Wavelet-Based Analysis for Multispectral Fractal Processes

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overheads for talk available at

`http://www.staff.washington.edu/dbp/talks.html`

joint work with:

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Introduction and Overview

- motivation: ABL aerothermal turbulence data
  - Fig. 1: 7.5 million points (100 point averages)
  - spatial resolution $\approx$ 1.83 cm
- will model using time-varying stochastic process
- basic idea: combine wavelets with stochastic fractals
  - wavelets give time/scale decomposition
    (yields multiscale approach to modeling)
  - fractals describe connections across scales
    (will use fractionally differenced processes)

Outline of Talk

- overview of discrete wavelet transform (DWT)
- overview of fractionally differenced (FD) processes
- basic properties of DWT of an FD process
  (DWT acts as decorrelator of FD processes)
- DWT-based estimation of parameters for FD process
  - maximum likelihood and least squares estimators
- application to ABL data
- future work

Overview of DWT: I

- let $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ be observed time series (for convenience, assume N integer multiple of 2^{J_0})
- let $\mathcal{W}$ be $N \times N$ orthonormal DWT matrix
- $\mathbf{W} = \mathcal{W}\mathbf{X}$ is vector of DWT coefficients
- orthonormality says $\mathbf{X} = \mathcal{W}^T\mathbf{W}$, so $\mathbf{X} \Leftrightarrow \mathbf{W}$
- can partition $\mathbf{W}$ as follows:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \vdots \\ \mathbf{W}_{J_0} \\ \mathbf{V}_{J_0} \end{bmatrix}$$

- $\mathbf{W}_j$ contains $N_j = N/2^j$ wavelet coefficients
  - related to changes of averages at scale $\tau_j = 2^{j-1}$ (τ_j is jth 'dyadic' scale)
  - related to times spaced 2^j units apart
- $\mathbf{V}_{J_0}$ contains $N_{J_0} = N/2^{J_0}$ scaling coefficients
  - related to averages at scale $\lambda_{J_0} = 2^{J_0}$
  - related to times spaced 2^{J_0} units apart
- Fig. 2: DWT of small segment of ABL data

Overview of DWT: II

- obtain DWT via filtering with subsampling
- filter $X_0, X_1, \ldots, X_{N-1}$ to obtain

$$2^{j/2}\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} h_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \ldots, N-1$$

  - $h_{j,l}$ is jth level wavelet filter
  - width of $h_{1,l}$ is $L_j = (2^j - 1)(L-1) + 1$
  - $\widetilde{W}_{j,t}$ part of 'maximal overlap' DWT (MODWT)
- subsample to obtain DWT wavelet coefficients:

$$W_{j,t} = 2^{j/2}\widetilde{W}_{j,2^j(t+1)-1}, \quad t = 0, 1, \ldots, N_j - 1,$$

  where $W_{j,t}$ is tth element of $\mathbf{W}_j$
- Fig. 3: Haar & 'least asymmetric' (LA) wavelet filters
- jth filter is band-pass with pass-band $[\frac{1}{2^{j+1}}, \frac{1}{2^j}]$
- similarly, scaling filters yield $\mathbf{V}_{J_0}$
- Fig. 3: Haar & LA(8) scaling filters
- J_0th filter is low-pass with pass-band $[0, \frac{1}{2^{J_0+1}}]$

Overview of FD Processes

- X_t called fractionally differenced (FD) process if it has a spectral density function (SDF) given by
$$S_X(f) = \frac{\sigma_\epsilon^2}{|2\sin(\pi f)|^{2\delta}},$$
where $\sigma_\epsilon^2 > 0$ and $-\infty < \delta < \infty$
- Fig. 4: for small f, have $S_X(f) \approx C/|f|^{2\delta}$; i.e., '$1/f$ type,' 'power law' or 'fractal' process
- also called ARFIMA(0,δ,0) process
- special cases
  - stationary if $\delta < \frac{1}{2}$
    * white noise if $\delta = 0$
    * has 'long memory' if $0 < \delta < \frac{1}{2}$
      · autocorrelation sequence $s_{X,\tau} \approx C_s \tau^{-1+2\delta}$
      · quite similar to fractional Gaussian noise
  - has stationary increments if $\delta \geq \frac{1}{2}$
    * random walk if $\delta = 1$
    * like fractional Brownian motion if $\frac{1}{2} < \delta < \frac{3}{2}$
    * like $-\frac{5}{3}$ power law (Kolmogorov) if $\delta = \frac{5}{6}$

DWT of FD Processes

- Fig. 5: DWT of realization of FD process ($\delta = 0.4$)
- sample ACSs suggest random variables (RVs) in $\mathbf{W}_j$ are approximately uncorrelated
- ignoring 'boundary' coefficients, $\mathbf{W}_j$ is stationary
- Fig. 6: SDFs for $\mathbf{W}_j$, $j = 1, 2, 3, 4$
  - quite close to white noise
  - remaining structure close to SDF for first or second order autoregressive process
- $\mathbf{W}_j$ & $\mathbf{W}_{j'}$, $j \neq j'$, approximately uncorrelated (can improve approximation by increasing L)
- DWT acts as a whitening transform (basis for wavelet-based maximum likelihood scheme)
- have $\nu_X^2(\tau_j) \equiv \operatorname{var}\{\widetilde{W}_{j,t}\} \propto \tau_j^{2\delta-1}$ approximately
  - implies $\log(\nu_X^2(\tau_j)) \approx \zeta + (2\delta - 1)\log(\tau_j)$
  - $\nu_X^2(\tau_j)$ called wavelet variance (note: based on MODWT $\widetilde{W}_{j,t}$ rather than DWT $W_{j,t}$)
  - basis for wavelet-based least squares scheme

ML Estimation for FD Processes: I

- suppose we are given $U_0, \ldots, U_{N-1}$ such that

$$U_t = T_t + X_t$$

  where $T_t \equiv \Sigma_{j=0}^{r} a_j t^j$ is polynomial trend & X_t is FD process

- width L wavelet filter has embedded differencing operation of order $L/2$

- if $\frac{L}{2} \geq r + 1$, reduces polynomial trend to 0

- can partition DWT coefficients as

$$\mathbf{W} = \mathbf{W}_s + \mathbf{W}_b + \mathbf{W}_w$$

  where

  - $\mathbf{W}_s$ has scaling coefficients and 0s elsewhere
  - $\mathbf{W}_s$ has boundary-dependent wavelet coefficients
  - $\mathbf{W}_w$ has boundary-independent wavelet coefficients

ML Estimation for FD Processes: II

- since $\mathbf{U} = \mathcal{W}^T\mathbf{W}$, can write
$$\mathbf{U} = \mathcal{W}^T(\mathbf{W}_s + \mathbf{W}_b) + \mathcal{W}^T\mathbf{W}_w \equiv \widehat{\mathbf{T}} + \widehat{\mathbf{X}}$$
- can use values in $\mathbf{W}_w$ to form likelihood:
$$L(\delta, \sigma_\epsilon^2) \equiv \prod_{j=1}^{J_0} \prod_{t=1}^{N'_j} \frac{1}{\left(2\pi\sigma_j^2\right)^{1/2}} e^{-W^2_{j,t+L'_j-1}/(2\sigma_j^2)}$$
where
$$\sigma_j^2 \equiv \int_{-1/2}^{1/2} \mathcal{H}_j(f) \frac{\sigma_\epsilon^2}{|2\sin(\pi f)|^{2\delta}}\, df;$$
and $\mathcal{H}_j(f)$ is squared gain for $h_{j,l}$
- leads to maximum likelihood estimator $\hat{\delta}^{(ml)}$ for δ
- $\hat{\delta}^{(ml)}$ asymptotically normal with mean δ and
$$\operatorname{var}\{\hat{\delta}^{(ml)}\} = 2\Big[\sum_{j=1}^{J_0} N'_j\gamma_j^2 - \frac{1}{N'}\big(\sum_{j=1}^{J_0} N'_j\gamma_j\big)^2\Big]^{-1},$$
where $N' \equiv \sum_{j=1}^{J_0} N'_j$ and
$$\gamma_j \equiv \frac{\frac{d \operatorname{var}\{W_{j,t}\}}{d\delta}}{\operatorname{var}\{W_{j,t}\}} = -\frac{4\sigma_\epsilon^2}{\operatorname{var}\{W_{j,t}\}} \int_0^{1/2} \mathcal{H}_j(f) \frac{\log\left(2\sin(\pi f)\right)}{[2\sin(\pi f)]^{2\delta}}\, df$$
- works well in Monte Carlo simulations

LS Estimation for FD Processes: I

- define unbiased estimator of wavelet variance $\nu_X^2(\tau_j)$:

  $$\hat{\nu}_X^2(\tau_j) \equiv \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2, \text{ where } M_j \equiv N - L_j + 1$$

- $\hat{\nu}_X^2(\tau_j)$ is approximately distribution as $\nu_X^2(\tau_j)\chi_{\eta_j}^2/\eta_j$, where
  - $\chi_{\eta_j}^2$ is chi-square RV with η_j degrees of freedom
  - can approximate η_j by $\max\{M_j/2^j, 1\}$
- using $\log(\nu_X^2(\tau_j)) \approx \zeta + \beta \log(\tau_j)$ with $\beta \equiv 2\delta - 1$, can formulate regression model; with

  $$Y(\tau_j) \equiv \log(\hat{\nu}_X^2(\tau_j)) - \psi(\tfrac{\eta_j}{2}) + \log(\tfrac{\eta_j}{2}),$$

  have $Y(\tau_j) = \zeta + \beta \log(\tau_j) + e_j$, where

  $$e_j \equiv \log\left(\frac{\hat{\nu}_X^2(\tau_j)}{\nu_X^2(\tau_j)}\right) - \psi(\tfrac{\eta_j}{2}) + \log(\tfrac{\eta_j}{2})$$

  has distribution $\log(\chi_{\eta_j}^2) - \psi(\frac{\eta_j}{2}) - \log(2)$
- have $E\{e_j\} = 0$ and $\text{var}\{e_j\} = \psi'(\frac{\eta_j}{2})$, where $\psi'(\cdot)$ is trigamma function
- e_j approximately Gaussian if $\eta_j \geq 10$

LS Estimation for FD Processes: II

- weighted least squares (LS) estimator for β:

  $$\hat{\beta}^{(wls)} = \frac{\sum w_j \sum w_j \log{(\tau_j)} Y(\tau_j) - \sum w_j \log{(\tau_j)} \sum w_j Y(\tau_j)}{\sum w_j \sum w_j \log^2(\tau_j) - \left(\sum w_j \log{(\tau_j)}\right)^2},$$

  where $w_j \equiv 1/\psi'(\frac{\eta_j}{2})$
- have

  $$\operatorname{var}\{\hat{\beta}^{(wls)}\} = \frac{\sum w_j}{\sum w_j \sum w_j \log^2(\tau_j) - \left(\sum w_j \log{(\tau_j)}\right)^2}$$

- use $\delta = \frac{1}{2}(\beta + 1)$ to get $\hat{\delta}^{(wls)} \equiv \frac{1}{2}(\hat{\beta}^{(wls)} + 1)$ with $\operatorname{var}\{\hat{\delta}^{(wls)}\} = \frac{1}{4}\operatorname{var}\{\hat{\beta}^{(wls)}\}$
- works well in Monte Carlo simulations

Analysis of ABL Data: I

- initial approach: divide into nonoverlapping blocks
  - each block has 10,000 points
  - blocks are contiguous
  - allows analysis out to $\tau_{10} = 9.37$ meters
- Fig. 7: wavelet variance estimates for 'typical' block
  - based upon LA(8) wavelet filter
  - single δ (i.e., power law) inadequate
  - will combine 3 adjacent scales via separate FD models
- Fig. 8, lower left-hand portion: scatter plots for $\log(\hat{\nu}^2_{X,b}(\tau_j))$
  - b is block index
  - $\log(\hat{\nu}^2_{X,b}(\tau_j))$ versus $\log(\hat{\nu}^2_{X,b}(\tau_k))$ for different j, k
  - lines shows expected pattern if σ^2_ϵ held fixed, but δ is changing across blocks
  - reasonable agreement at higher scales when $k = j \pm 1$ or $k = j \pm 2$

Analysis of ABL Data: II

- Fig. 8, upper right-hand portion: ‘slope differential’ plots
  - plot for $(j, j+1)$ with $(k, k+1)$ defined as

$$\frac{\log\left(\hat{\nu}_{X,b}^2(\tau_{j+1})\right) - \log\left(\hat{\nu}_{X,b}^2(\tau_j)\right)}{\log\left(\hat{\nu}_{X,b}^2(\tau_{k+1})\right) - \log\left(\hat{\nu}_{X,b}^2(\tau_k)\right)} - 1 \text{ versus } b$$

  - above is zero if estimated slopes are identical
  - box plots assess significance of deviations from 0
- conclusion: reasonable to combine scales as suggested by ‘typical’ block
- Fig. 9: blocked WLS estimates of power law exponent $\alpha \equiv -2\delta$
  - scale τ_4 has periodic burst (artifact?)
  - scales τ_5, τ_6, τ_7 swing from $\alpha = 0$ to $-\frac{5}{3}$
  - 95% confidence intervals say variations in α are significant

Analysis of ABL Data: III

- Fig. 10: comparison of WLS estimates for scales τ_5, τ_6, τ_7 and $\tau_8, \tau_9, \tau_{10}$
  - two groups do not track each other
  - largest scales generally consistent with $-\frac{5}{3}$ power law, but show significant deviations at times
- Figs. 11–2: corresponding plots for ML estimates
  - very good agreement with WLS estimates (except for τ_1, τ_2, τ_3 – not surprising)
  - 95% confidence intervals similar to those for WLS (but now block dependent)

Future Work

- 'instantaneous' LS and ML estimates
  - designed to get away from block dependence
  - Fig. 13: use MODWT coefficients co-located across scales (one coefficient per scale)
  - easy to modify LS and ML estimators
  - Fig. 14: preliminary LS results for scales τ_5, τ_6, τ_7
    - individual estimates very noisy, so have smoothed
    - good agreement with blocked estimates
  - need to study distributional properties of instantaneous estimates
  - need to study ways to denoise instantaneous estimates (waveshrink)
- need to study ways to model evolution of α
- need to study ways of combining multiscale models

Papers, Thesis and Book

- P. F. Craigmile, D. B. Percival and P. Guttorp (2000), 'Wavelet-Based Parameter Estimation for Trend Contaminated Fractionally Differenced Processes,' submitted to *JTSA*; TRS #47 at

`http://www.nrcse.washington.edu/research/reports/reports.asp`

- P. F. Craigmile, D. B. Percival and P. Guttorp (2000), 'Decorrelation Properties of Wavelet Based Estimators for Fractionally Differenced Processes,' *Proceedings of the 3ecm*; TRS #49 at above.
- W. Constantine, D. B. Percival and P. G. Reinhall (2000), 'Modeling aerothermal turbulence using fractionally differenced processes,' MathSoft Tech. Rep.
- P. F. Craigmile (2000), 'Wavelet Based Estimation for Trend Contaminated Long Memory Processes,' Ph.D. dissertation, Department of Statistics, University of Washington, Seattle
- D. B. Percival and A. T. Walden (2000), *Wavelet Methods for Time Series Analysis*, Cambridge, UK: Cambridge University Press

  `http://www.staff.washington.edu/dbp/wmtsa.html`