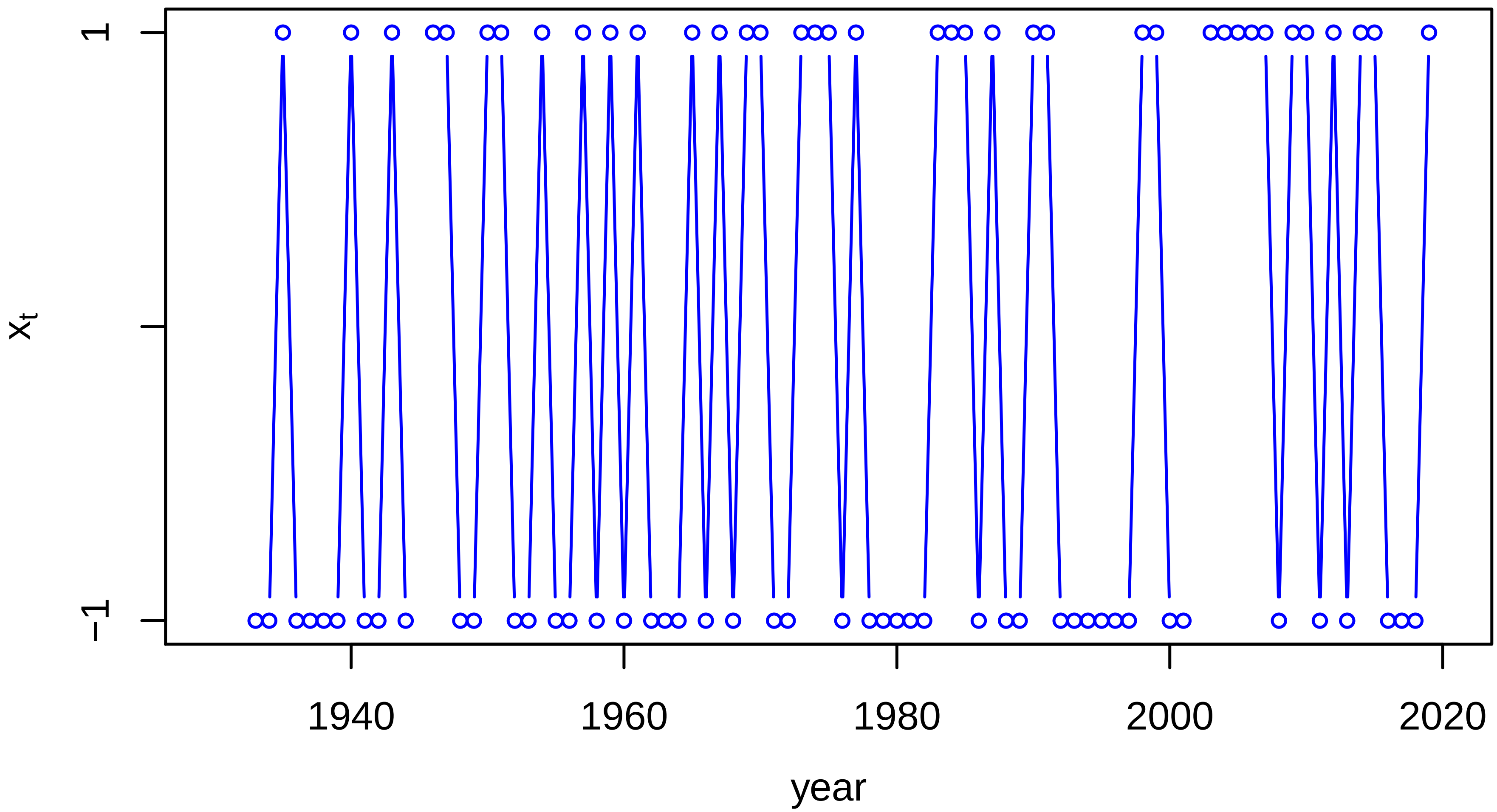
  for which $\mu = E\{X_t\} = p \times 1 + (1-p) \times (-1) = 2p - 1$

- reconsider first all-star baseball game in each year, with 1 and -1 indicating, respectively, AL and NL victories (regard 2002 tie as a gap, along with actual gap in 1945)
- as of Jan 2020, have been 43 AL victories and 42 NL victories, so entertain hypothesis $p = 43/85 \doteq 0.506$
- compare actual time series with realizations of binary-valued IID noise with $p = 43/85$

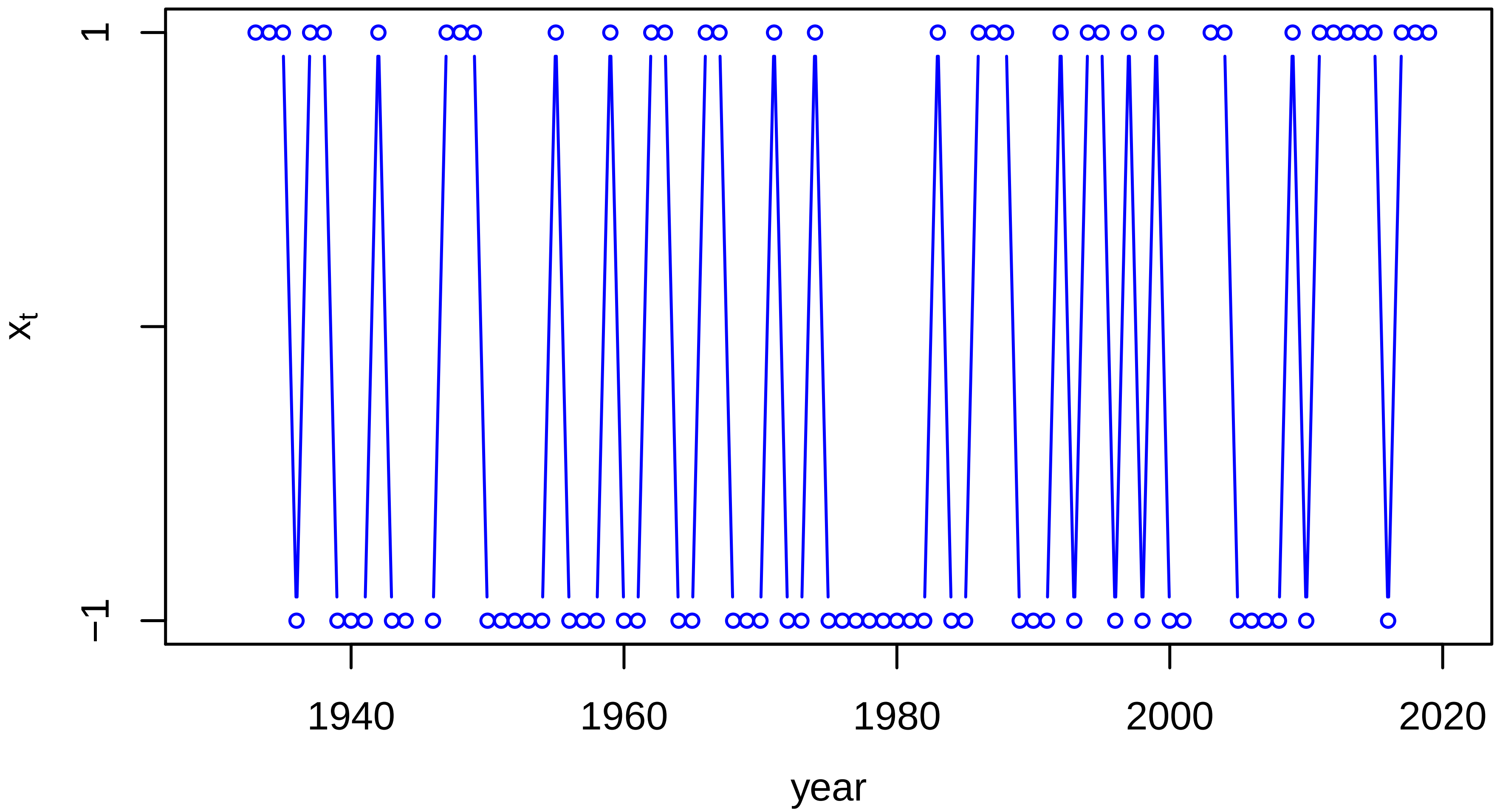


All-Star Baseball Games (First in Each Year)
AL win
tie
NL win
outcome of game
1940
1960
1980
2000
2020
year

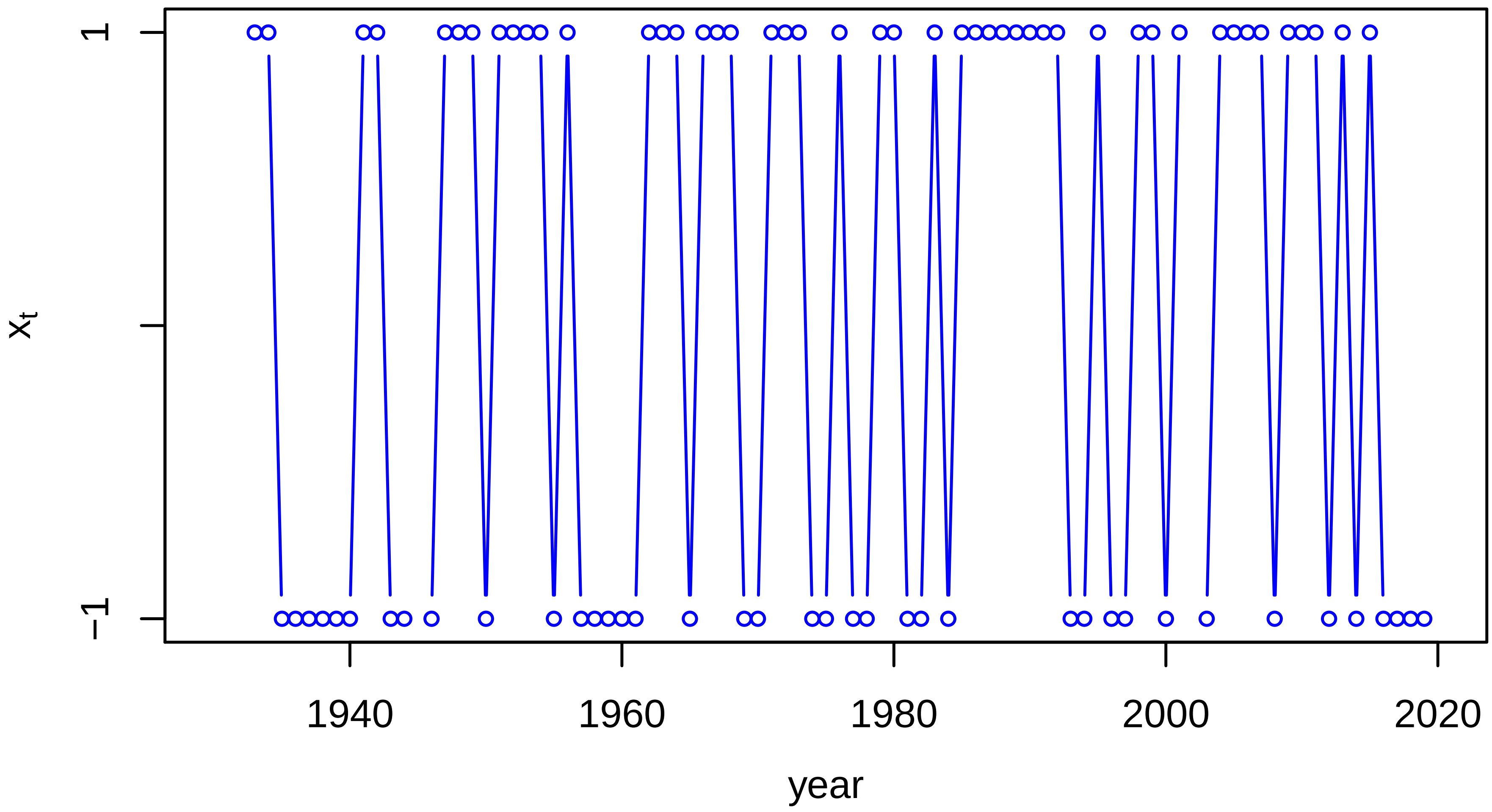
Binary-valued IID Noise ($p = 43/85$)



Binary-valued IID Noise ($p = 43/85$)



Binary-valued IID Noise ($p = 43/85$)



Random Walk Process: I

- can use IID noise as building block for other processes of interest
- suppose that $\{X_t\}$ is IID noise such that $E\{X_t\} = 0$
- for $t \geq 1$, construct new process

$$S_t = \sum_{u=1}^{t} X_u;$$

thus

$$\begin{aligned}
S_1 &= X_1 = S_0 + X_1 \quad \text{(if we define } S_0 \text{ to be 0)} \\
S_2 &= X_1 + X_2 = S_1 + X_2 \\
S_3 &= X_1 + X_2 + X_3 = S_2 + X_3 \\
&\vdots \\
S_t &= X_1 + X_2 + \cdots + X_{t-1} + X_t = S_{t-1} + X_t
\end{aligned}$$

Random Walk Process: II

- for $t \geq 1$, can recover X_t from first backward difference of S_t:

$$S_t - S_{t-1} = X_t$$

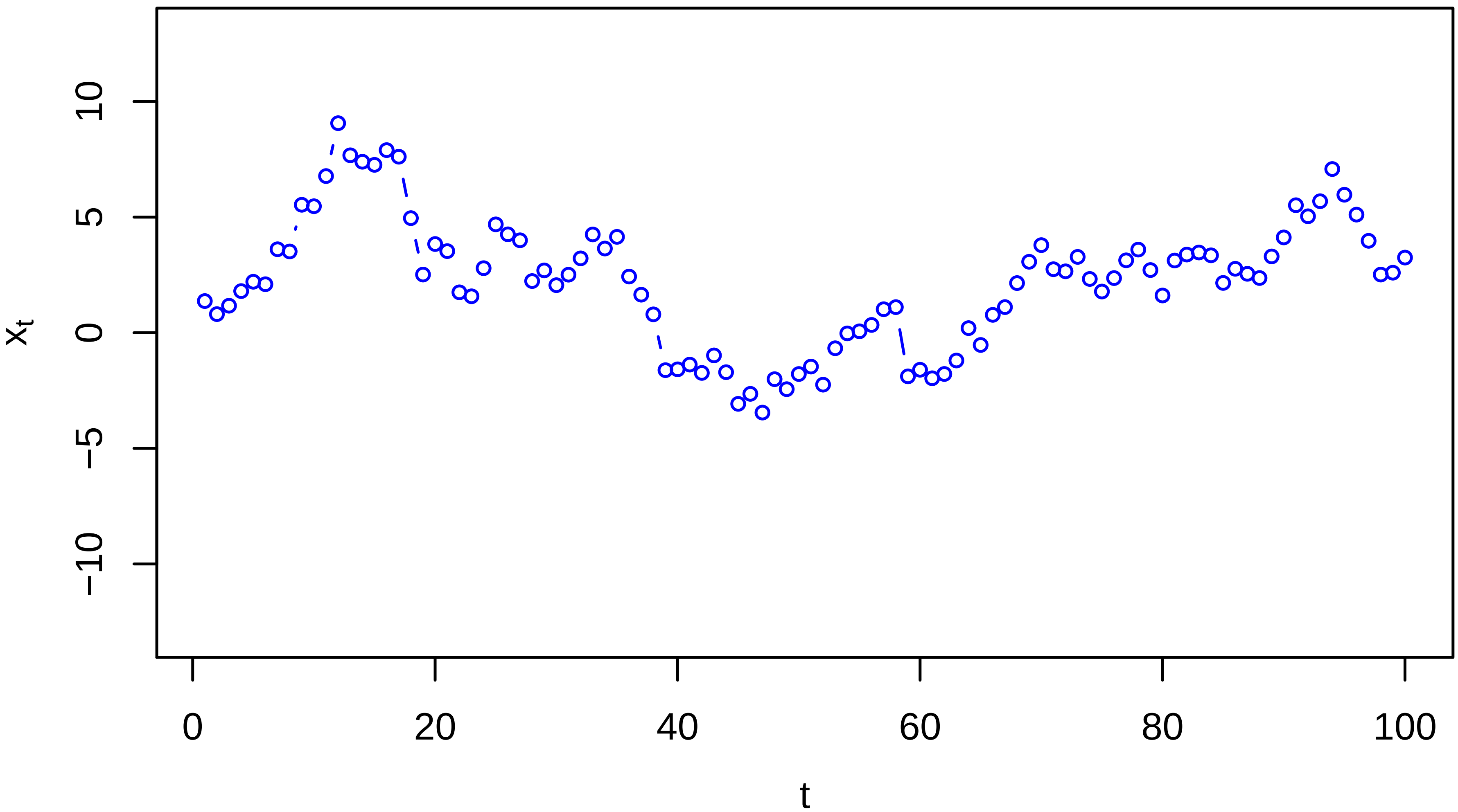
since

$$\begin{aligned} S_t &= X_1 + X_2 + \cdots + X_{t-2} + X_{t-1} + X_t \\ S_{t-1} &= X_1 + X_2 + \cdots + X_{t-2} + X_{t-1} \end{aligned}$$

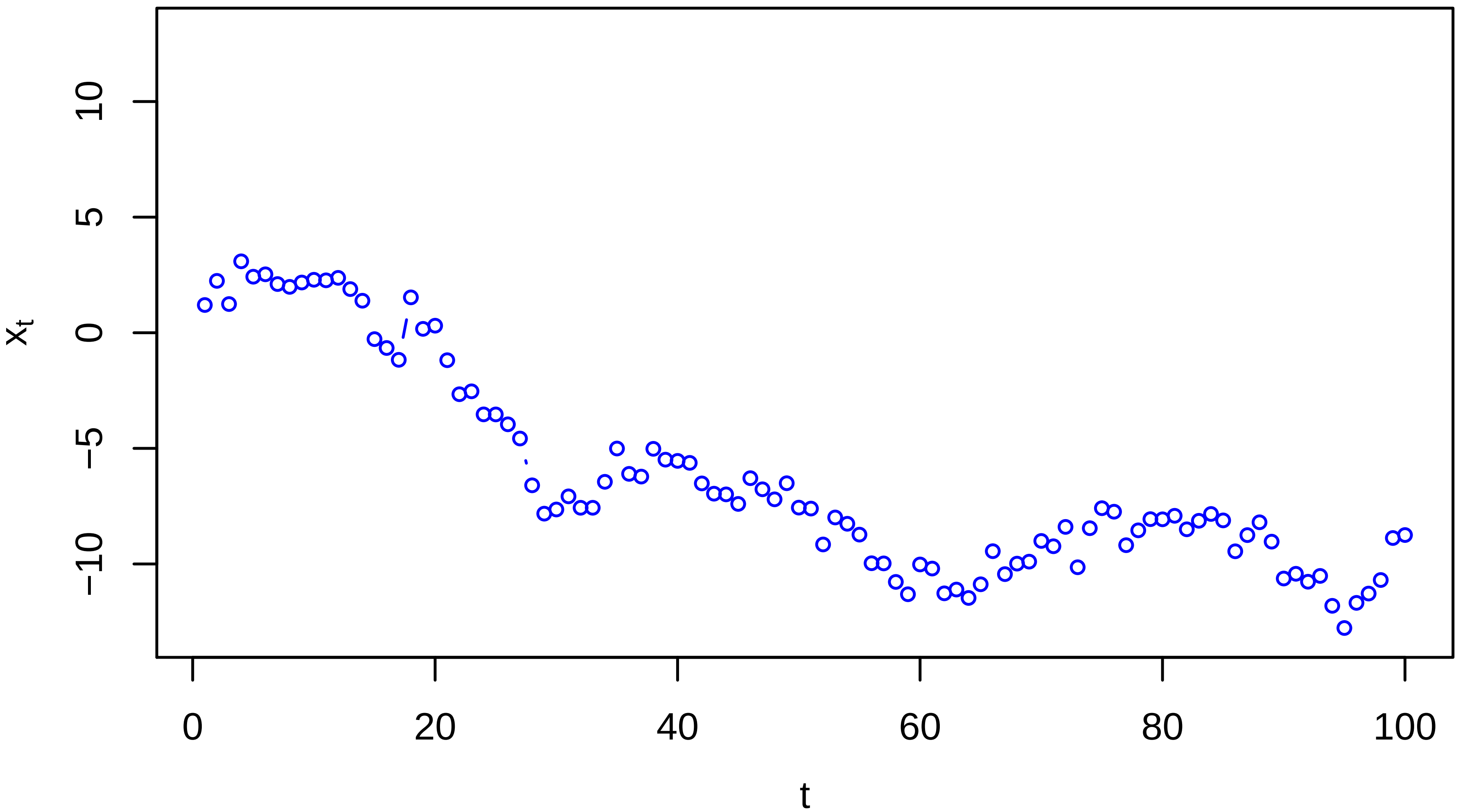
(works for $t = 1$ also since $S_0 = 0$ by definition)

- $\{S_t\}$ is called a zero-mean random walk process
- consider realizations of two examples of random walks, for which
  - $\{X_t\}$ is a standard Gaussian IID process
  - $\{X_t\}$ is a binary-valued IID process (1 or -1 with $p = 0.5$; Brockwell & Davis call this a simple symmetric random walk)

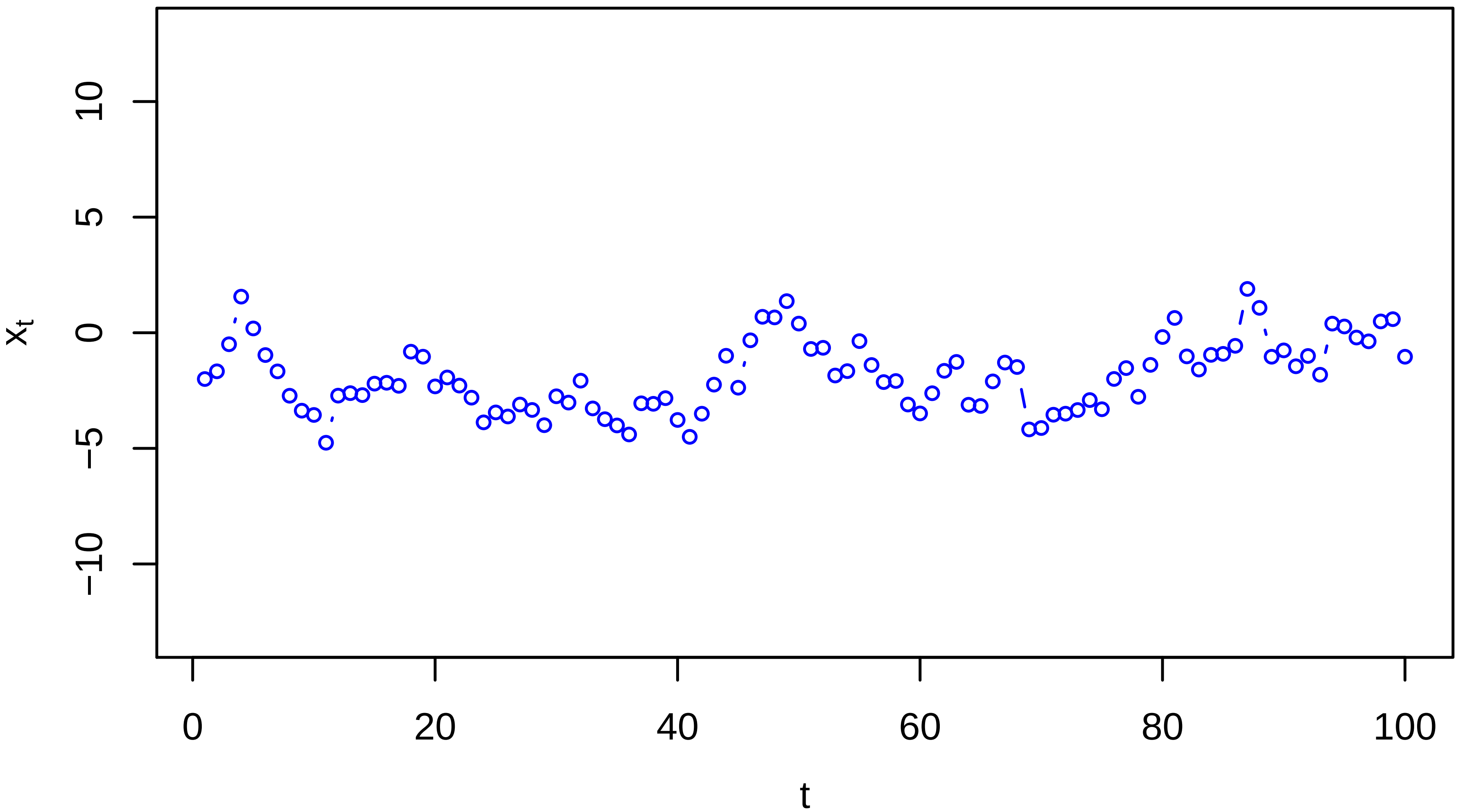
Gaussian Random Walk Process

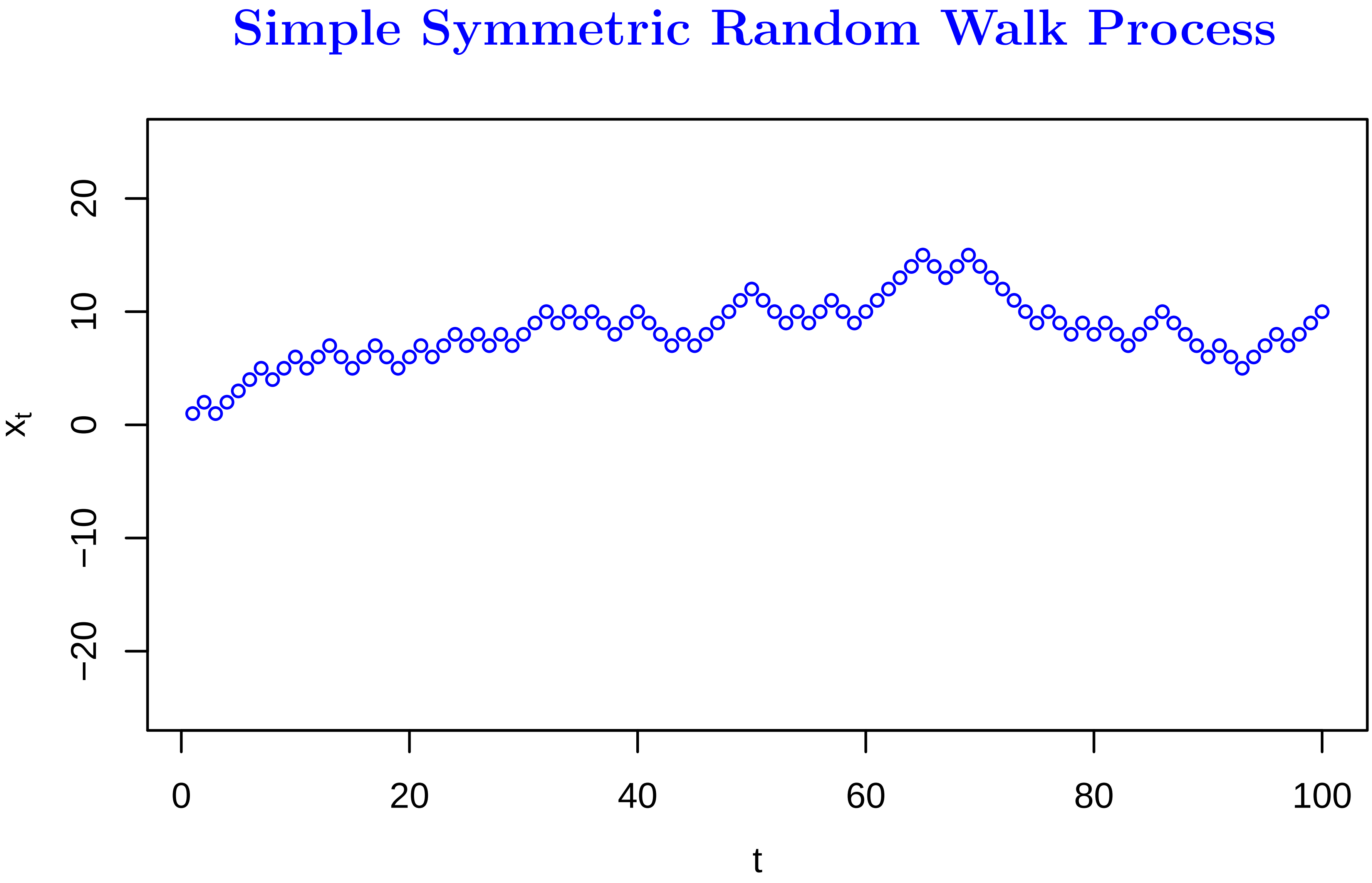


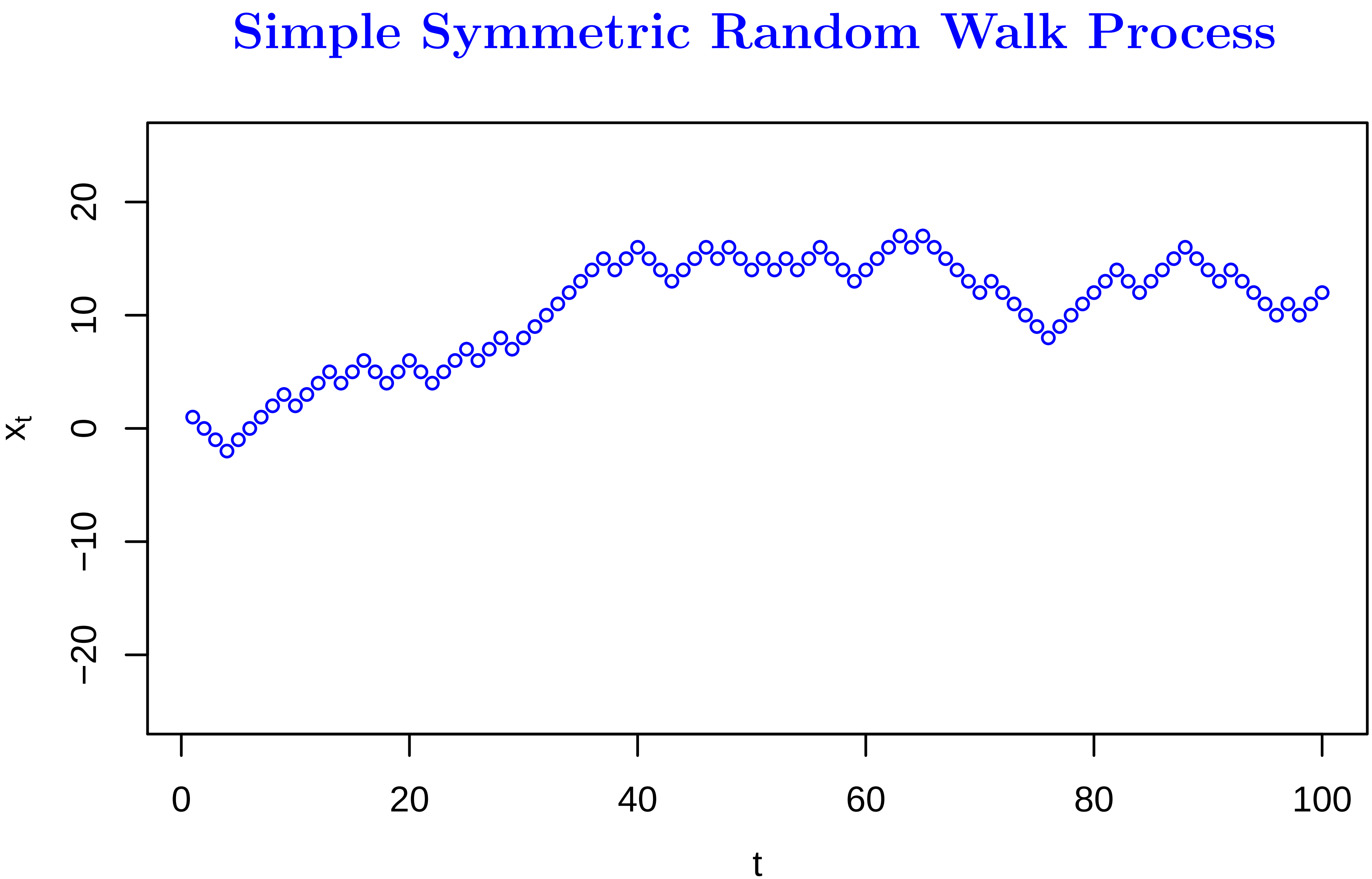
Gaussian Random Walk Process



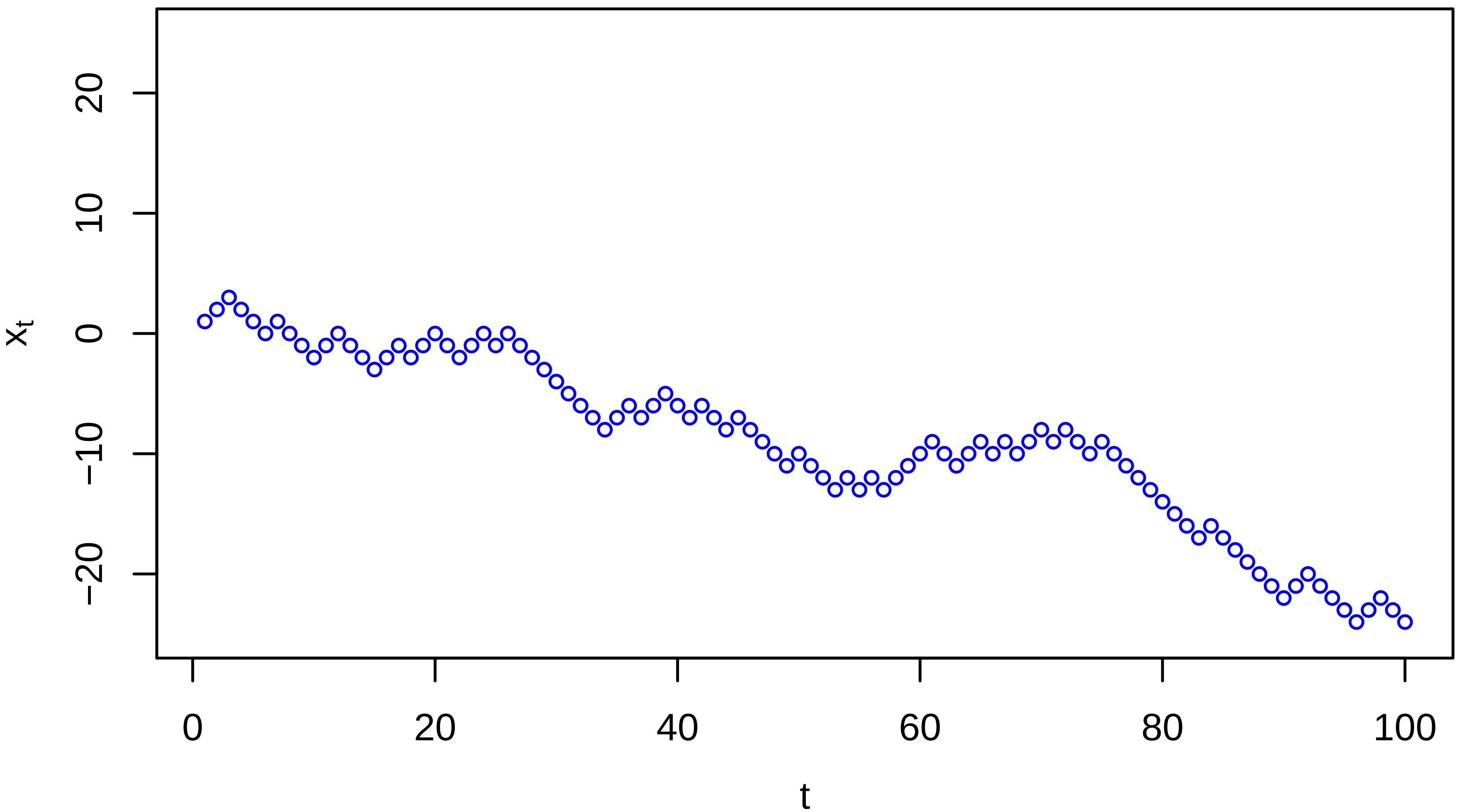
Gaussian Random Walk Process




Simple Symmetric Random Walk Process
x_t
20
10
0
−10
−20
0
20
40
60
80
100
t


Simple Symmetric Random Walk Process
20
10
0
−10
−20
x_t
0
20
40
60
80
100
t

Simple Symmetric Random Walk Process



Stationary Models: I

- let $\{X_t : t \in \mathbb{Z}\}$ be a time series (stochastic process) such that $E\{X_t^2\} < \infty$ for all t, where $\mathbb{Z} \stackrel{\text{def}}{=} \{\ldots, -1, 0, 1, \ldots\}$
- mean function for $\{X_t\}$ is defined to be
$$\mu_X(t) = E\{X_t\}$$
- covariance function for $\{X_t\}$ is defined to be
$$\gamma_X(r, s) = \text{cov}\,\{X_r, X_s\} = E\{[X_r - \mu_X(r)][X_s - \mu_X(s)]\}$$
for all integers r and s
- $\{X_t\}$ is said to be (weakly) stationary if
  1. $\mu_X(t)$ is independent of t
  2. $\gamma_X(r, s)$ depends on just $r - s$, i.e., the lag (spacing in time) between X_r & X_s, but not on r & s (thus $\gamma_X(t + h, t)$ is independent of t for each lag h)

Stationary Models: II

- stationary says that first- and second-order properties of
$$X_t, X_{t+1}, \ldots, X_{t+m} \qquad (*)$$
are the same as those for
$$X_{t+h}, X_{t+h+1}, \ldots, X_{t+h+m} \qquad (**)$$
for all $t \in \mathbb{Z}$, all $m \geq 0$ and all $h \in \mathbb{Z}$
- weak stationarity is also called second-order stationarity, covariance stationarity and wide-sense stationarity
- another notion is strict stationarity, which says that the RVs in $(*)$ have the same joint distribution as the RVs in $(**)$
- if $\{X_t\}$ is strictly stationary and $E\{X_t^2\} < \infty$, then $\{X_t\}$ is also weakly stationary (converse not true in general)
- henceforth will take 'stationary' to mean 'weakly stationary'

Autocovariance and Autocorrelation Functions: I

- since $\gamma_X(t+h,t)$ depends on h but not t, covariance function for a stationary process can be taken to be a function of one variable (the lag h)
- accordingly define the autocovariance function $\gamma_X(\cdot)$ (ACVF) via
$$\gamma_X(h) = \gamma_X(t+h,t) = \operatorname{cov}\{X_{t+h}, X_t\}$$
- note: $\gamma_X(0) = \operatorname{cov}\{X_t, X_t\} = \operatorname{var}\{X_t\}$ (the process variance)
- ACVF is symmetric about $h=0$: letting $t' = t+h$, have

$$\gamma_X(-h) = \operatorname{cov}\{X_{t'-h}, X_{t'}\} = \operatorname{cov}\{X_t, X_{t+h}\} = \operatorname{cov}\{X_{t+h}, X_t\} = \gamma_X(h)$$

since $\operatorname{cov}\{U,V\} = \operatorname{cov}\{V,U\}$

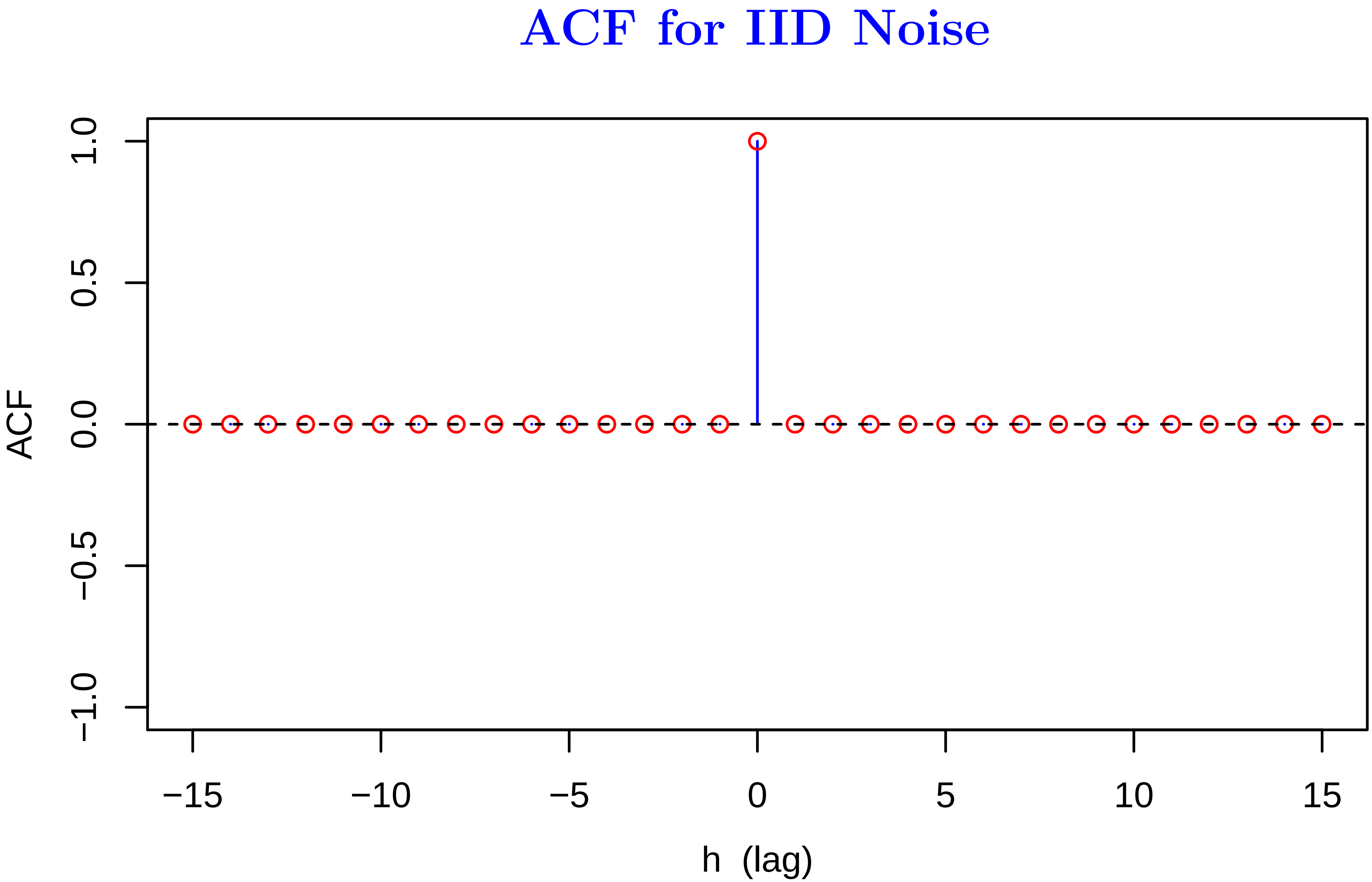
Autocovariance and Autocorrelation Functions: II

- when $\gamma_X(0) > 0$, define the corresponding autocorrelation function $\rho_X(\cdot)$ (ACF) via

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \frac{\operatorname{cov}\{X_{t+h}, X_t\}}{\operatorname{var}\{X_t\}} = \frac{\operatorname{cov}\{X_{t+h}, X_t\}}{\sqrt{\operatorname{var}\{X_{t+h}\} \operatorname{var}\{X_t\}}}$$

Example – IID Noise: I

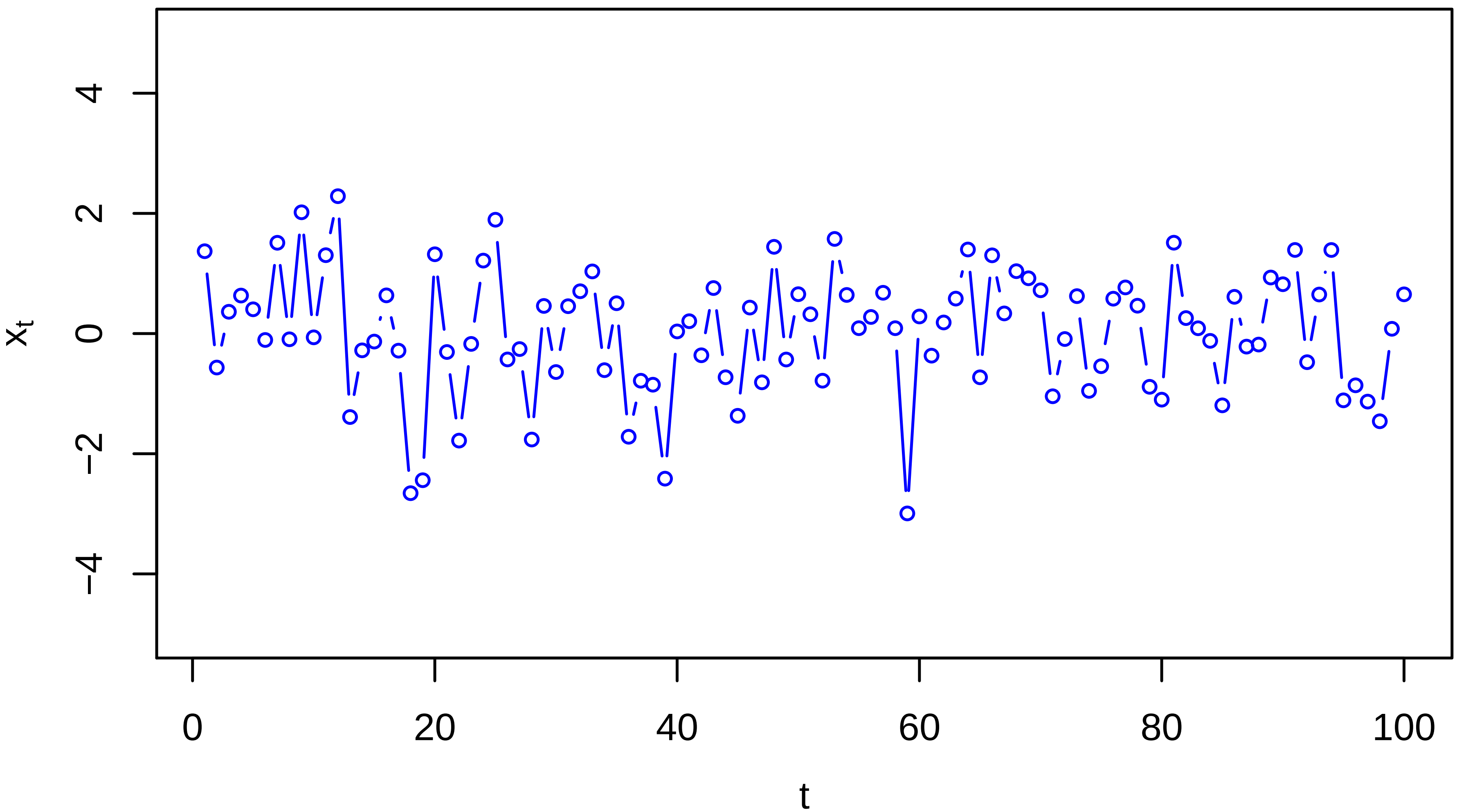
- if $\{X_t\}$ is IID noise such that $E\{X_t^2\} < \infty$, then $E\{X_t\} = \mu$ (a constant independent of t) and $\operatorname{cov}\{X_{t+h}, X_t\} = 0$ for all $h \neq 0$
- hence IID noise with a finite variance, say σ^2, is a stationary process with ACVF
$$\gamma_X(h) = \begin{cases} \sigma^2, & h = 0, \\ 0, & \text{otherwise,} \end{cases}$$
and, when $\sigma^2 > 0$, ACF
$$\rho_X(h) = \begin{cases} 1, & h = 0, \\ 0, & \text{otherwise} \end{cases}$$
- use notation $\{X_t\} \sim \mathrm{IID}(\mu, \sigma^2)$ to denote IID noise with finite variance σ^2


ACF for IID Noise
ACF
1.0
0.5
0.0
−0.5
−1.0
−15
−10
−5
0
5
10
15
h (lag)

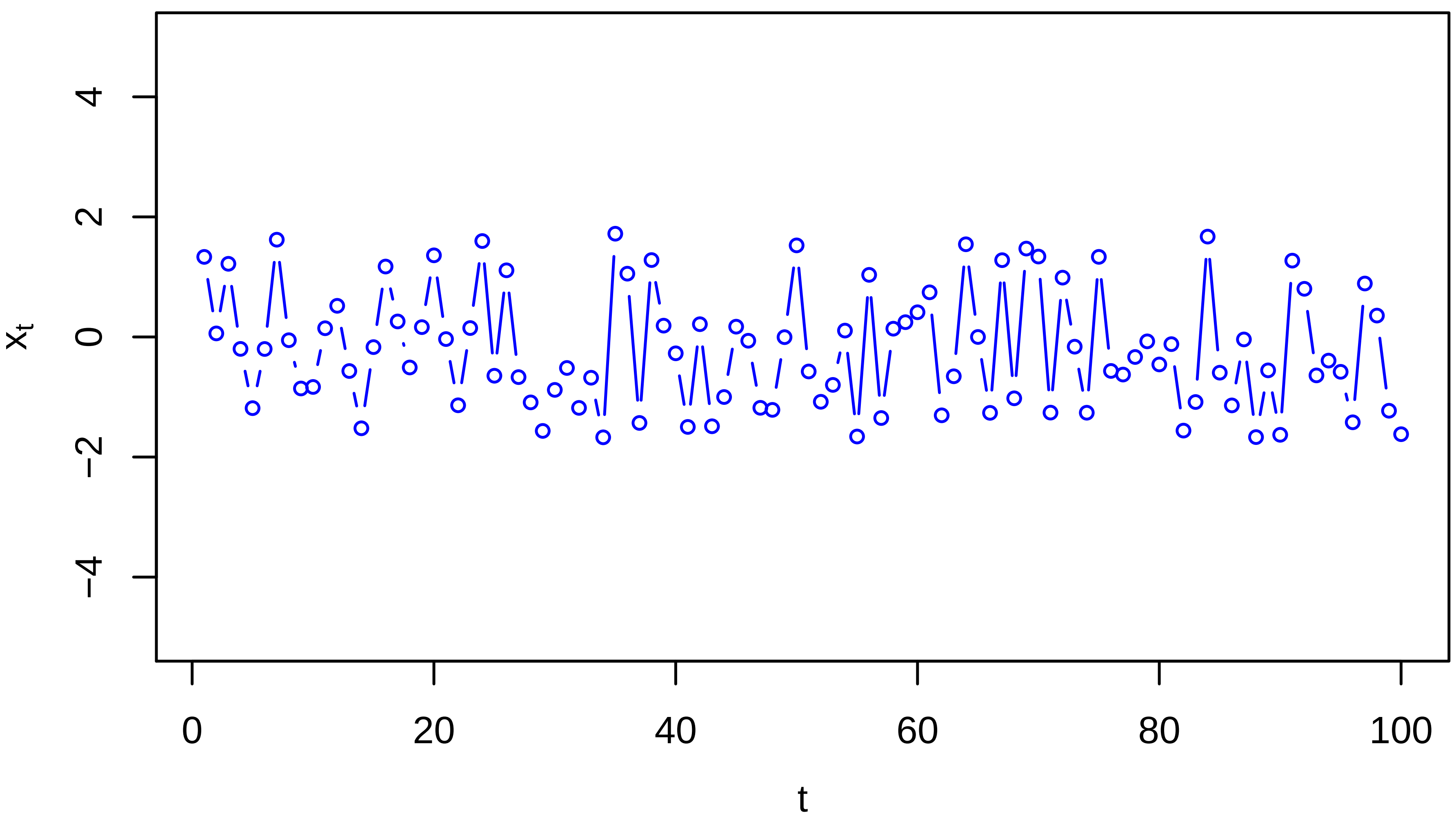
Example – IID Noise: II

- following four plots show examples of IID(0,1) noise with these distributions:
  1. standard normal (Gaussian)
  2. uniform over interval $[-\sqrt{3}, \sqrt{3}]$
  3. double exponential with probability density function (PDF) given by $f(x) = \exp(-|x|\sqrt{2})/\sqrt{2}$
  4. discrete distribution that assumes values -5, 0 and 5 with probabilities 0.02, 0.96 and 0.02, respectively

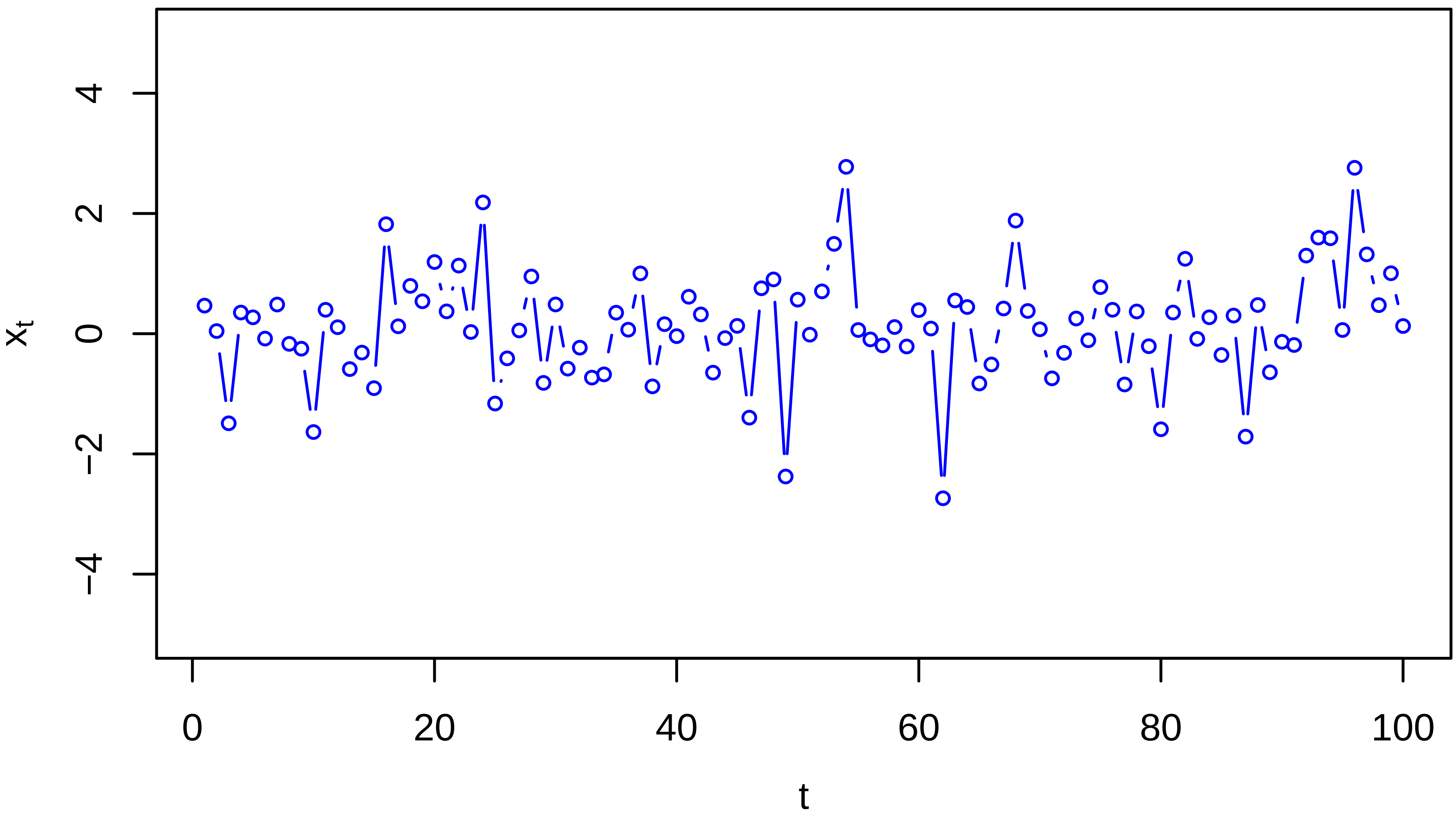
IID(0,1) Noise from Gaussian Distribution



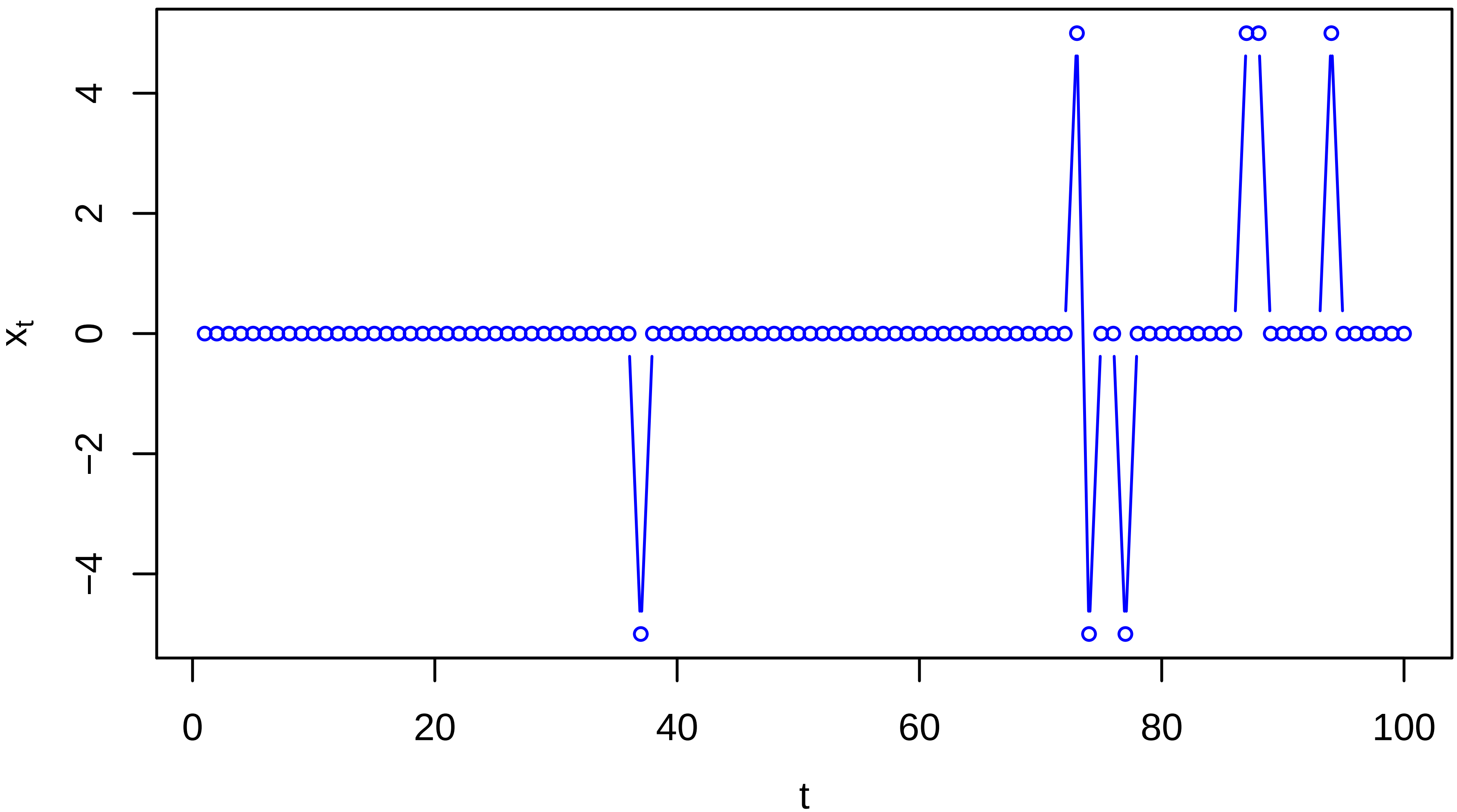
IID(0,1) Noise from Uniform Distribution



IID(0,1) Noise from Double Exponential Distribution



IID(0,1) Noise from Discrete Distribution



Example – White Noise: I

- by definition $\{X_t\}$ is a white noise process if its RVs are uncorrelated (i.e., $\text{cov}\,\{X_r, X_s\} = 0$ as long as $r \neq s$) and have the same mean μ and the same variance σ^2 (assumed to be finite)
- $\{X_t\} \sim \text{WN}(\mu, \sigma^2)$ denotes a white noise process
- a $\text{WN}(\mu, \sigma^2)$ process is a stationary process with ACVF

$$\gamma_X(h) = \begin{cases} \sigma^2, & h = 0, \\ 0, & \text{otherwise}, \end{cases}$$

and, when $\sigma^2 > 0$, ACF

$$\rho_X(h) = \begin{cases} 1, & h = 0, \\ 0, & \text{otherwise} \end{cases}$$

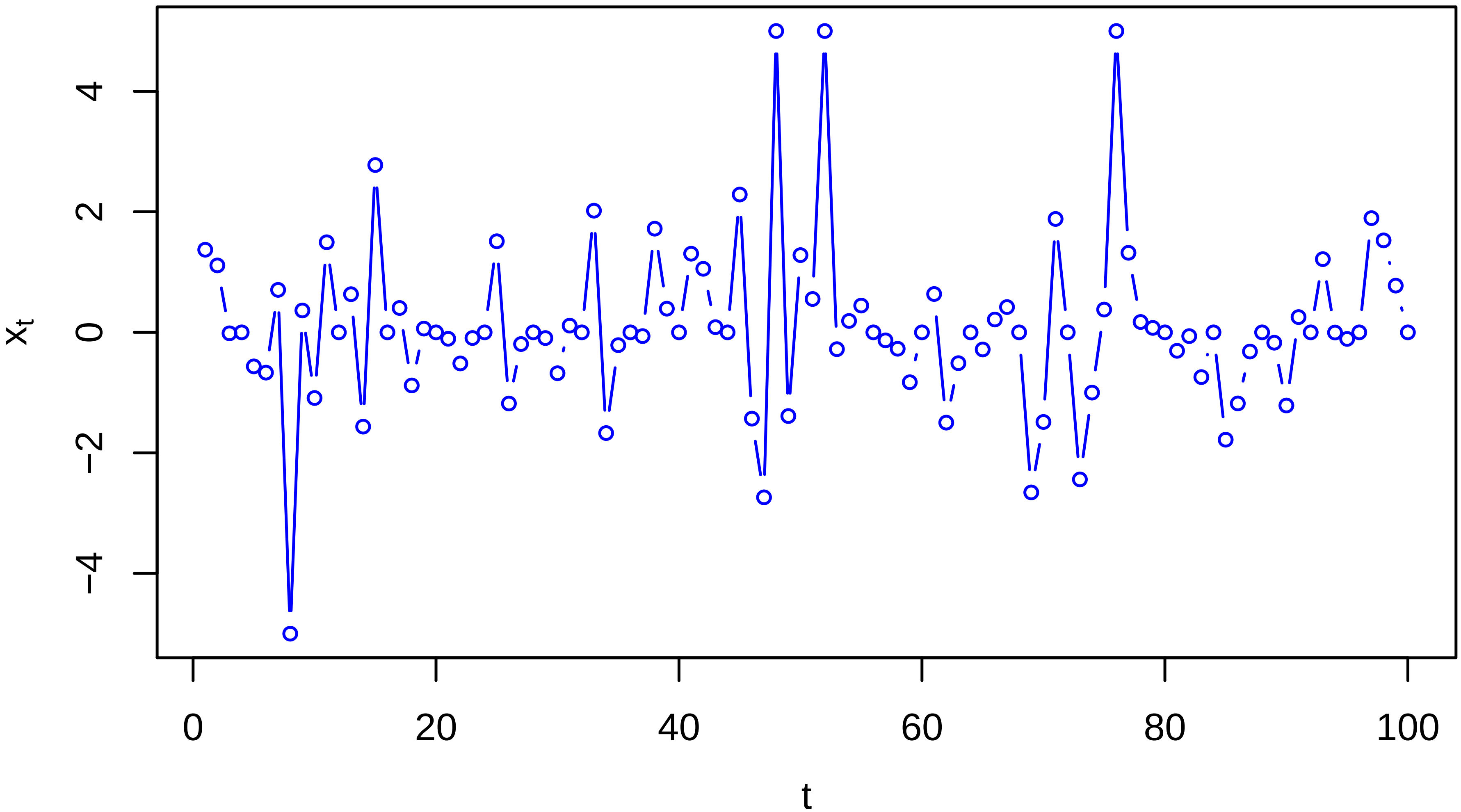
(same ACVF and ACF as for IID noise)

Example – White Noise: II

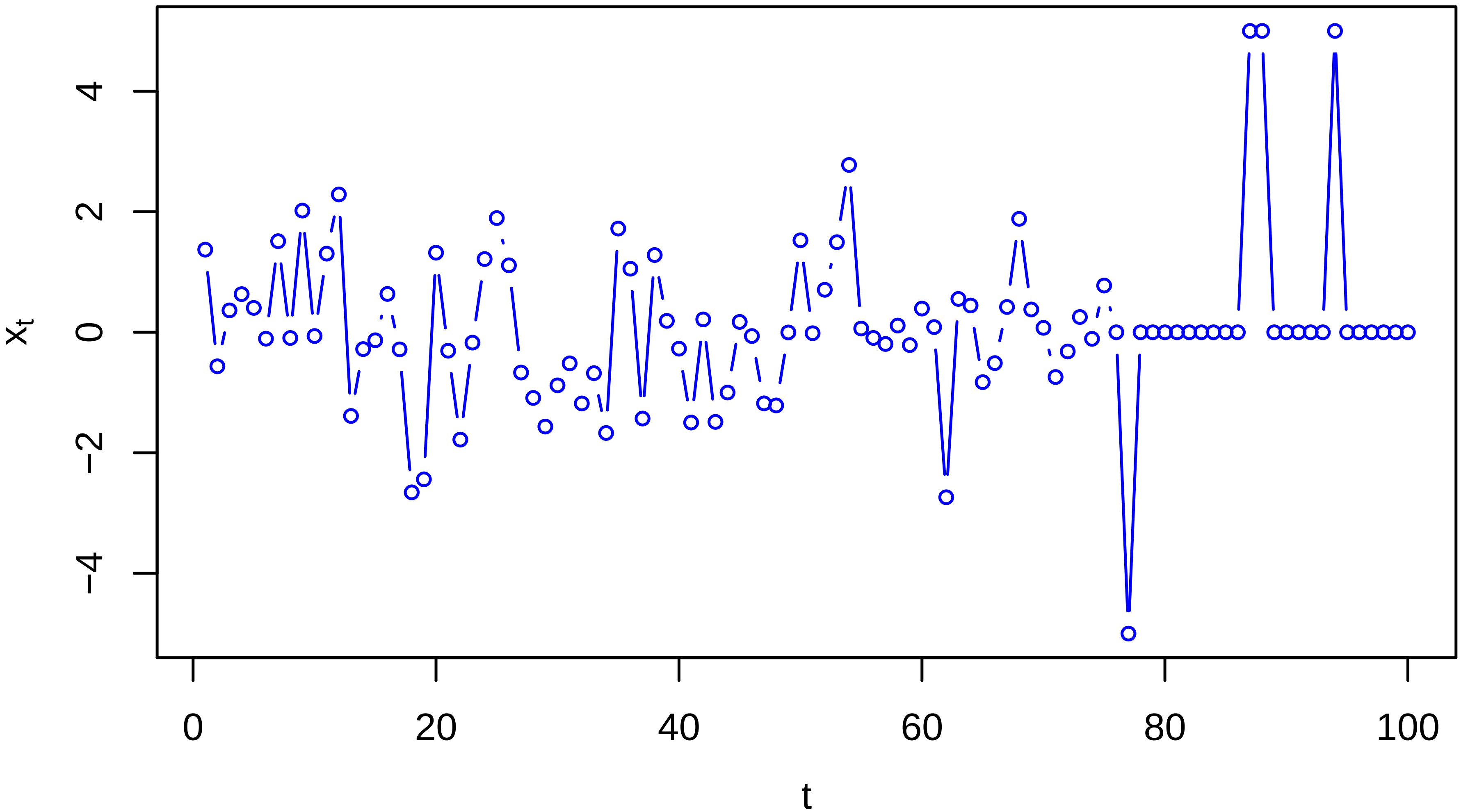
- every IID(μ, σ^2) process is also a WN(μ, σ^2) process (hence examples of Gaussian, uniform, double exponental and discrete IID(0,1) noise also serve as examples of WN(0,1) processes)
- converse is not true, as the following two examples show:
  1. mishmash white noise: choose at random from RV with
     - Gaussian distribution if $t = 1, 5, 9, \ldots$
     - uniform distribution if $t = 2, 6, 10, \ldots$
     - double exponental distribution if $t = 3, 7, 11, \ldots$
     - discrete distribution if $t = 4, 8, 12, \ldots$

     (each assumed to have zero mean and unit variance)
  2. blocky white noise: paste together blocks from Gaussian, uniform, double exponental and discrete IID(0,1) time series

Mishmash White Noise



Blocky White Noise



Example – Random Walk: I

- suppose that $\{X_t\}$ is IID$(0,\sigma^2)$ noise, and construct corresponding random walk process:

$$S_t = \sum_{u=1}^{t} X_u, \quad t \geq 1$$

- since

$$E\{S_t\} = \sum_{u=1}^{t} E\{X_u\} = 0,$$

first of two conditions for stationarity holds

Example – Random Walk: II

- in order for second condition to hold, the variance of $\{S_t\}$ cannot depend on t; however, due to independence,

$$\mathrm{var}\,\{S_t\} = \mathrm{var}\left\{\sum_{u=1}^{t} X_u\right\} = \sum_{u=1}^{t} \mathrm{var}\,\{X_u\} = t\sigma^2,$$

  which *does* depend on t unless $\sigma^2 = 0$

- conclusion: $\{S_t\}$ need *not* be a stationary process, but it is sometimes called *intrinsically stationary of unit order* because its first-order backward differences $S_t - S_{t-1} = X_t$ form a stationary process

Example – Random Walk: III

- recall the linearity property of covariances:

$$\text{cov}\{aX + bY + c, Z\} = a\,\text{cov}\{X, Z\} + b\,\text{cov}\{Y, Z\},$$

where X, Y & Z are RVs such that $E\{X^2\}$, $E\{Y^2\}$ & $E\{Z^2\}$ are all finite, and a, b & c are arbitrary real-valued constants

- covariance function for $\{S_t\}$ is such that, for $t \geq 1$ and $h \geq 1$,

$$\begin{aligned}
\gamma_S(t+h, t) &= \text{cov}\{S_{t+h}, S_t\} \\
&= \text{cov}\{S_t + X_{t+1} + \cdots + X_{t+h}, S_t\} \\
&= \text{cov}\{S_t, S_t\} + \sum_{l=1}^{h} \text{cov}\{X_{t+l}, S_t\} \\
&= t\sigma^2 + \sum_{l=1}^{h} \text{cov}\{X_{t+l}, X_1 + \cdots + X_t\} = t\sigma^2
\end{aligned}$$

Example – First-Order Moving Average Process: I

- suppose that $\{Z_t : t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma^2)$, and define

$$X_t = Z_t + \theta Z_{t-1}, \quad t \in \mathbb{Z},$$

  where θ is a real-valued constant

- since

$$E\{X_t\} = E\{Z_t\} + \theta E\{Z_{t-1}\} = 0,$$

  first of two conditions for stationarity holds

- to establish stationarity, we need to show the second condition holds, namely, that $\gamma_X(t+h, t)$ does not depend on t

Example – First-Order Moving Average Process: II

- now

$$\begin{aligned}\gamma_X(t+h,t) &= \text{cov}\left\{Z_{t+h}+\theta Z_{t+h-1}, Z_t+\theta Z_{t-1}\right\} \\ &= \text{cov}\left\{Z_{t+h}, Z_t\right\}+\theta\,\text{cov}\left\{Z_{t+h}, Z_{t-1}\right\} \\ &\quad +\theta\,\text{cov}\left\{Z_{t+h-1}, Z_t\right\}+\theta^2\,\text{cov}\left\{Z_{t+h-1}, Z_{t-1}\right\}\end{aligned}$$

- when $h=0$, the 1st and 4th cov's are equal to σ^2, while the 2nd and 3rd are zero, yielding $\gamma_X(t,t) = \text{var}\{X_t\} = \sigma^2(1+\theta^2)$
- when $h=1$, the 3rd cov is equal to σ^2, while the rest are all zero, yielding $\gamma_X(t+1,t) = \theta\sigma^2$
- when $h=-1$, the 2nd cov is equal to σ^2, while the rest are all zero, yielding $\gamma_X(t-1,t) = \theta\sigma^2$
- when $|h| \geq 2$, all four cov's are zero, yielding $\gamma_X(t+h,t) = 0$ for $h \neq 0, 1$ or -1

Example – First-Order Moving Average Process: III

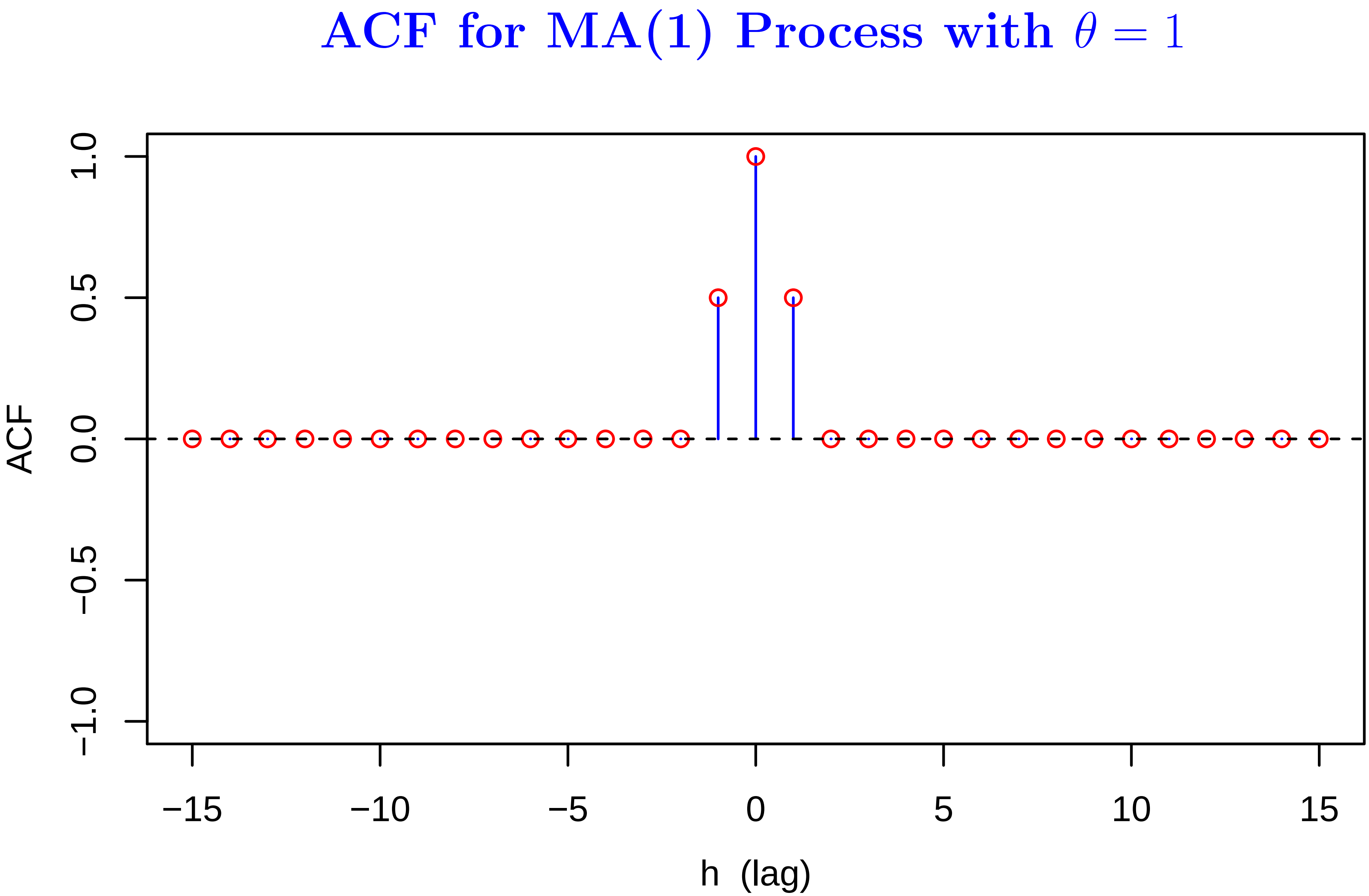
- since $\gamma_X(t+h,t)$ is independent of t for all h, the process $\{X_t\}$ is stationary with ACVF

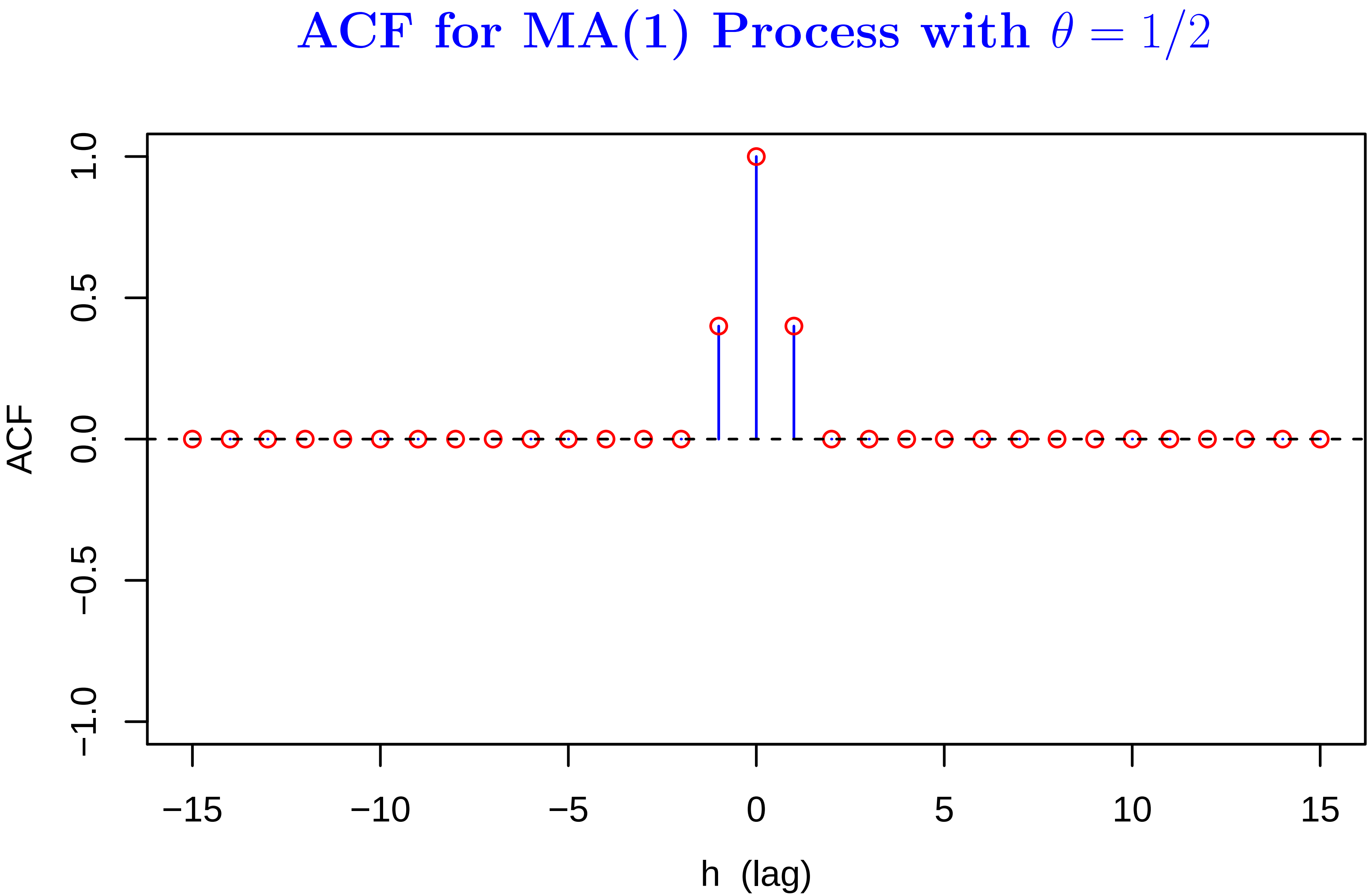
$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2), & h = 0, \\ \sigma^2\theta, & h = \pm 1, \\ 0, & \text{otherwise,} \end{cases}$$

and, when $\sigma^2 > 0$, ACF

$$\rho_X(h) = \begin{cases} 1, & h = 0, \\ \theta/(1+\theta^2), & h = \pm 1, \\ 0, & \text{otherwise} \end{cases}$$

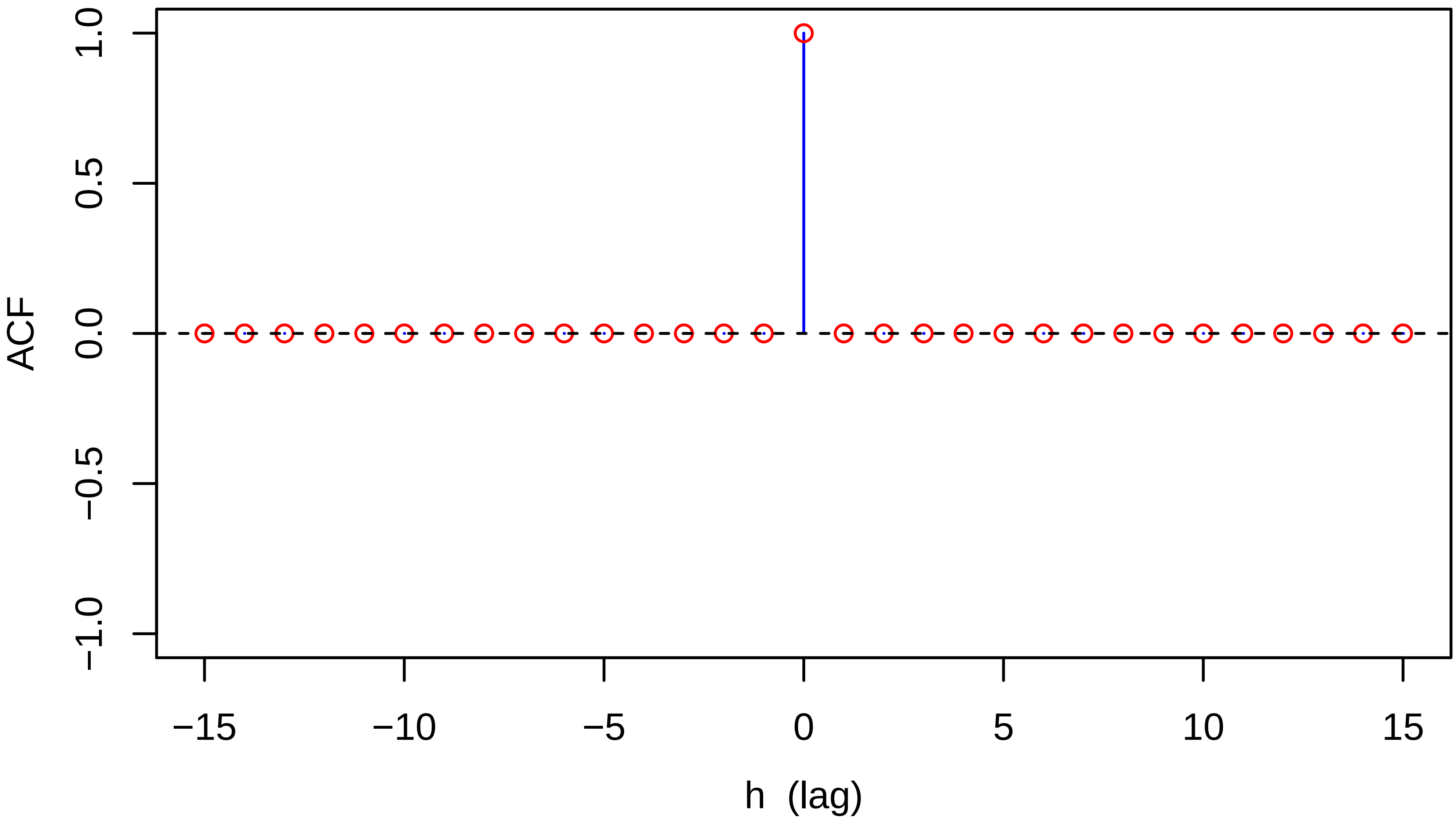
- $\{X_t\}$ is called a first-order moving average or MA(1) process


ACF for MA(1) Process with $\theta = 1$
ACF
1.0
0.5
0.0
−0.5
−1.0
−15
−10
−5
0
5
10
15
h (lag)

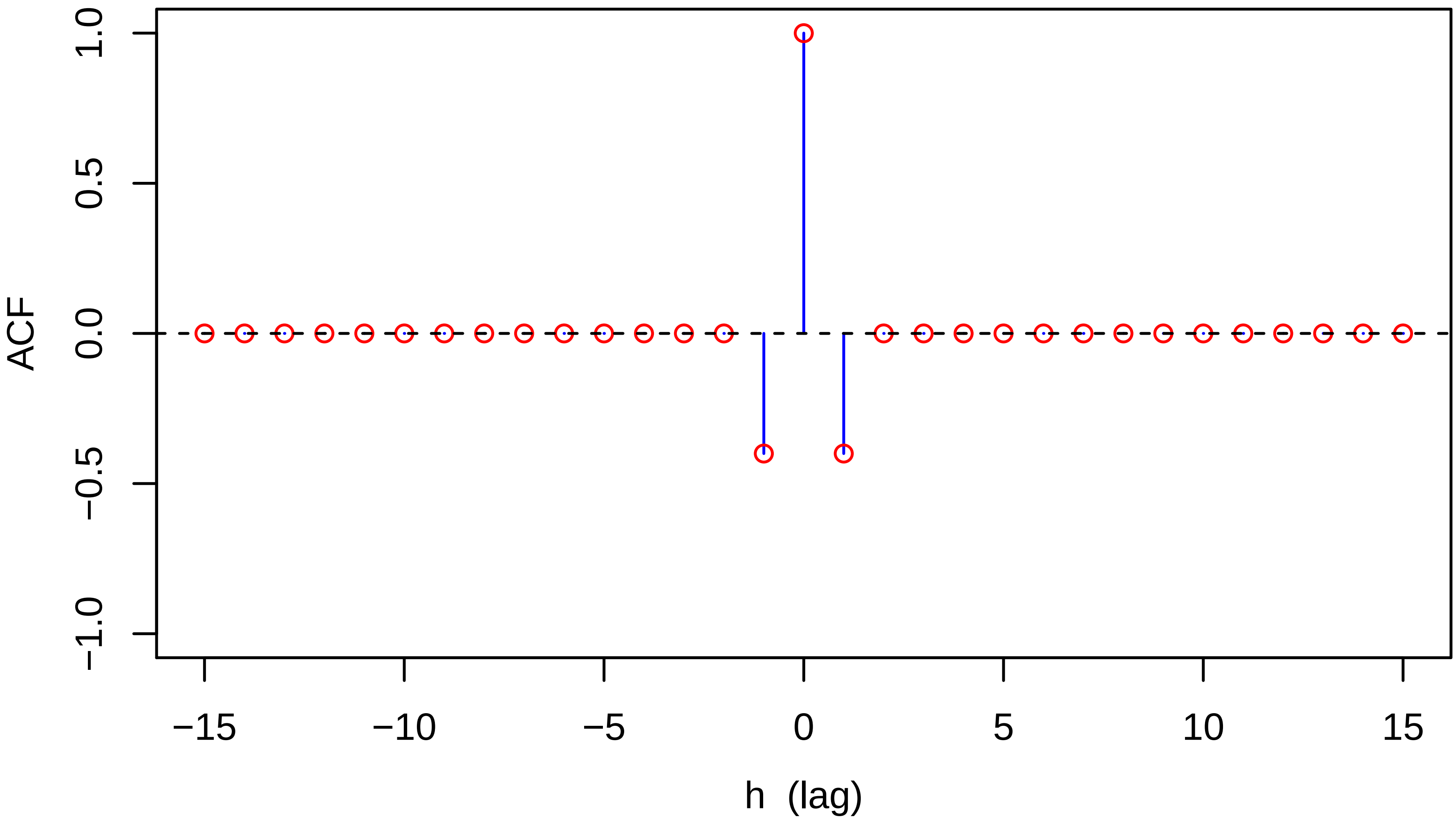


ACF for MA(1) Process with $\theta = 1/2$
ACF
1.0
0.5
0.0
−0.5
−1.0
−15
−10
−5
0
5
10
15
h (lag)

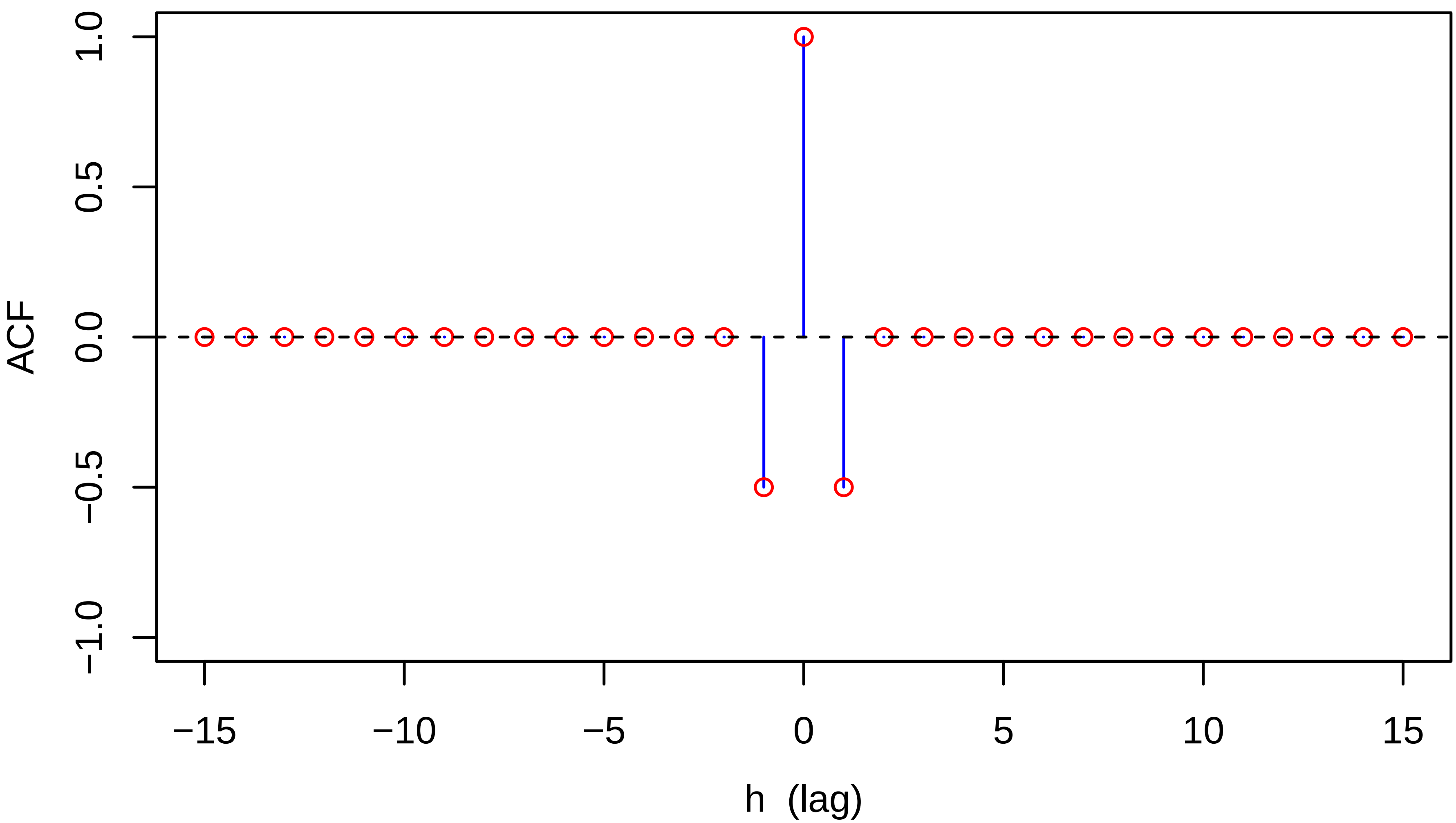
ACF for MA(1) Process with $\theta = 0$



ACF for MA(1) Process with $\theta = -1/2$



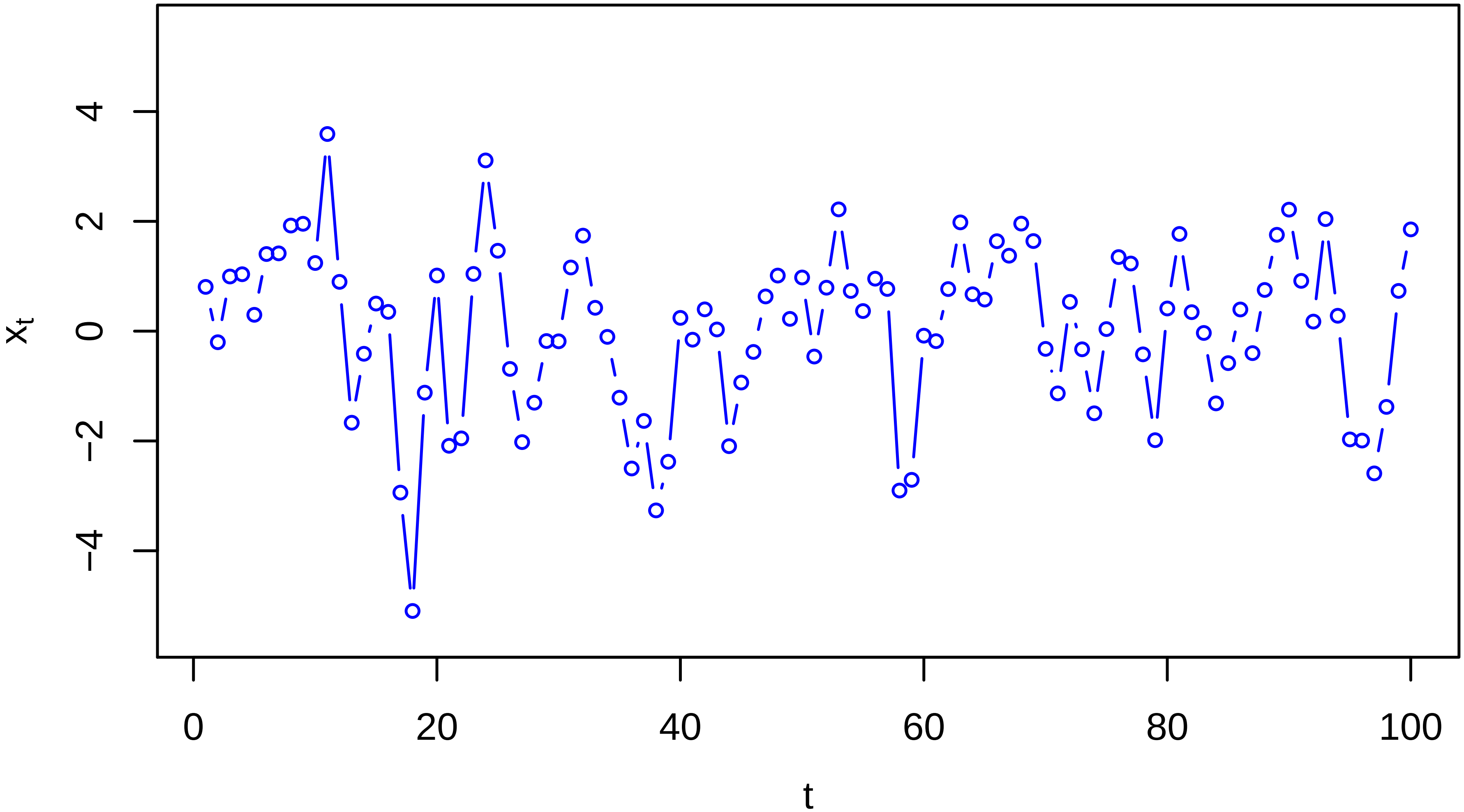
ACF for MA(1) Process with $\theta = -1$



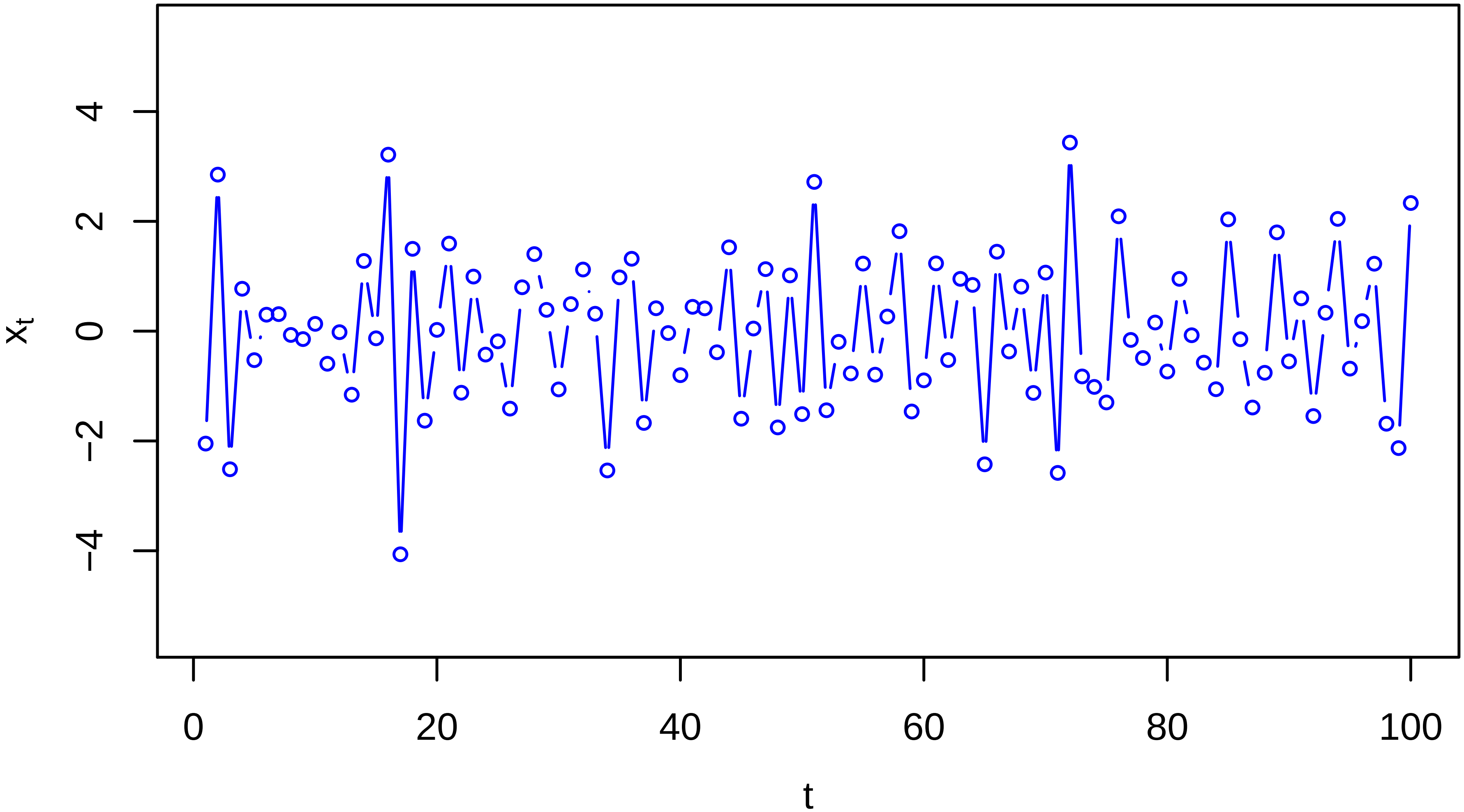
Example – First-Order Moving Average Process: IV

- as examples, generate realizations of MA(1) process with $\theta = 1$ and $\theta = -1$ using WN(0,1) processes with Gaussian, uniform, double exponential and discrete distributions

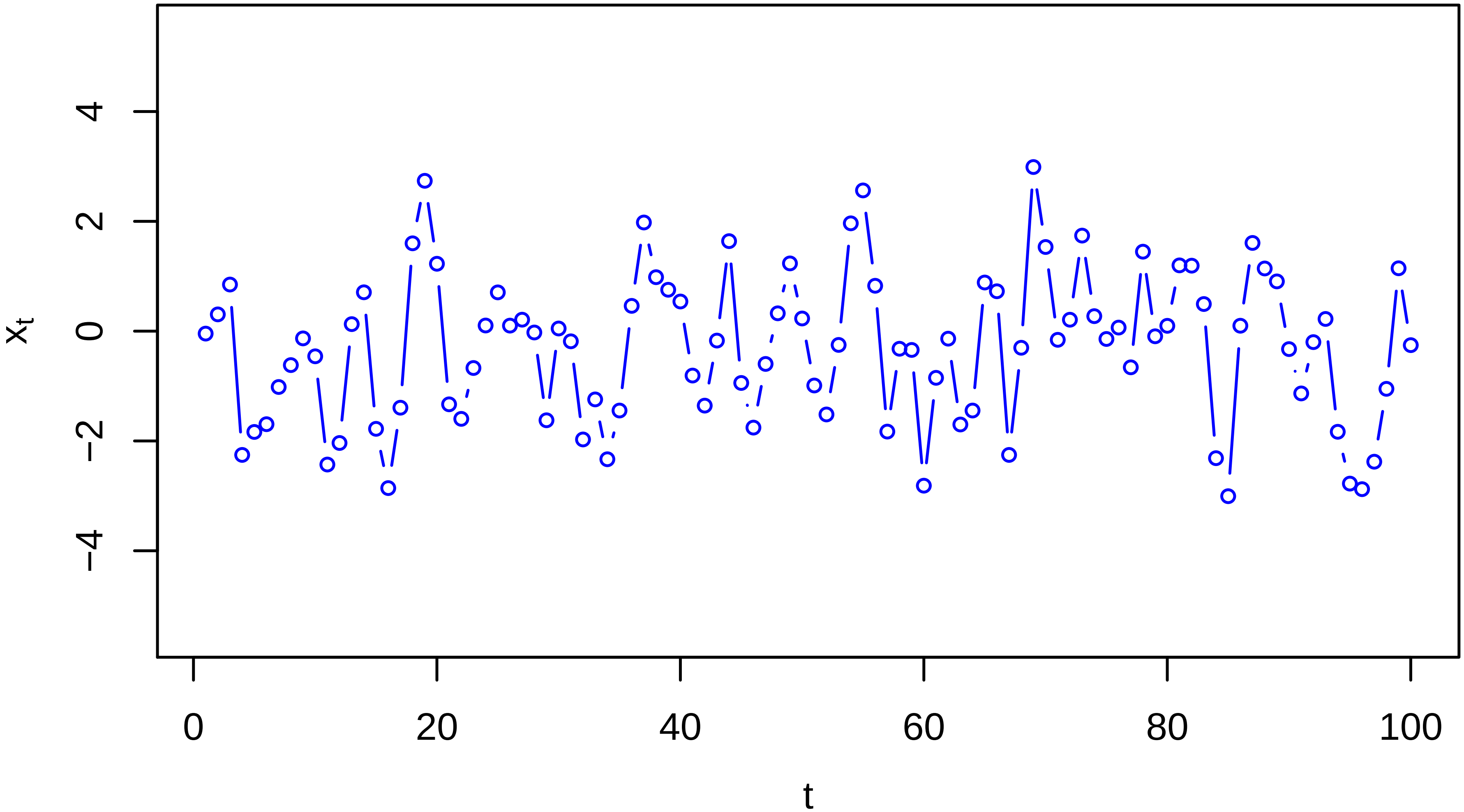
$\theta = 1$ MA(1) x_t from Gaussian WN(0,1)



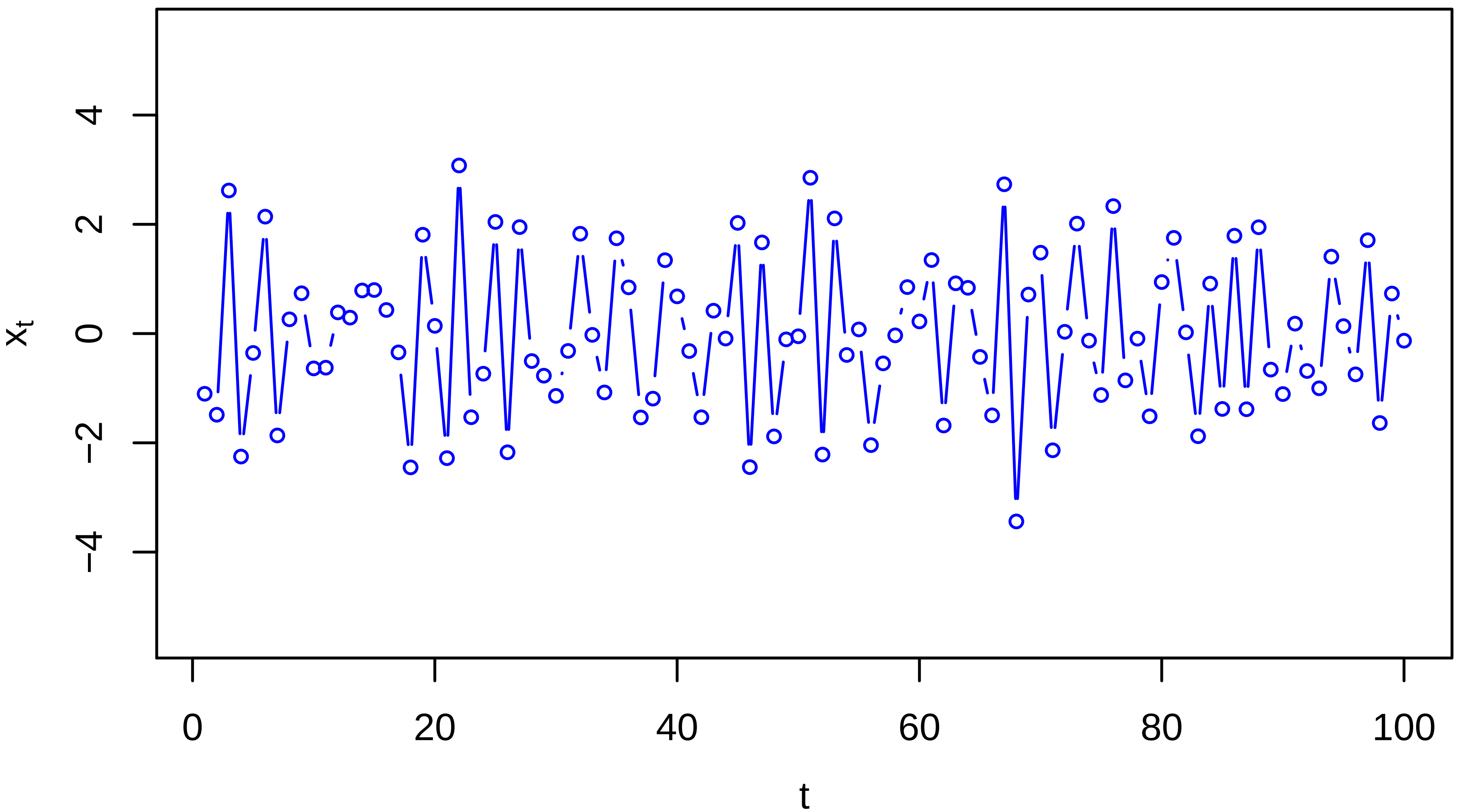
$\theta = -1$ MA(1) x_t from Gaussian WN(0,1)



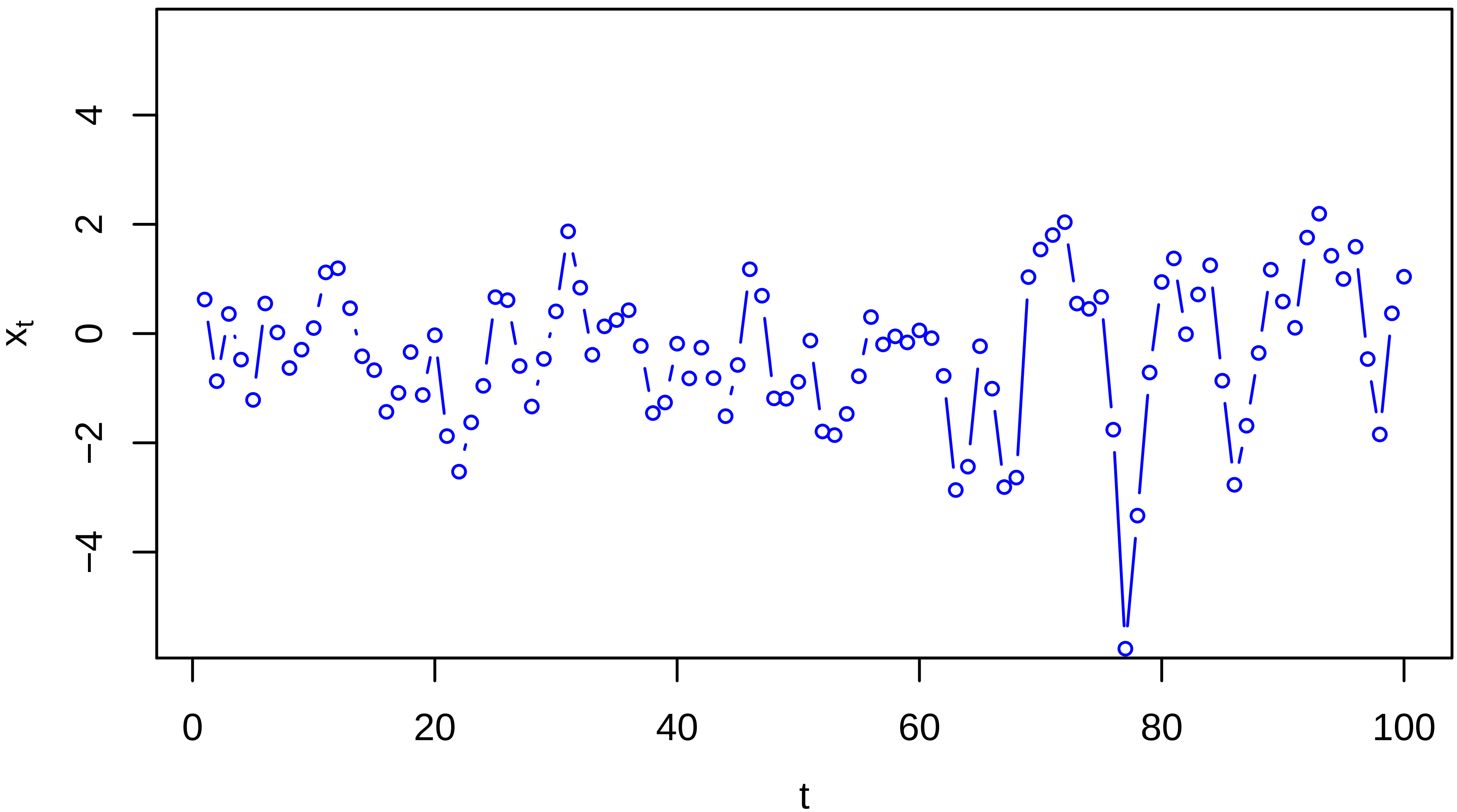
$\theta = 1$ **MA(1)** x_t **from Uniform WN(0,1)**



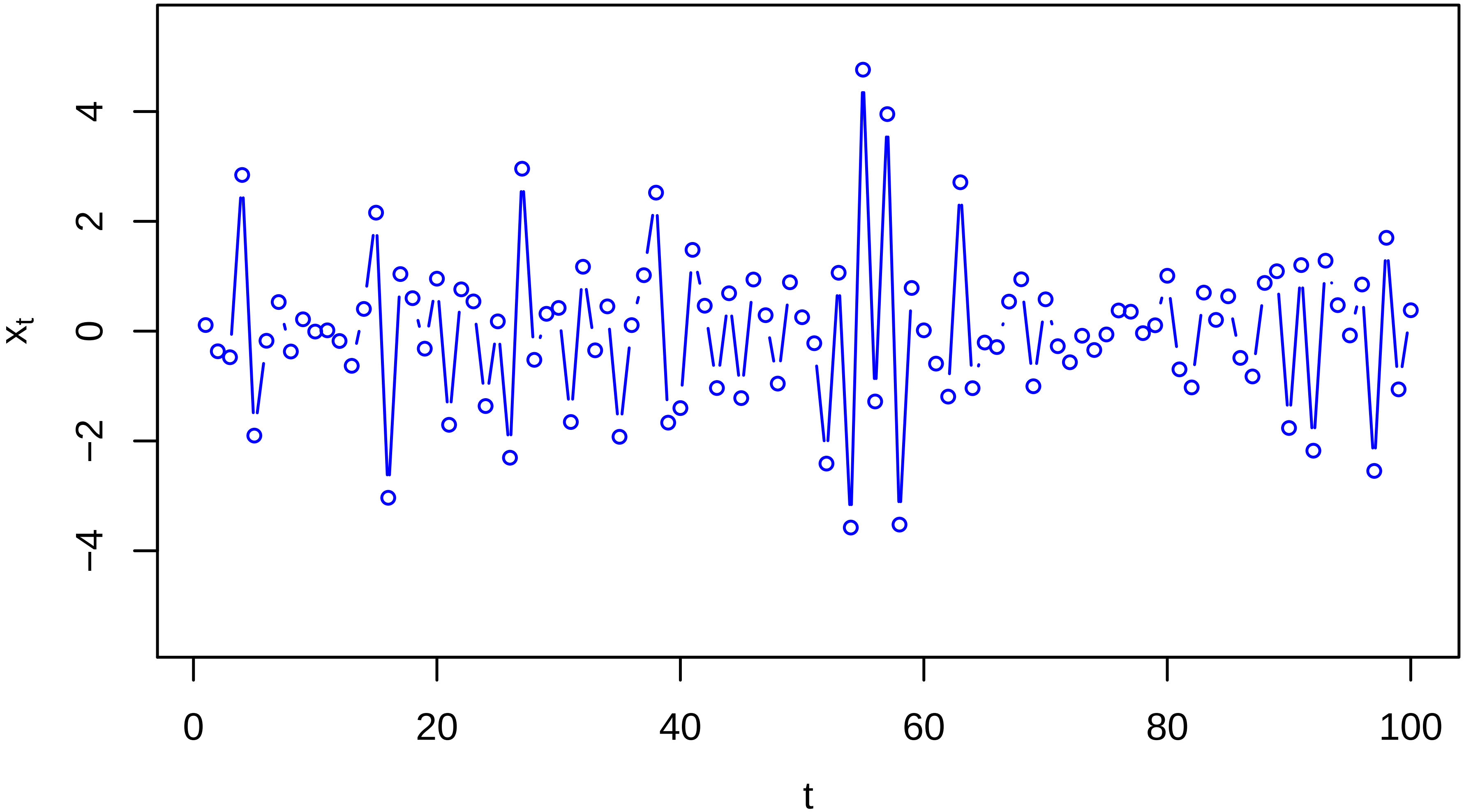
$\theta = -1$ MA(1) x_t from Uniform WN(0,1)

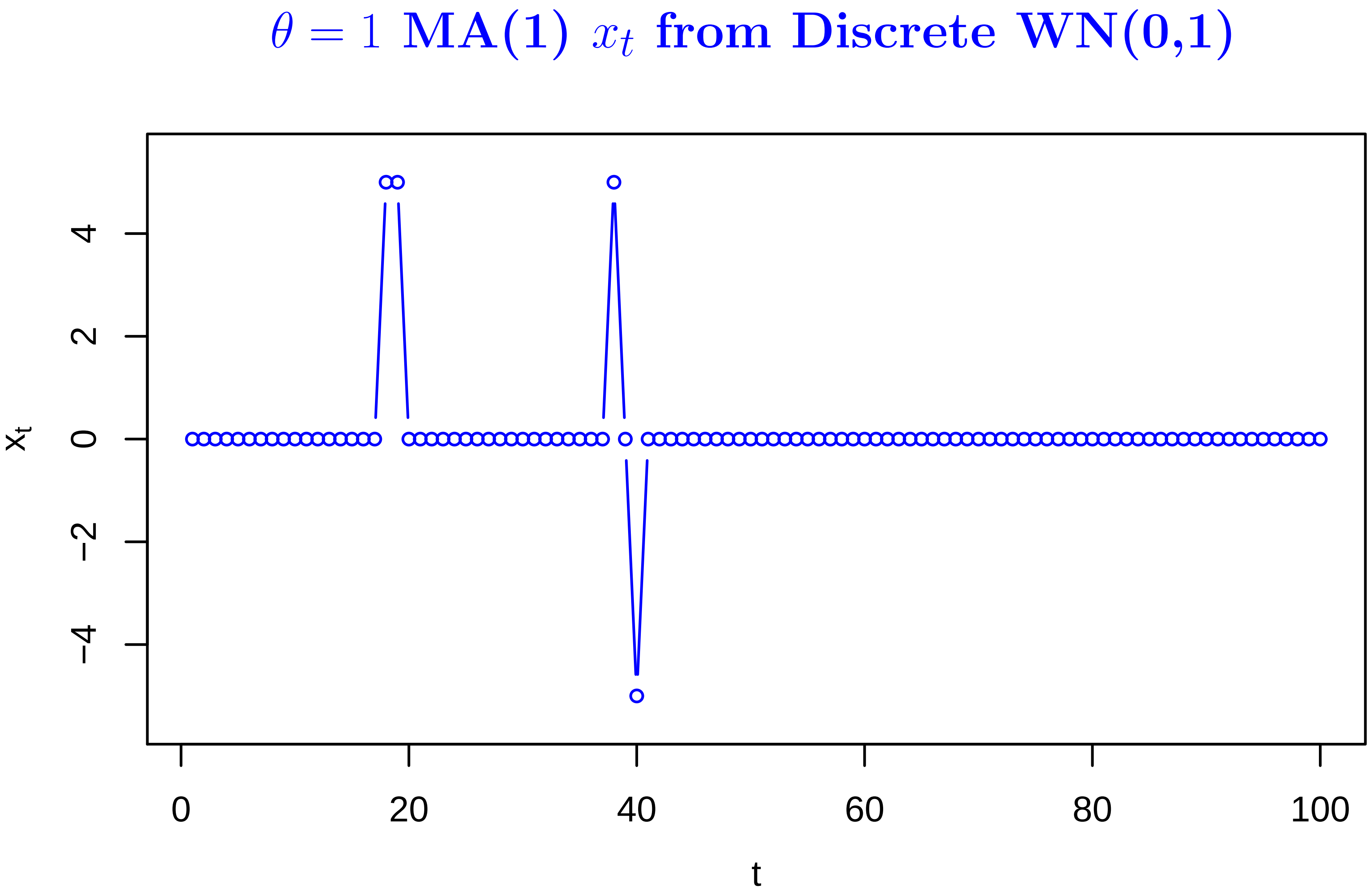


$\theta = 1$ MA(1) x_t from Double Exponential WN(0,1)

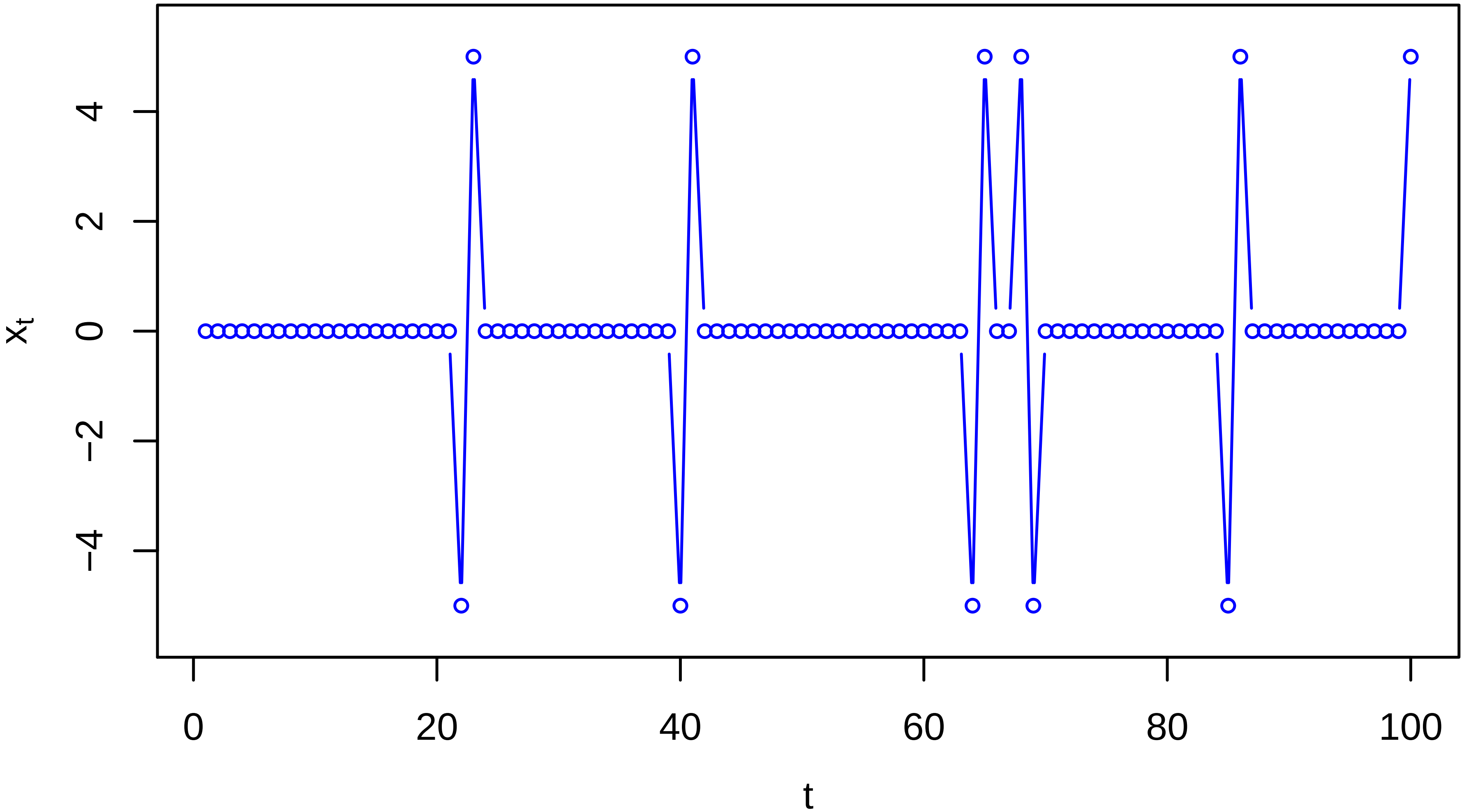


$\theta = -1$ **MA(1)** x_t **from Double Exponential WN(0,1)**




$\theta = 1$ MA(1) x_t from Discrete WN(0,1)
x_t
4
2
0
−2
−4
0
20
40
60
80
100
t

$\theta = -1$ MA(1) x_t from Discrete WN(0,1)



Example – First-Order Autoregressive Process: I

- assume there exists a stationary process $\{X_t\}$ satisfying

$$X_t = \mu + \phi(X_{t-1} - \mu) + Z_t, \quad t \in \mathbb{Z},$$

where μ and ϕ are real-valued constants with $|\phi| < 1$, and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ with cov $\{X_s, Z_t\} = 0$ for all $s < t$ (will verify existence of such a process later on in course)

- since

$$E\{X_t\} = \mu + \phi(E\{X_{t-1}\} - \mu) + E\{Z_t\}$$

implies

$$E\{X_t\} - \mu = \phi(E\{X_t\} - \mu)$$

we can conclude that $\mu = E\{X_t\}$ if $\phi \neq 0$; if in fact $\phi = 0$, then $\mu = E\{X_t\}$ follows immediately from $X_t = \mu + Z_t$

Example – First-Order Autoregressive Process: II

- to find the ACVF, subtract μ from both sides of

$$X_t = \mu + \phi(X_{t-1} - \mu) + Z_t,$$

multiply each side by $X_{t-h} - \mu$ for $h > 0$ and take expectations

$$E\{(X_t - \mu)(X_{t-h} - \mu)\} = \phi E\{(X_{t-1} - \mu)(X_{t-h} - \mu)\} + E\{(X_{t-h} - \mu)Z_t\}$$

to get

$$\text{cov}\,\{X_t, X_{t-h}\} = \phi\,\text{cov}\,\{X_{t-1}, X_{t-h}\} + \text{cov}\,\{X_{t-h}, Z_t\};$$

however, since $\text{cov}\,\{X_{t-h}, Z_t\} = 0$ by assumption, we have

$$\text{cov}\,\{X_t, X_{t-h}\} = \phi\,\text{cov}\,\{X_{t-1}, X_{t-h}\},$$

i.e.,

$$\gamma_X(h) = \phi\gamma_X(h-1)$$

BD: 15; CC: 66; SS: 76–78

Example – First-Order Autoregressive Process: III

- repetitive use of $\gamma_X(h) = \phi\gamma_X(h-1)$ yields

$$\gamma_X(h) = \phi\gamma_X(h-1) = \phi^2\gamma_X(h-2) = \cdots = \phi^h\gamma_X(0)$$

- since ACVF is symmetric about $h = 0$, must have

$$\gamma_X(h) = \phi^{|h|}\gamma_X(0)$$

  for all $h \in \mathbb{Z}$

- assuming $\gamma_X(0) > 0$, corresponding ACF is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \phi^{|h|}, \quad h \in \mathbb{Z}$$

Example – First-Order Autoregressive Process: IV

- in addition, we have

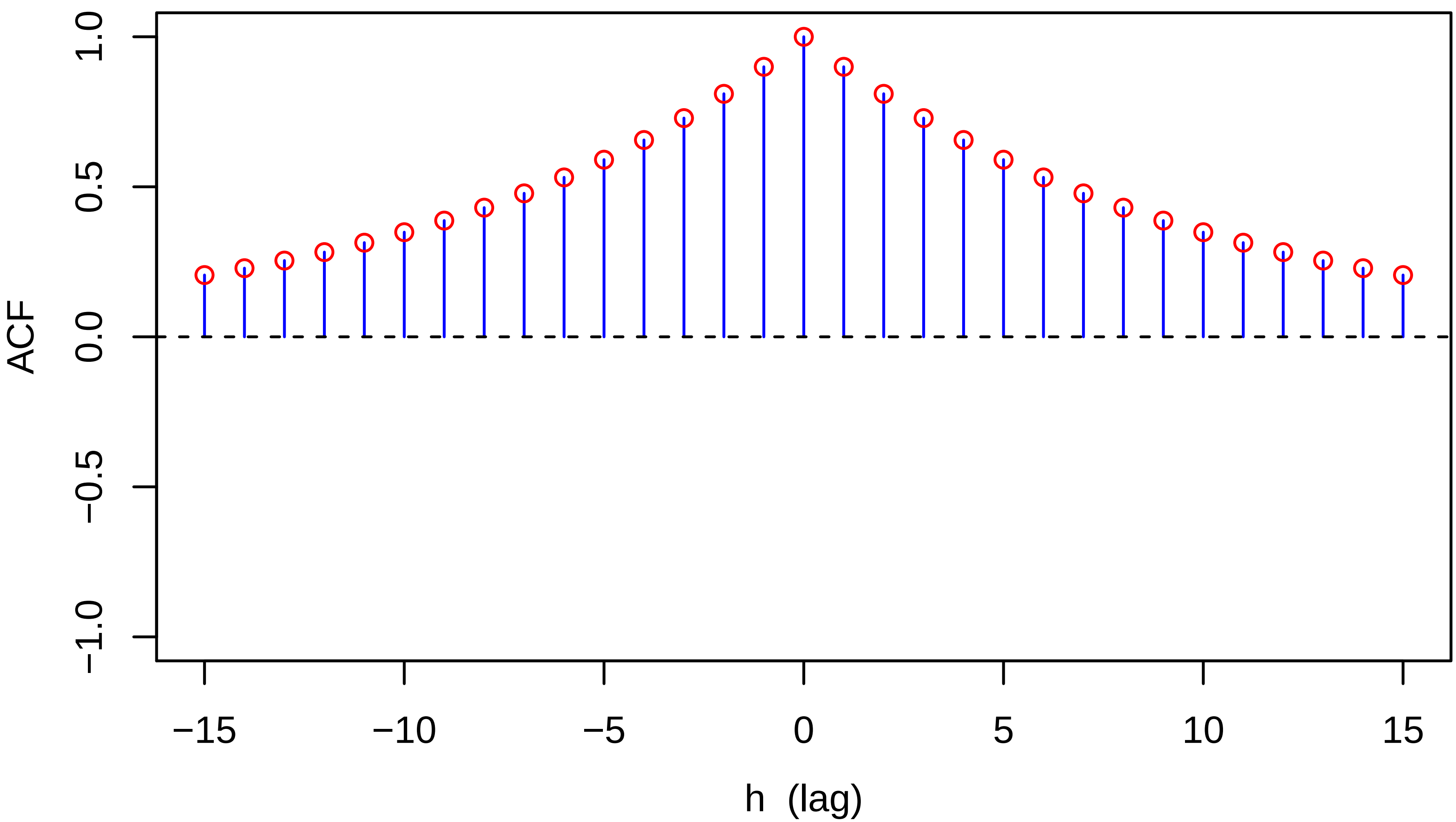
$$\begin{aligned}\gamma_X(0) &= \text{cov}\,\{X_t, X_t\} \\ &= \text{cov}\,\{\mu + \phi(X_{t-1} - \mu) + Z_t, \mu + \phi(X_{t-1} - \mu) + Z_t\} \\ &= \phi^2 \,\text{cov}\,\{X_{t-1}, X_{t-1}\} + \phi \,\text{cov}\,\{X_{t-1}, Z_t\} \\ &\quad + \phi \,\text{cov}\,\{Z_t, X_{t-1}\} + \text{cov}\,\{Z_t, Z_t\} \\ &= \phi^2 \gamma_X(0) + \sigma^2,\end{aligned}$$

  from which we can conclude

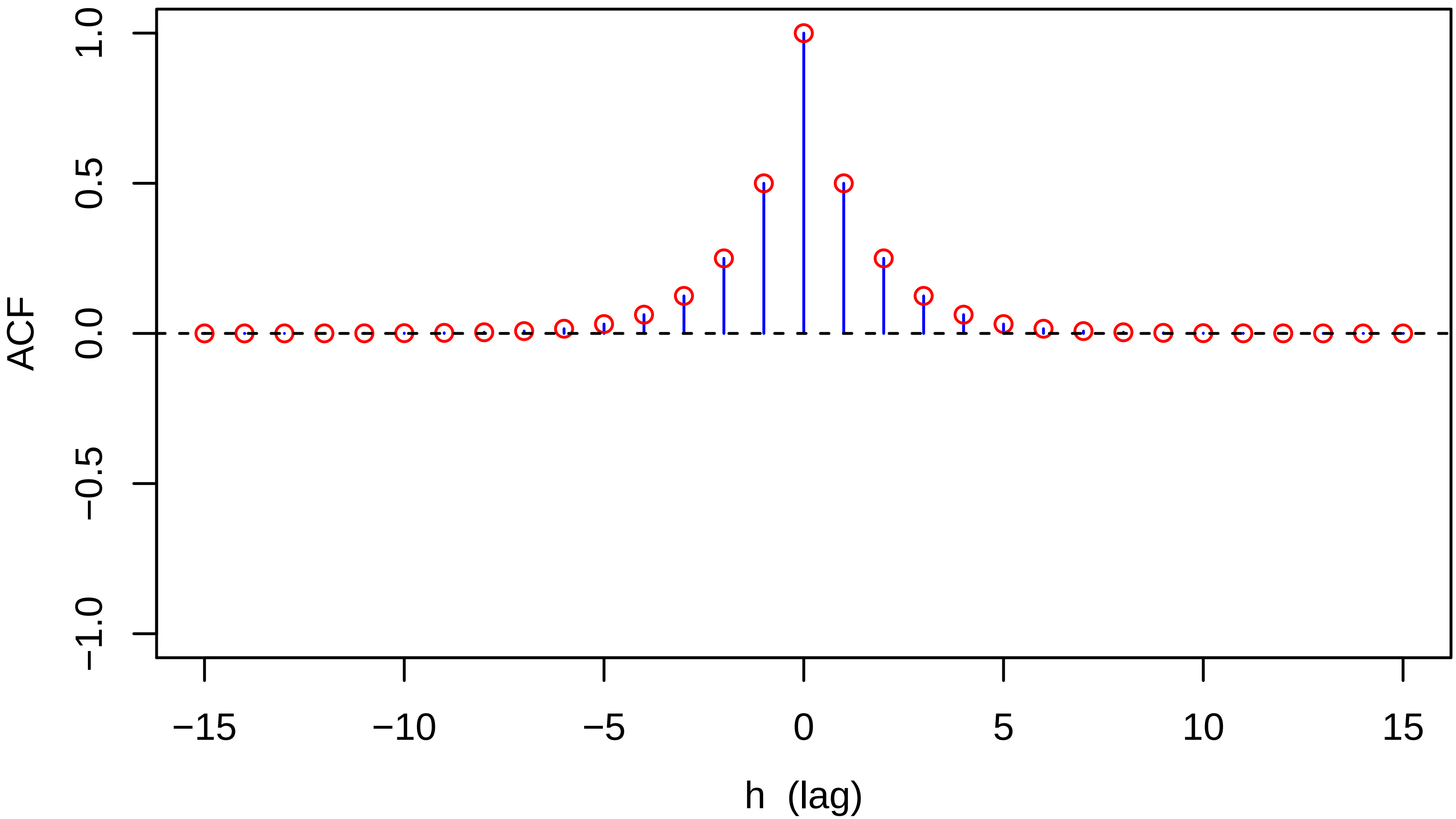
$$\gamma_X(0) = \frac{\sigma^2}{1 - \phi^2}$$

- process $\{X_t\}$ is called a first-order autoregressive process or AR(1) process (when $\phi > 0$, called ‘red noise’ in the geophysical literature)

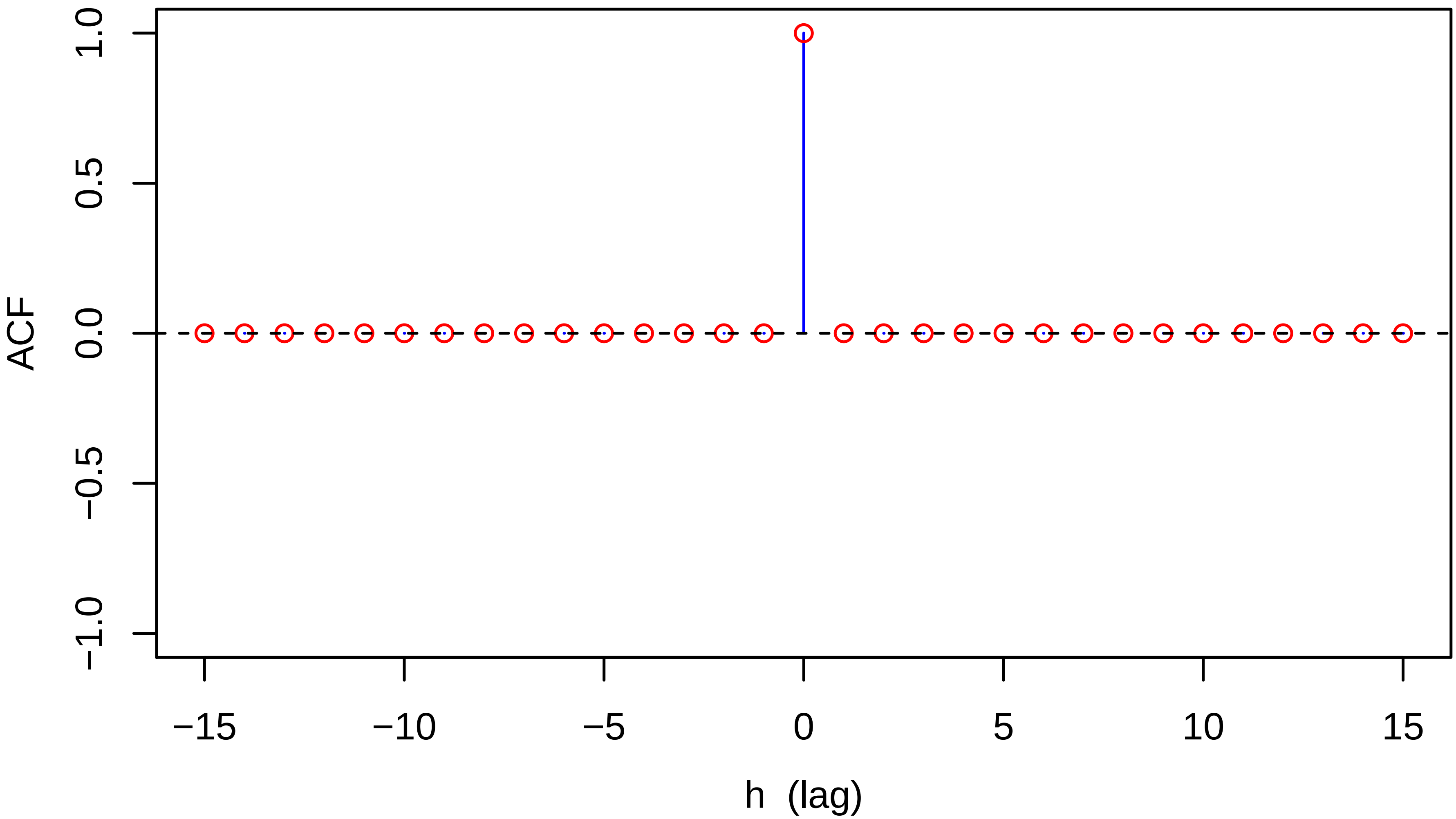
ACF for AR(1) Process with $\phi = 0.9$



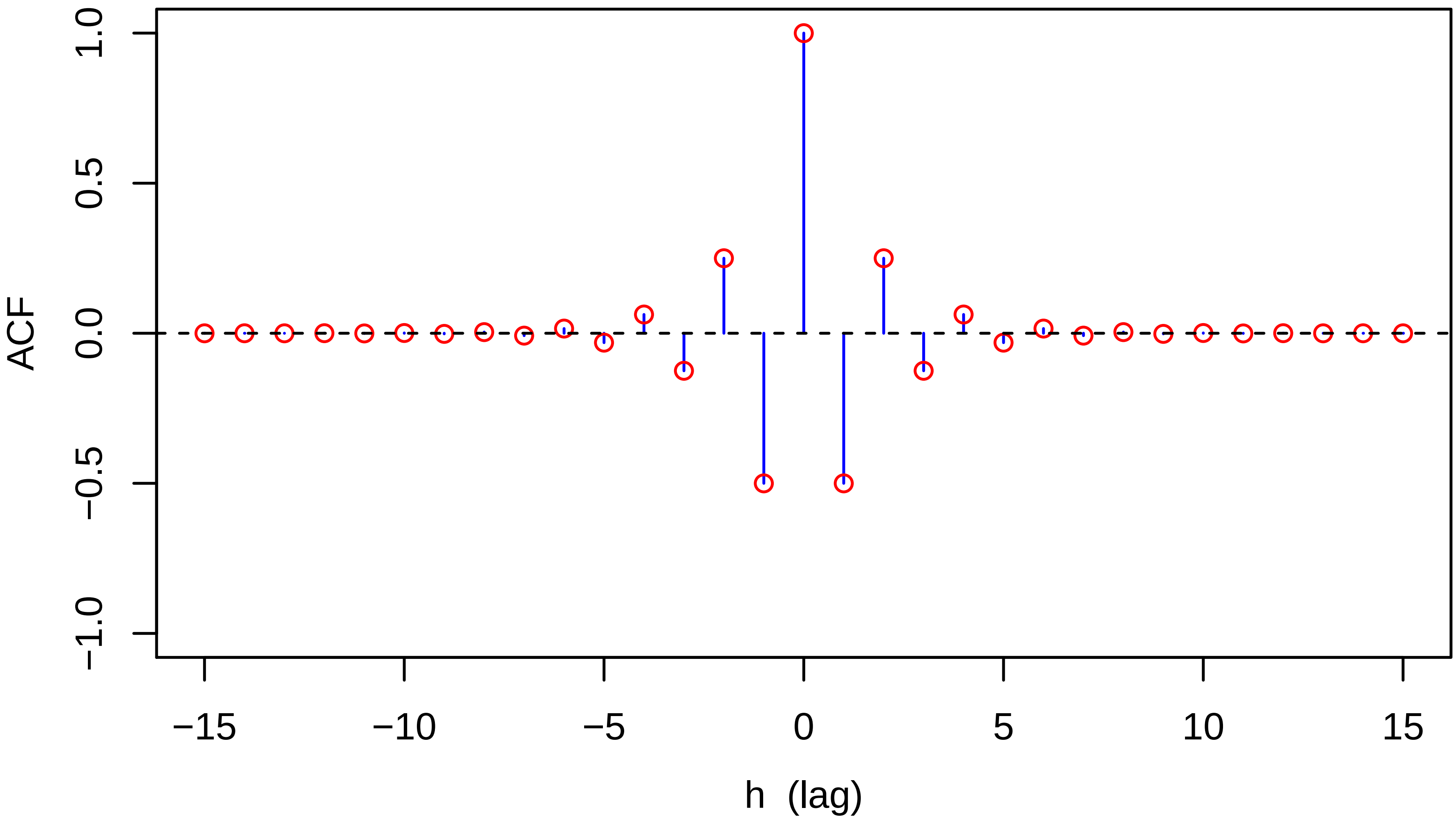
ACF for AR(1) Process with $\phi = 0.5$

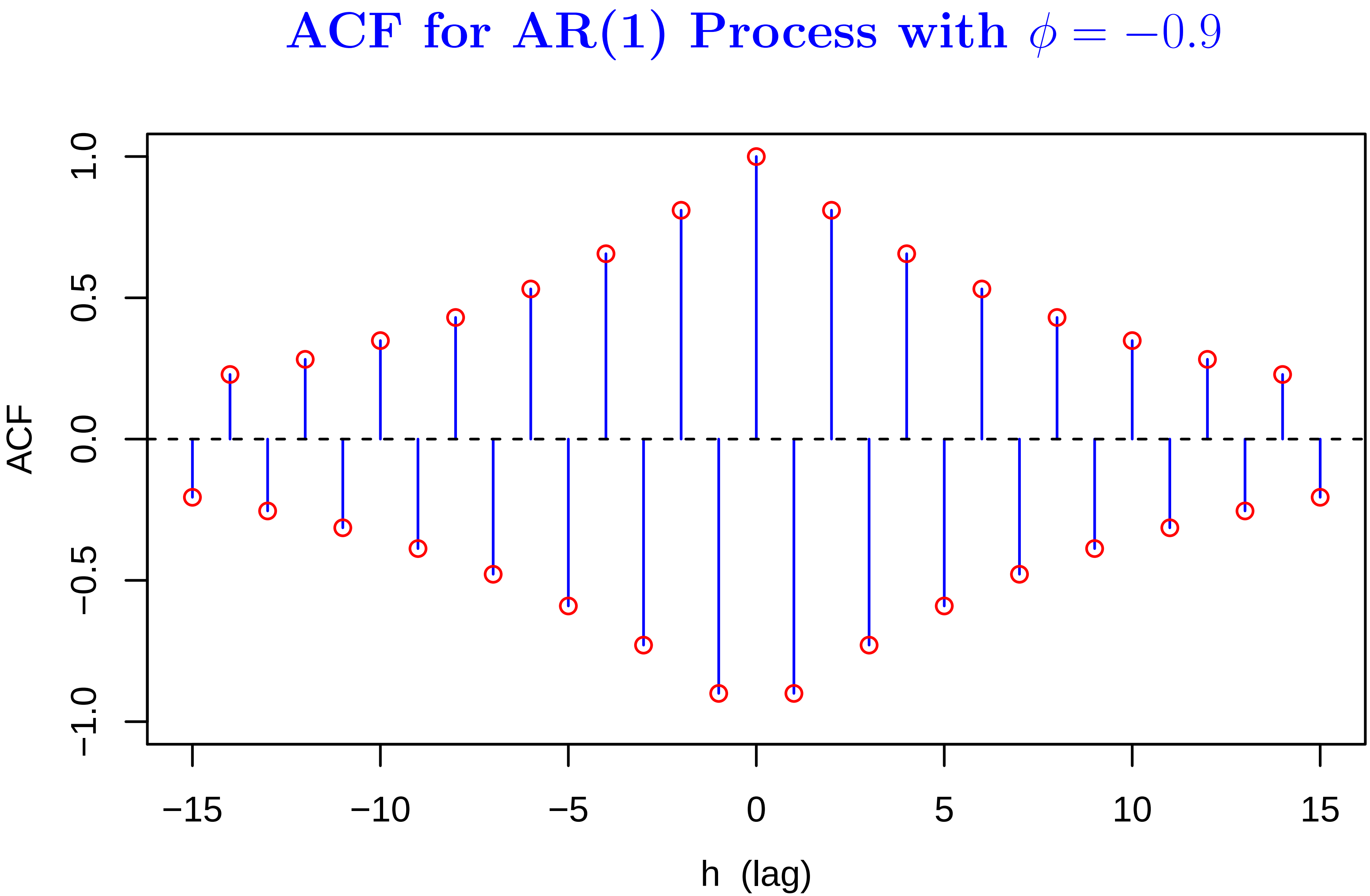


ACF for AR(1) Process with $\phi = 0$



ACF for AR(1) Process with $\phi = -0.5$



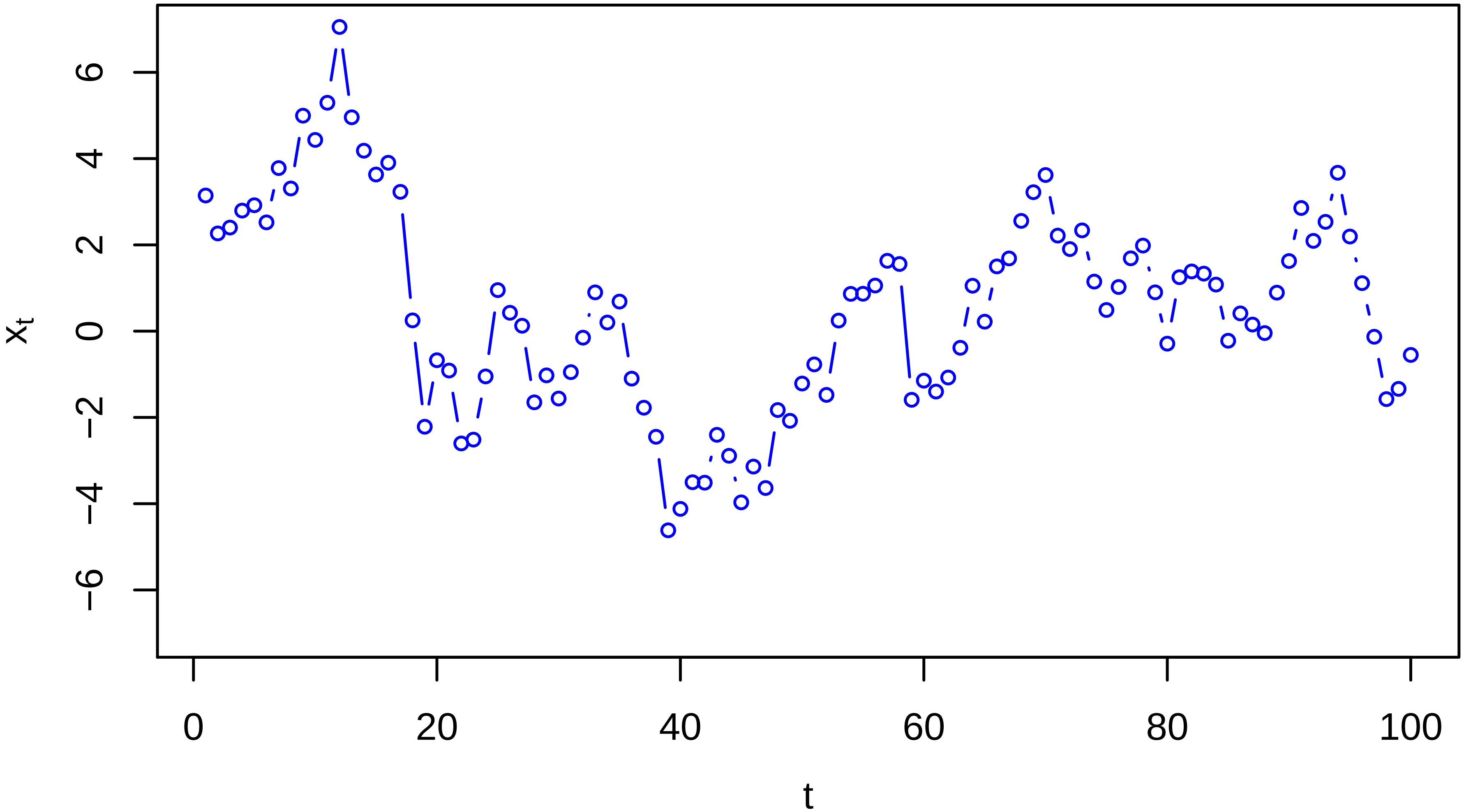

ACF for AR(1) Process with $\phi = -0.9$
1.0
0.5
0.0
−0.5
−1.0
ACF
−15
−10
−5
0
5
10
15
h (lag)

Example – First-Order Autoregressive Process: V

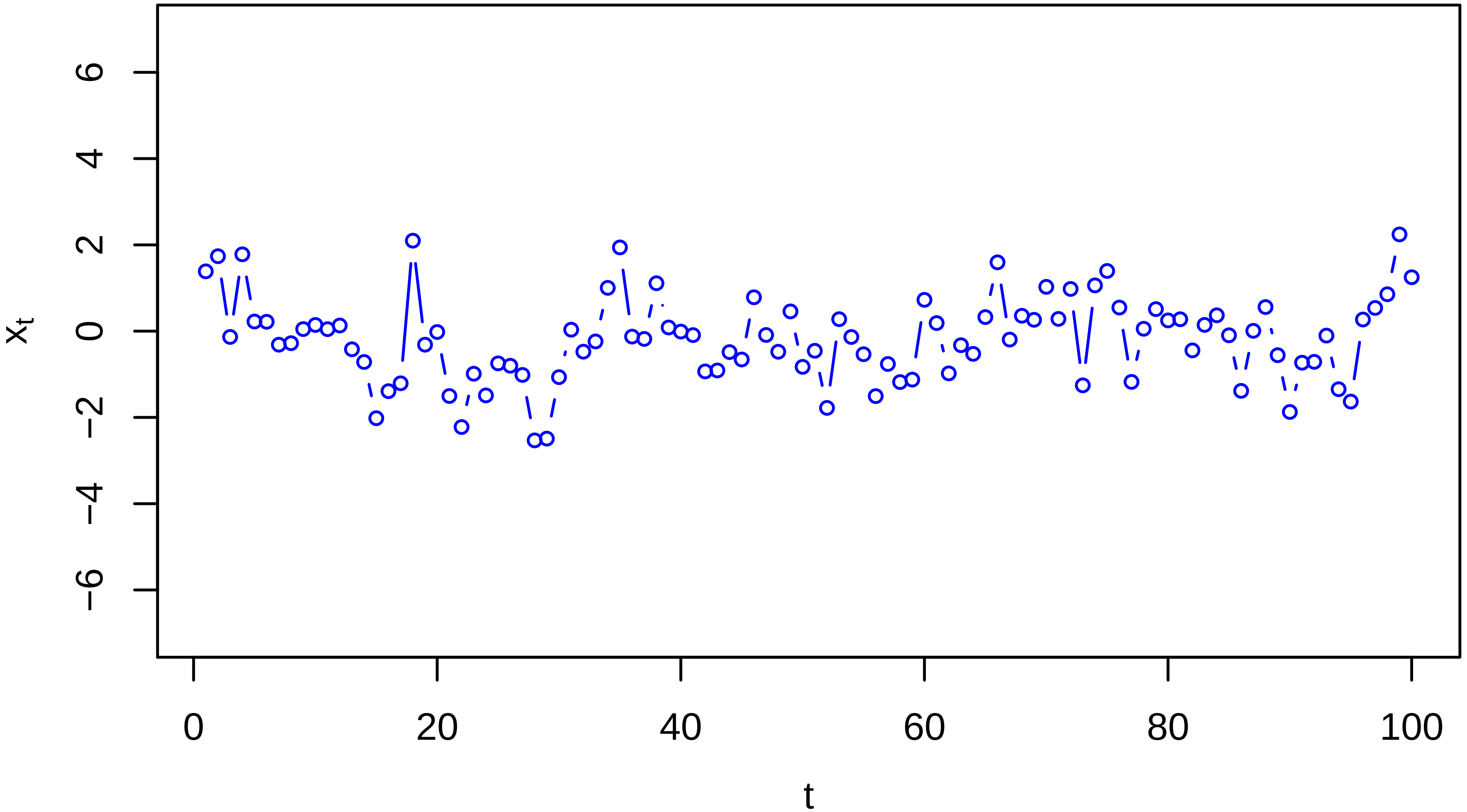
- as examples, let's generate some realizations of AR(1) processes $\{X_t\}$ with $\mu = 0$ and with $\{Z_t\}$ taken to be Gaussian WN(0,1)
- to do so, note that X_1 is Gaussian with zero mean and variance $1/(1-\phi^2)$, which we can generate from a standard normal RV Z_1 using $X_1 = Z_1/\sqrt{(1-\phi^2)}$
- deviates X_2, X_3, … can be generated using the defining equation:

$$X_t = \phi X_{t-1} + Z_t, \quad t = 2, 3, \ldots$$

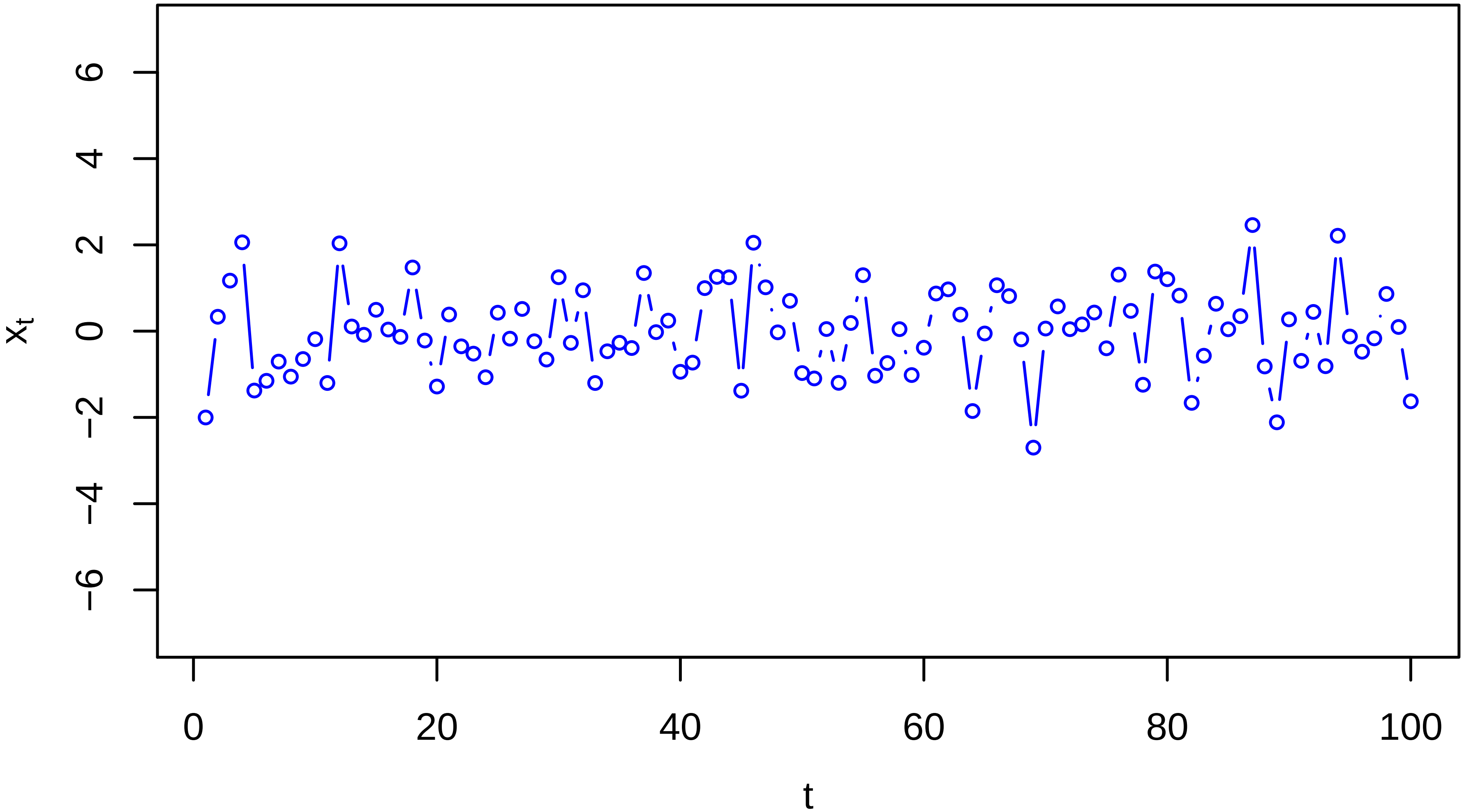
$\phi = 0.9$ **AR(1)** x_t **from Gaussian WN(0,1)**



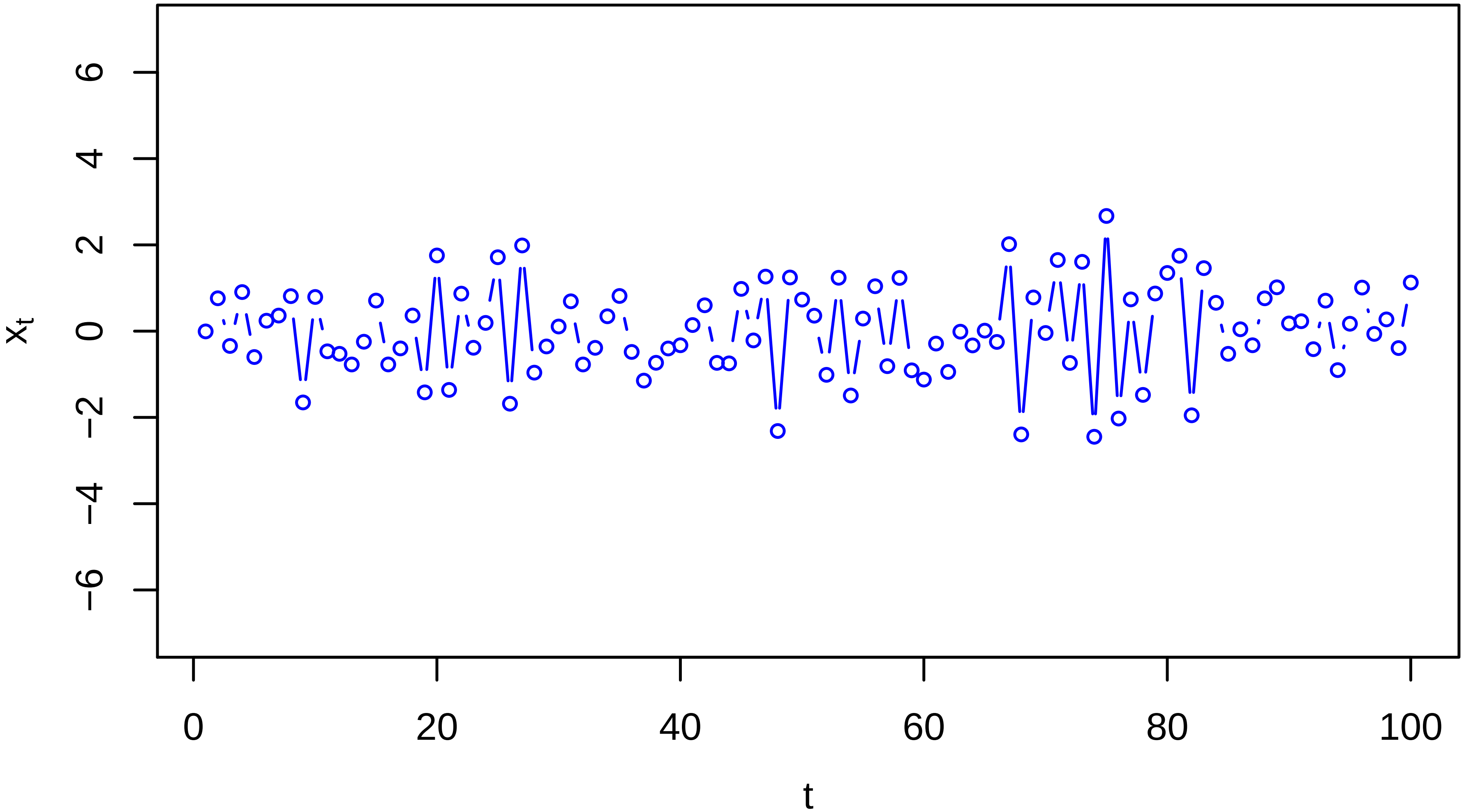
$\phi = 0.5$ **AR(1)** x_t **from Gaussian WN(0,1)**



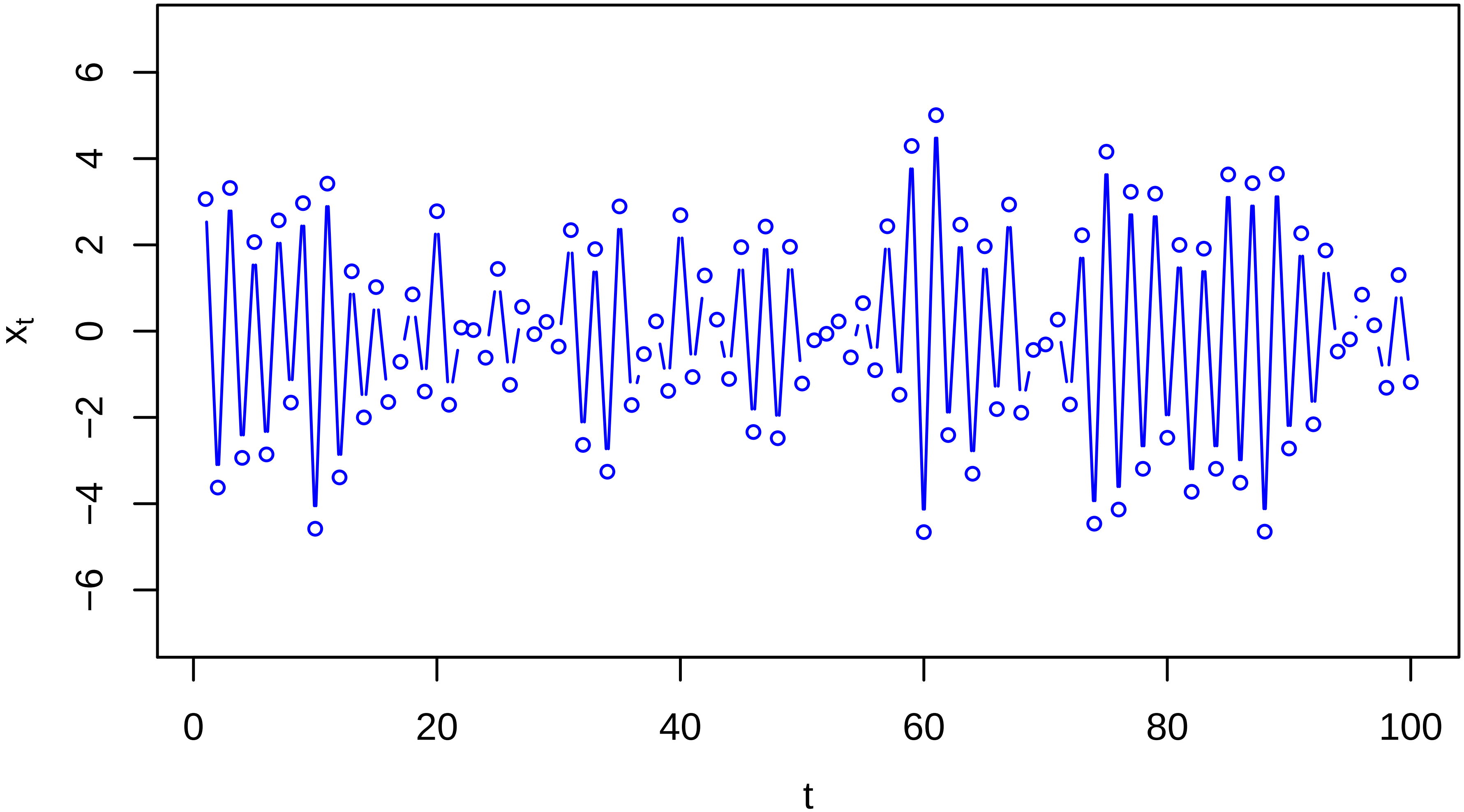
$\phi = 0$ AR(1) x_t from Gaussian WN(0,1)



$\phi = -0.5$ **AR(1)** x_t **from Gaussian WN(0,1)**



$\phi = -0.9$ AR(1) x_t from Gaussian WN(0,1)



Example – First-Order Autoregressive Process: VI

- three notes about $X_t = \phi X_{t-1} + Z_t$
  - X_t for $\phi = 0.9$ is reminiscent of a random walk (why?)
  - variance $\gamma_X(0) = 1/(1 - \phi^2)$ of AR(1) process increases as $|\phi|$ increases
  - if $\{Z_t\}$ is non-Gaussian, getting going is not so easy if we want marginal distributions for X_t to be the same for all t

Sample ACVFs and ACFs: I

- given a time series $x_1, x_2, \ldots, x_n$ that is presumed to be a realization of a portion $X_1, X_2, \ldots, X_n$ of a stationary process with mean μ and ACVF

$$\gamma_X(h) = \text{cov}\,\{X_{t+h}, X_t\} = E\{(X_{t+h} - \mu)(X_t - \mu)\},$$

  we can estimate its ACVF for lags h satisfying $-n < h < n$ using

$$\hat{\gamma}_X(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}), \text{ where } \bar{x} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{t=1}^{n} x_t$$

- $\hat{\gamma}_X(\cdot)$ is called the sample autocovariance function
- note: we are dividing by n rather than $n - |h|$ (?!)

Sample ACVFs and ACFs: II

- since the ACF is given by $\rho_X(h) = \gamma_X(h)/\gamma_X(0)$, the corresponding sample autocorrelation function is given by

$$\hat{\rho}_X(h) \stackrel{\text{def}}{=} \frac{\hat{\gamma}_X(h)}{\hat{\gamma}_X(0)} = \frac{\sum_{t=1}^{n-|h|}(x_{t+|h|} - \bar{x})(x_t - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}$$

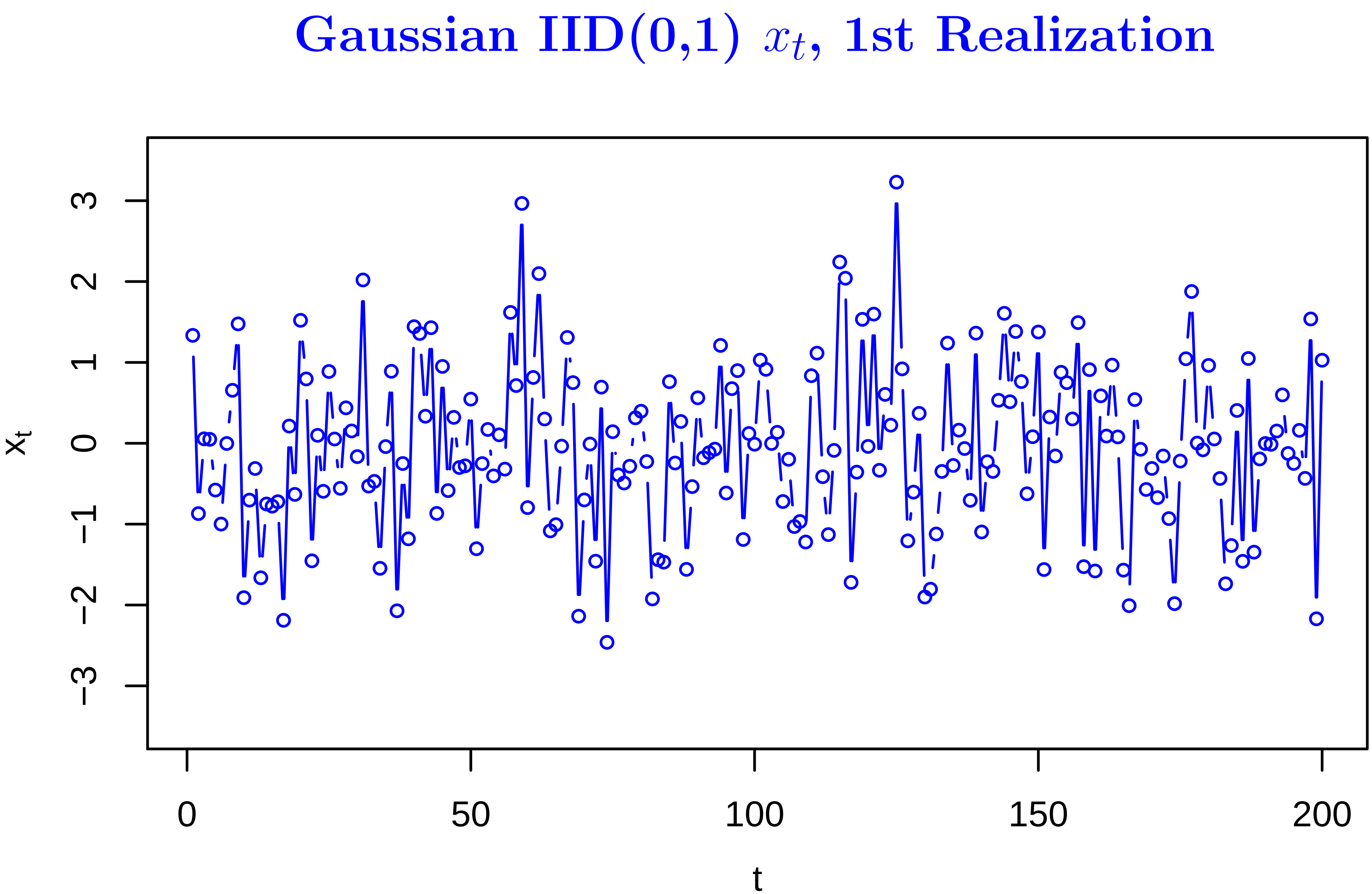
- sample correlation coefficient for u_t and v_t, $t = 1, \ldots, m$, is usually taken to be

$$\frac{\sum_{t=1}^{m}(u_t - \bar{u})(v_t - \bar{v})}{\sqrt{\left[\sum_{t=1}^{m}(u_t - \bar{u})^2 \sum_{t=1}^{m}(v_t - \bar{v})^2\right]}};$$

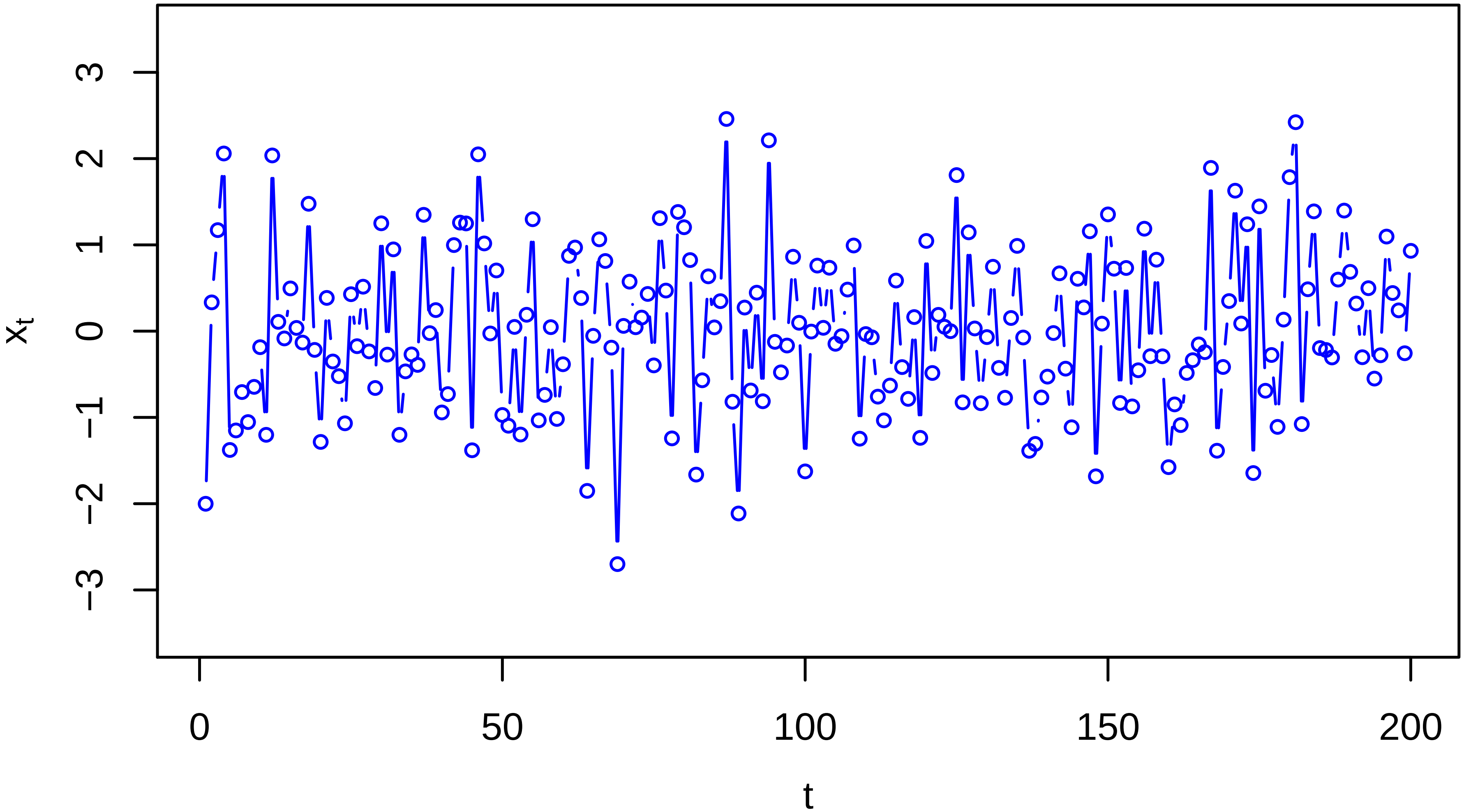
  note that letting $u_t = x_{t+|h|}$, $v_t = x_t$ and $m = n - |h|$ in the above does *not* lead to $\hat{\rho}_X(h)$, as a result of which it is possible to construct (non-pathological!) time series such that $\hat{\rho}_X(h)$ does *not* reflect the strength of the linear relationship between $\{x_1, x_2, \ldots, x_{n-|h|}\}$ and $\{x_{|h|+1}, x_{|h|+2}, \ldots, x_n\}$

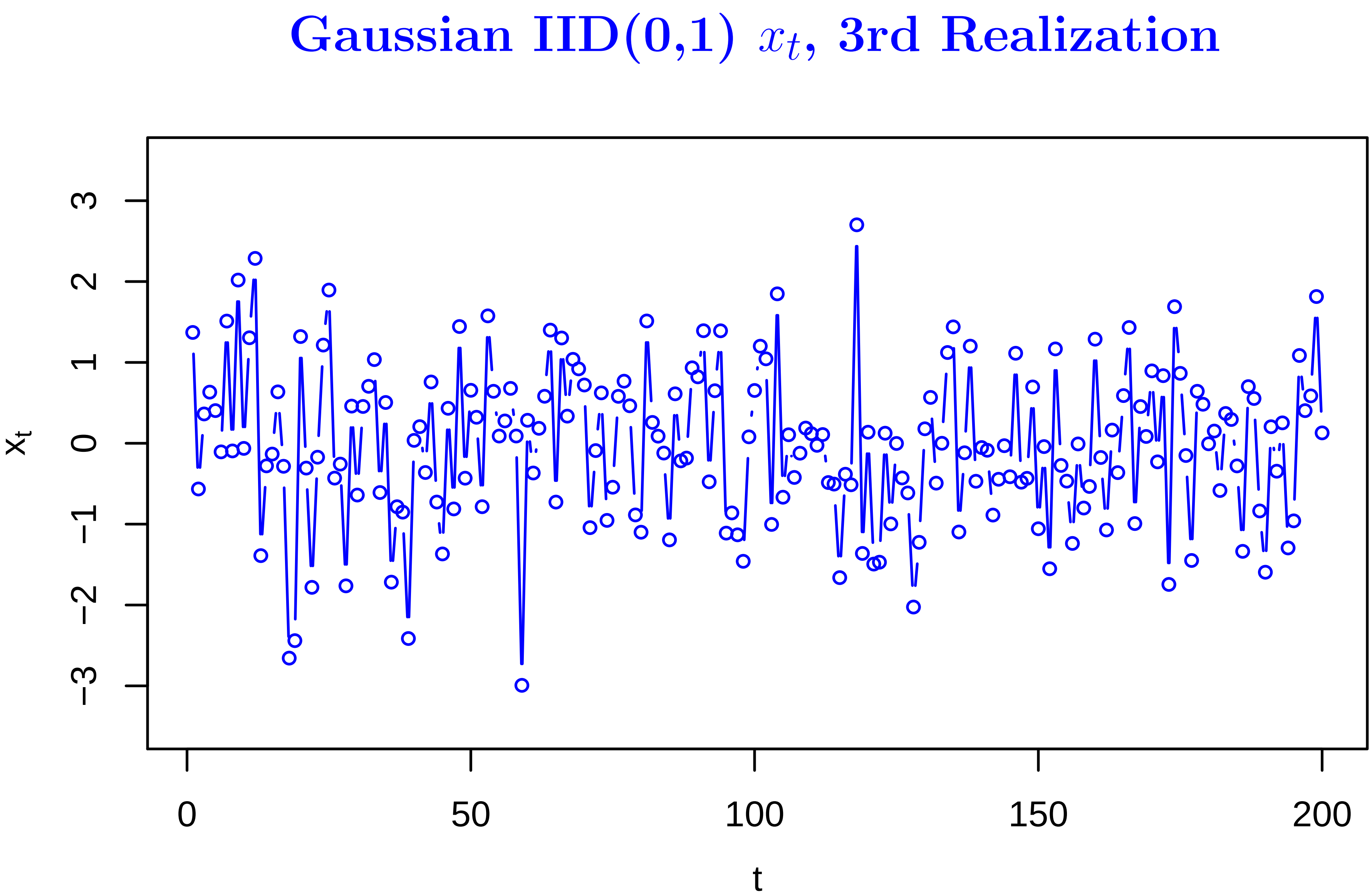
Distribution of Sample ACF for IID(μ, σ^2) Noise: I

- if $\{X_t\}$ is IID(μ, σ^2) process & n is large, can argue that RV
$$\hat{\rho}_X(h) = \frac{\sum_{t=1}^{n-|h|}(X_{t+|h|} - \overline{X})(X_t - \overline{X})}{\sum_{t=1}^{n}(X_t - \overline{X})^2} \text{ when } h \neq 0$$
is approximately $\mathcal{N}(0, 1/n)$, where $\mathcal{N}(\mu, \sigma^2)$ denotes RV with a normal (Gaussian) distribution, mean μ & variance σ^2
- for fixed h', $\hat{\rho}_X(1), \ldots, \hat{\rho}_X(h')$ are IID $\mathcal{N}(0, 1/n)$ as $n \to \infty$
- implies about 95% of $\hat{\rho}_X(h)$'s should fall within $\pm 1.96/\sqrt{n}$
- note: above approximation breaks down for h/n close to unity
- refined bounds are $\pm 1.96\left(\sqrt{n - |h|}\right)/n$ (Fuller, 1996, p. 336)
- let's see how these pan out on realizations of Gaussian IID(0,1) x_t's with $n = 200$ at lags $h = 1, \ldots, 40$ (note: 95% of 40 is 38)

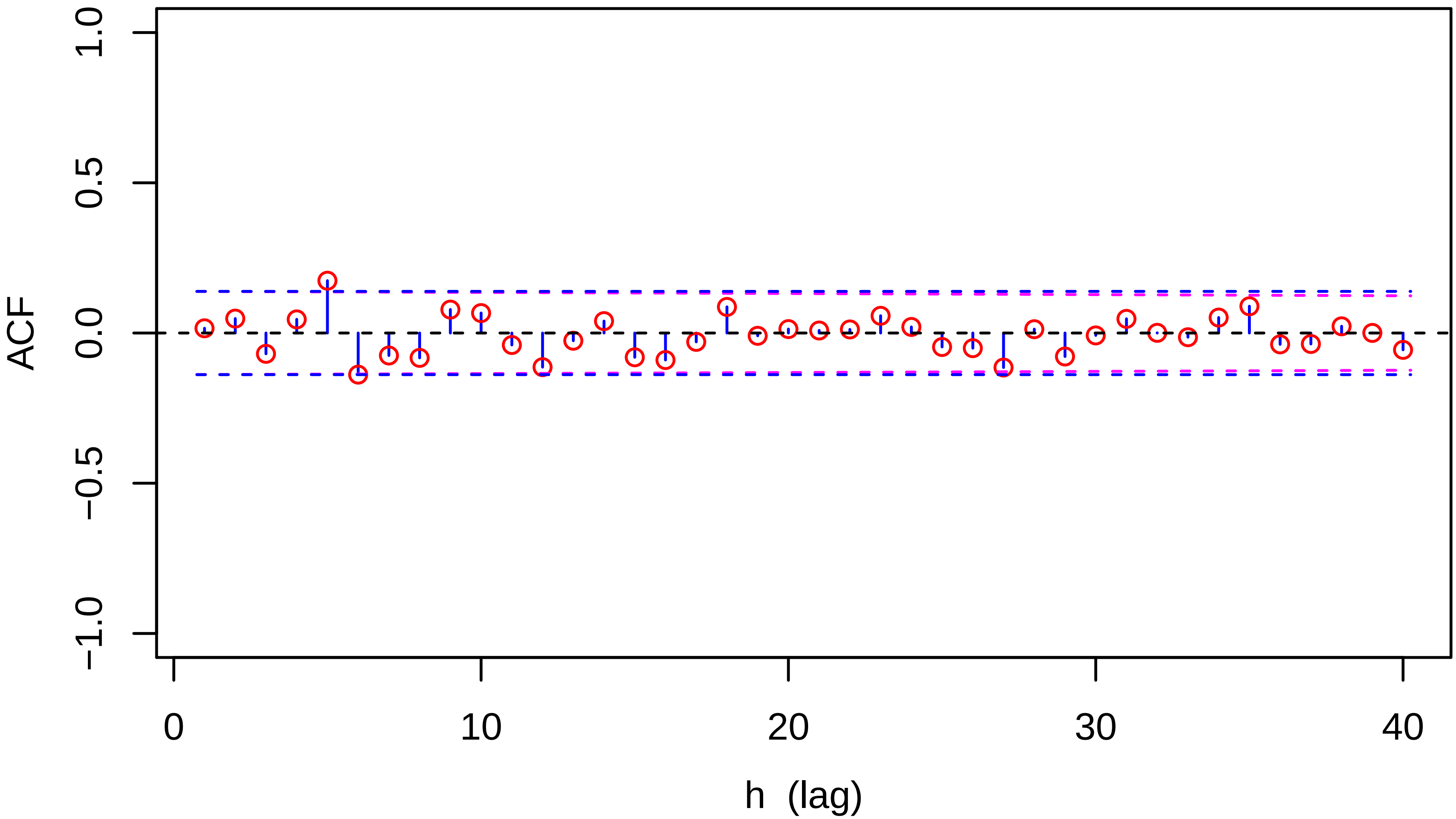

Gaussian IID(0,1) x_t, 1st Realization
x_t
3
2
1
0
−1
−2
−3
0
50
100
150
200
t

Gaussian IID(0,1) x_t, 2nd Realization

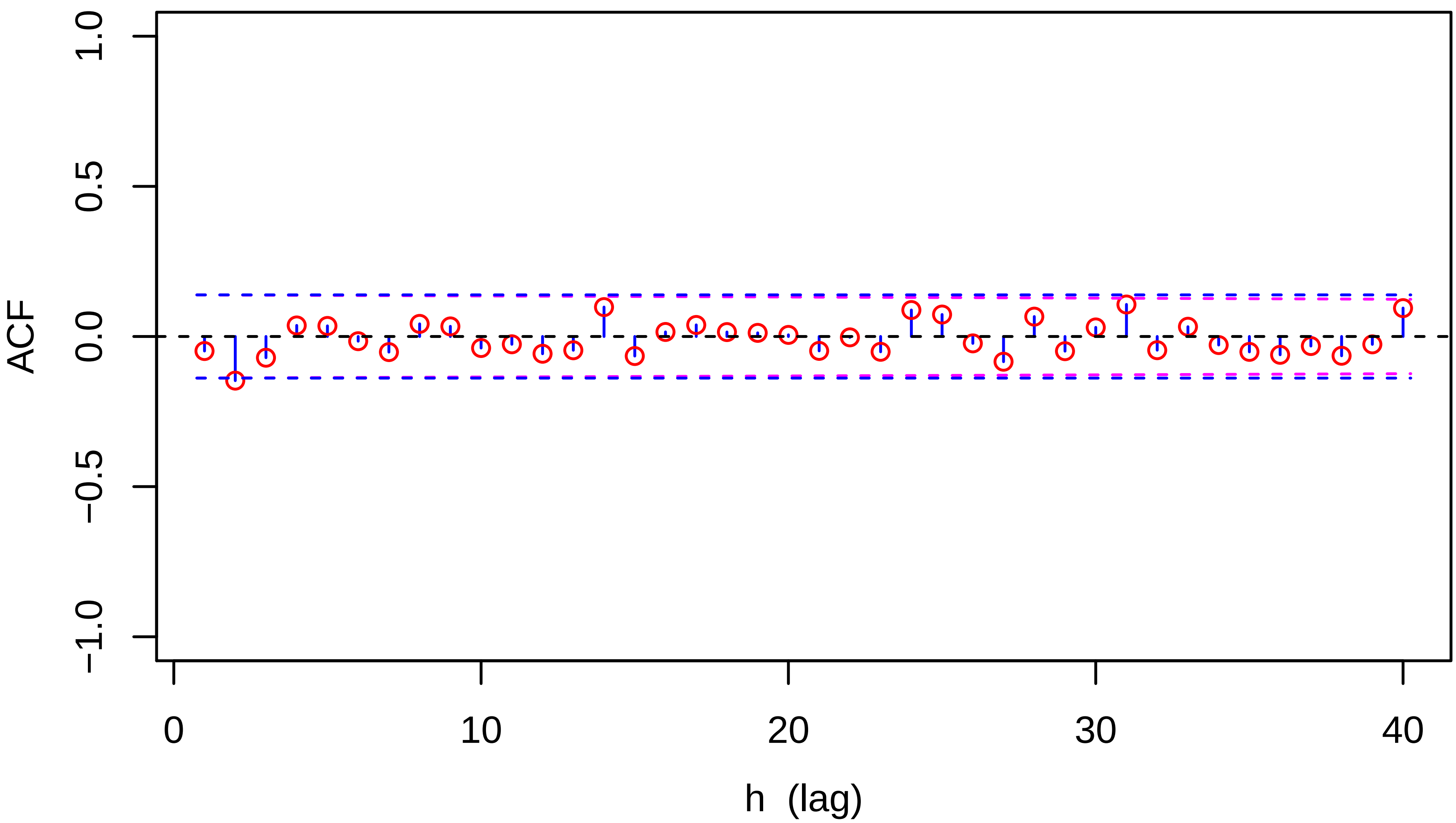


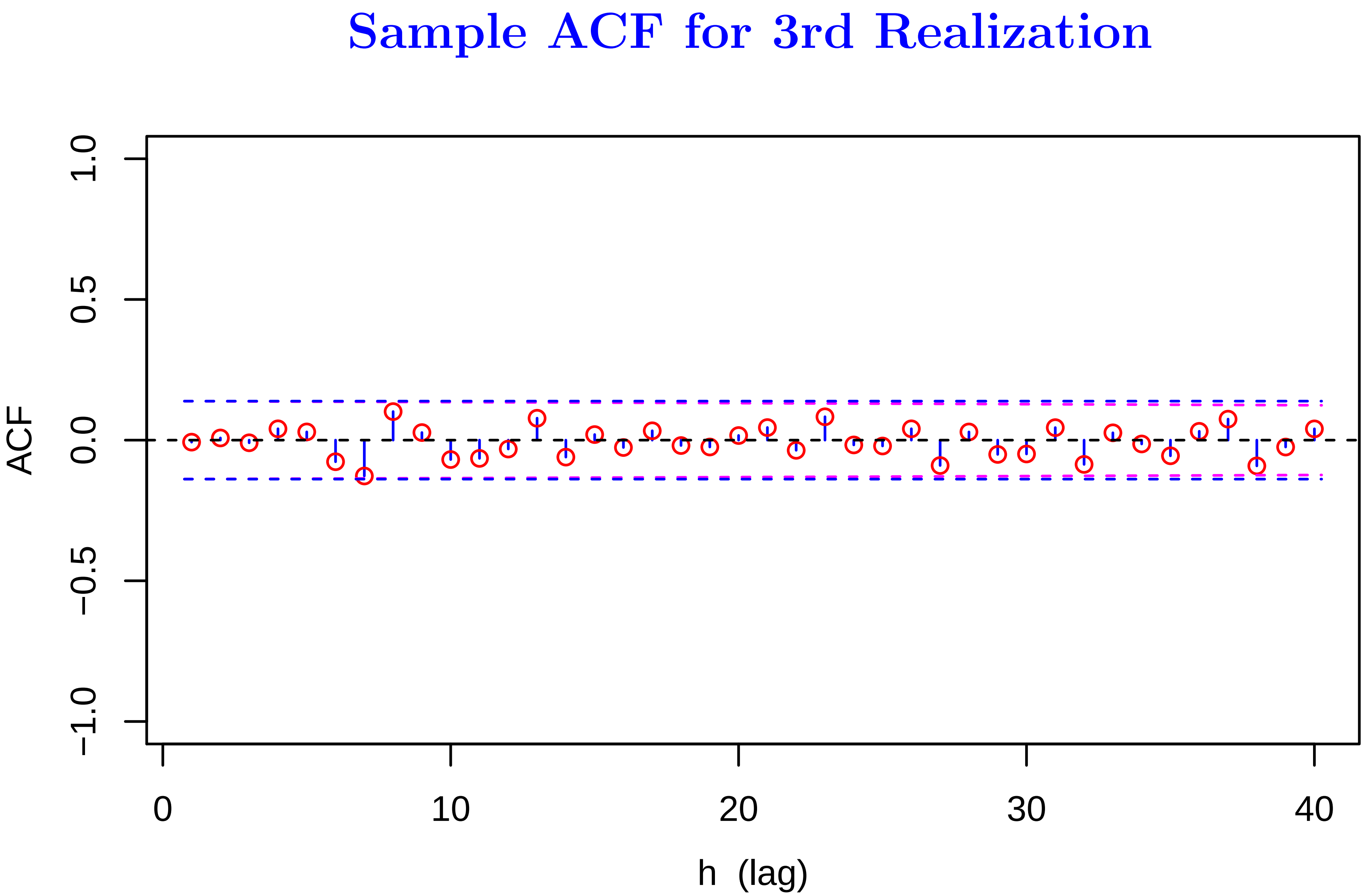

Gaussian IID(0,1) x_t, 3rd Realization
x_t
3
2
1
0
−1
−2
−3
0
50
100
150
200
t

Sample ACF for 1st Realization



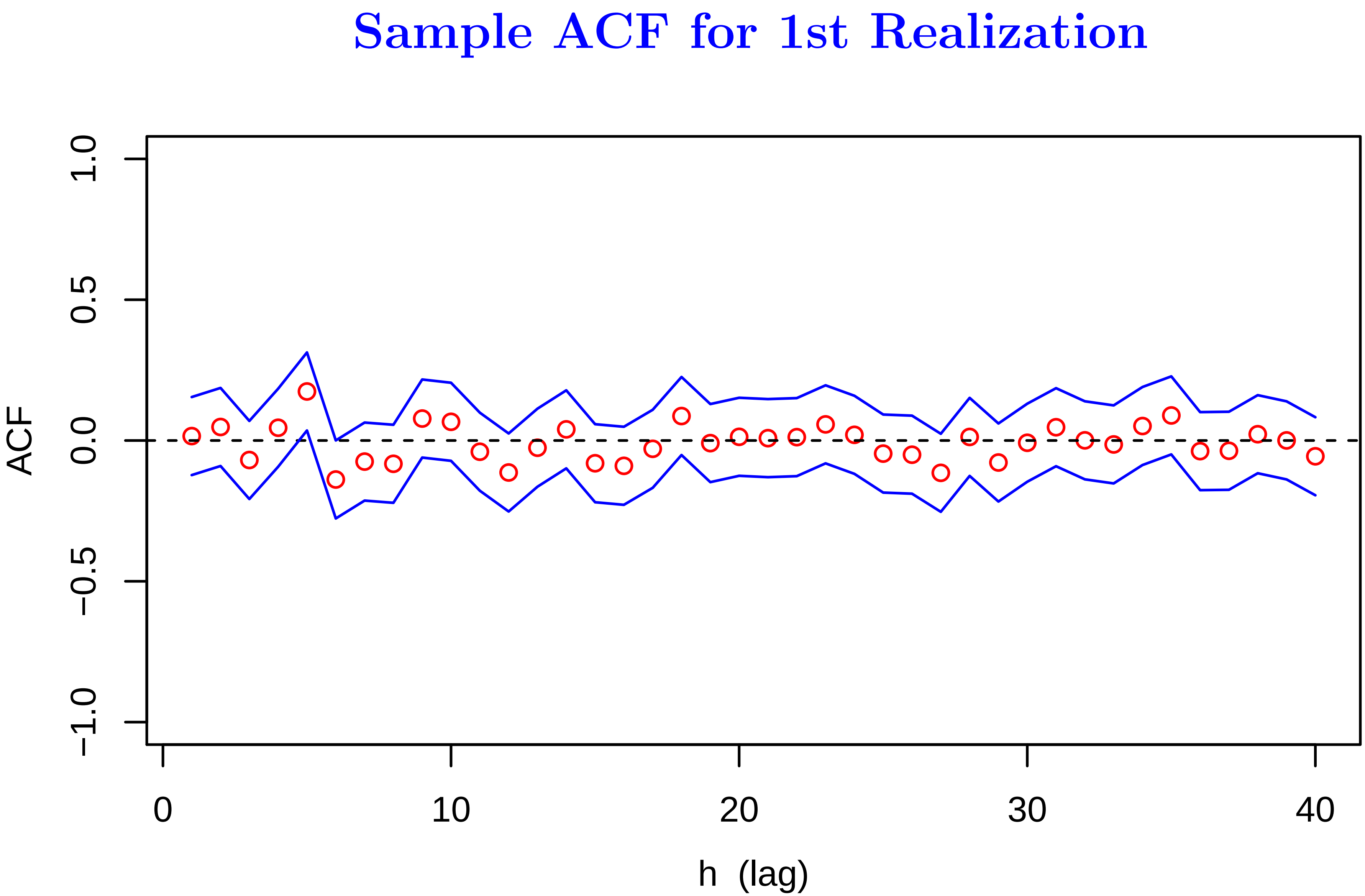
Sample ACF for 2nd Realization




Sample ACF for 3rd Realization
ACF
1.0
0.5
0.0
−0.5
−1.0
0
10
20
30
40
h (lag)

Distribution of Sample ACF for IID(μ, σ^2) Noise: II

- for the three realizations, $\pm 1.96/\sqrt{n}$ bounds and refined bounds tell same story (need not be true for other realizations)
- number of times that $\hat{\rho}_X(h)$ is outside bounds is
  - 1 for first realization
  - 1 for second
  - 0 for third
- if $\hat{\rho}_X(h)$ falls within $\pm 1.96/\sqrt{n}$, then $\hat{\rho}_X(h) \pm 1.96/\sqrt{n}$ must trap 0
- hence, can assess viability of null hypothesis of IID(μ, σ^2) noise by plotting $\hat{\rho}_X(h) \pm 1.96/\sqrt{n}$ versus h and considering cases that do not trap 0

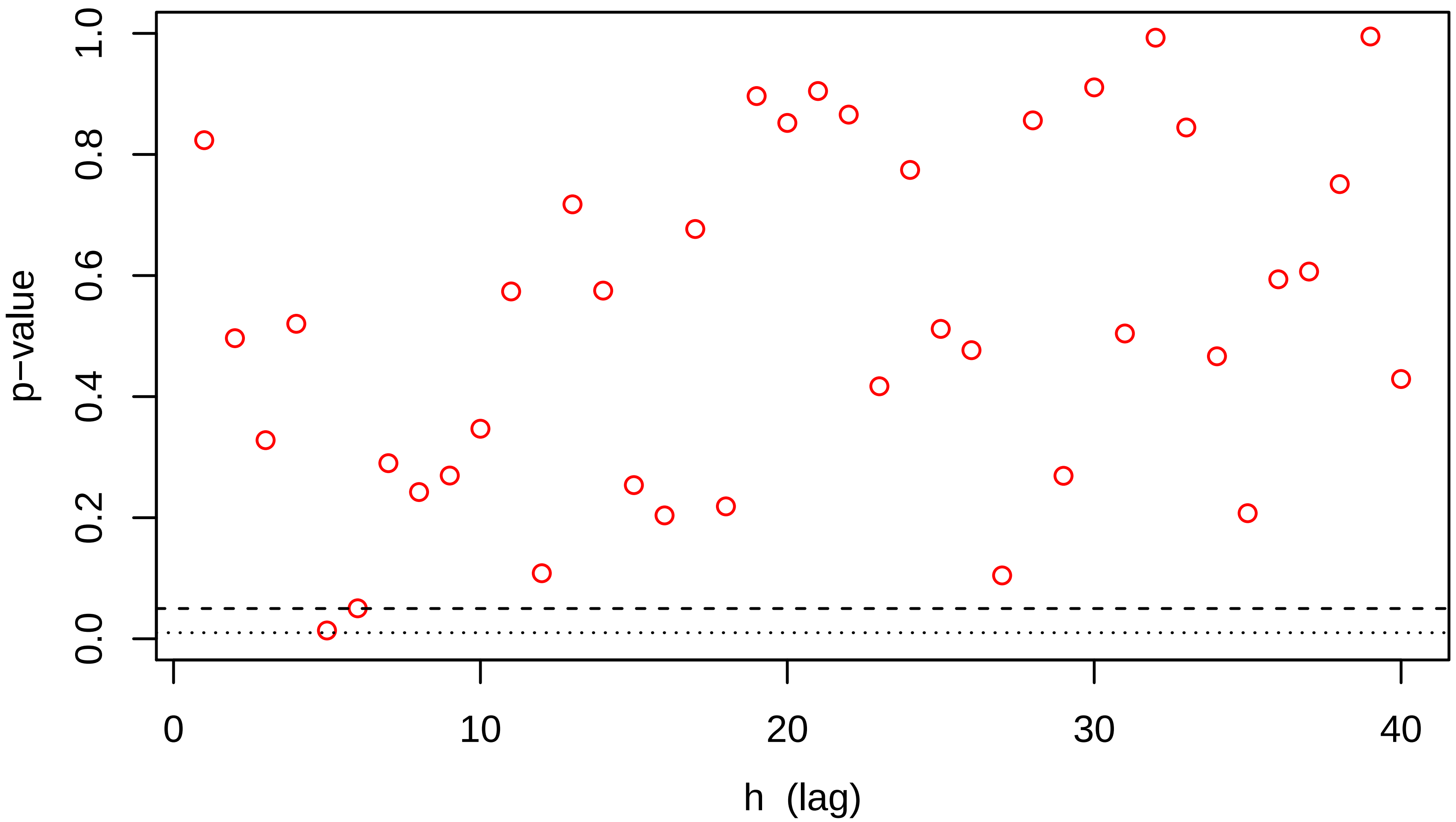


Sample ACF for 1st Realization
ACF
1.0
0.5
0.0
−0.5
−1.0
0
10
20
30
40
h (lag)

Distribution of Sample ACF for IID(μ, σ^2) Noise: III

- another option for assessing null hypothesis of IID(μ, σ^2) noise is to consider p-value for $\hat{\rho}_X(h)$, i.e., probability under null hypothesis of observing a value at least as extreme as what was actually observed
- taking W to be an $\mathcal{N}(0, 1/n)$ RV, plot $P[W \geq |\hat{\rho}_X(h)|]$ versus h and consider cases where p-value is small
- in following plot, a p-value of
  - 0.01 is indicated by a horizontal dotted line
  - 0.05 is indicated by a horizontal dashed line
- smallest p-value is 0.014 (occurs at lag $h = 5$)

Sample ACF-based p-values for 1st Realization



Reference

- W. A. Fuller (1996), *Introduction to Statistical Time Series* (Second Edition), New York: John Wiley & Sons, Inc.