Discrete Wavelet Transforms Based on Zero-Phase Daubechies Filters

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overheads for talk available at

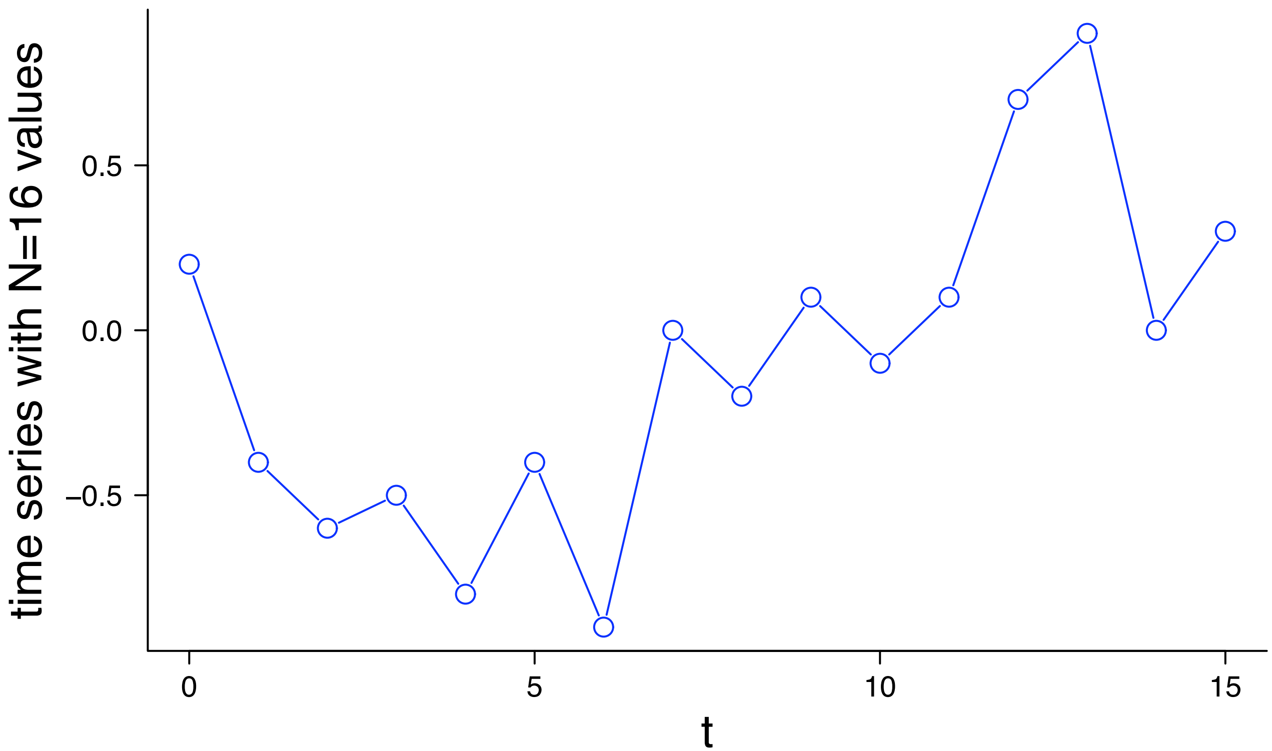
`http://faculty.washington.edu/dbp/talks.html`

Overview

- will discuss work in progress on the 'zephlet' transform, an orthonormal discrete wavelet transform (DWT) based on zero-phase filters
- will start by giving some background on the DWT as formulated in Daubechies (1992) – see, e.g., Percival & Walden (2000) or Gençay et al. (2002) for further details
- will then describe the zephlet transform and how it differs from the usual DWT, with an illustration of some of its properties

Background on DWT: I

- let $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ be a vector of N time series values (note: 'T' denotes transpose; i.e., $\mathbf{X}$ is a column vector)
- for simplicity, assume N is an even number



Background on DWT: II

- DWT is a linear transform of $\mathbf{X}$ yielding N DWT coefficients
- notation: $\mathbf{W} = \mathcal{W}\mathbf{X}$, where $\mathbf{W}$ is vector of DWT coefficients, and $\mathcal{W}$ is $N \times N$ *orthonormal* transform matrix
- orthonormality says $\mathcal{W}^T\mathcal{W} = I_N$ ($N \times N$ identity matrix)
- orthonormality is exploited heavily in, among other uses, DWT-based extraction of signals ('wavelet shrinkage')
- to focus discussion, will concentrate on so-called unit-level DWT, for which $\mathbf{W} = [\mathbf{W}_1^T, \mathbf{V}_1^T]^T$, where the two subvectors contain
  - wavelet coefficients $\mathbf{W}_1 = [W_{1,0}, W_{1,0}, \ldots, W_{1,\frac{N}{2}-1}]^T$ and
  - scaling coefficients $\mathbf{V}_1 = [V_{1,0}, V_{1,0}, \ldots, V_{1,\frac{N}{2}-1}]^T$
- higher-level DWTs use unit-level DWTs over and over again

The Wavelet Filter: I

- matrix $\mathcal{W}$ is rarely constructed explicitly, but rather is formed implicitly by use of a wavelet filter
- let $\{h_l : l = 0, \ldots, L-1\}$ be a real-valued filter of width L
- for convenience, will define $h_l = 0$ for $l < 0$ and $l \geq L$

The Wavelet Filter: II

- $\{h_l\}$ called a wavelet filter if it has these 3 properties

  1. summation to zero:
$$\sum_{l=0}^{L-1} h_l = 0$$
  2. unit 'energy' (i.e., squared Euclidean norm):
$$\sum_{l=0}^{L-1} h_l^2 = 1$$
  3. orthogonality to even shifts: for all nonzero integers n, have
$$\sum_{l=0}^{L-1} h_l h_{l+2n} = 0$$

- 2 and 3 together are called the *orthonormality property*

The Wavelet Filter: III

- summation to zero and unit energy relatively easy to achieve
- orthogonality to even shifts is key property & hardest to satisfy (implies L must be even; common choices are $2, 4, \ldots, 20$)
- define transfer function for wavelet filter, i.e., its discrete Fourier transform (DFT), along with its squared gain function:

$$H(f) \equiv \sum_{l=0}^{L-1} h_l e^{-i2\pi f l} \text{ and } \mathcal{H}(f) \equiv |H(f)|^2$$

- orthonormality property is equivalent to

$$\mathcal{H}(f) + \mathcal{H}(f + \tfrac{1}{2}) = 2 \quad \text{for all } f$$

(an elegant – but not obvious! – result)

The Wavelet Filter: IV

- simplest wavelet filter is Haar ($L = 2$): $h_0 = \frac{1}{\sqrt{2}}$ & $h_1 = -\frac{1}{\sqrt{2}}$
- note that $h_0 + h_1 = 0$ and $h_0^2 + h_1^2 = 1$, as required
- orthogonality to even shifts also readily apparent
- squared gain function is

$$\mathcal{H}(f) = 2\sin^2(\pi f),$$

for which

$$\begin{aligned} \mathcal{H}(f) + \mathcal{H}(f + \tfrac{1}{2}) &= 2\sin^2(\pi f) + 2\sin^2(\pi[f + \tfrac{1}{2}]) \\ &= 2\sin^2(\pi f) + 2\cos^2(\pi f) \\ &= 2, \end{aligned}$$

as required

Construction of Wavelet Coefficients: I

- given wavelet filter $\{h_l\}$ of width L & time series of even length, obtain wavelet coefficients as follows
- *circularly* filter $\mathbf{X}$ with wavelet filter to yield output

$$\sum_{l=0}^{L-1} h_l X_{t-l} = \sum_{l=0}^{L-1} h_l X_{t-l \bmod N}, \quad t = 0, \ldots, N-1;$$

i.e., if $t - l$ does not satisfy $0 \le t - l \le N-1$, interpret X_{t-l} as $X_{t-l \bmod N}$; for example, $X_{-1} = X_{N-1}$ and $X_{-2} = X_{N-2}$

- take every other value of filter output to define

$$W_{1,t} \equiv \sum_{l=0}^{L-1} h_l X_{2t+1-l \bmod N}, \quad t = 0, \ldots, \frac{N}{2} - 1;$$

$\mathbf{W}_1$ formed by *downsampling* filter output by a factor of 2

Construction of Wavelet Coefficients: II

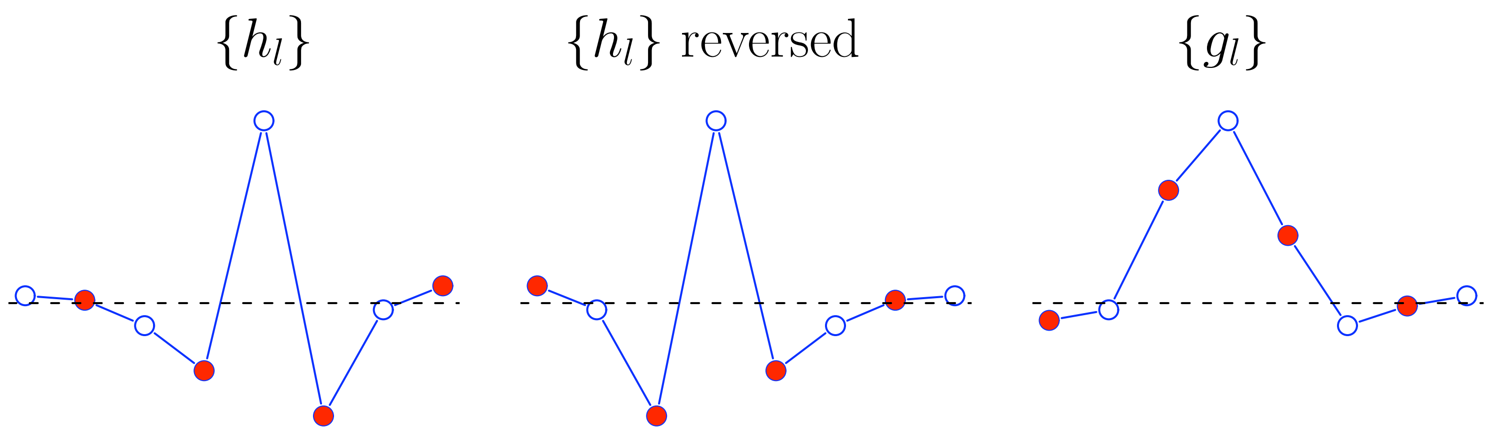
- can write $\mathbf{W}_1 = \mathcal{W}_1 \mathbf{X}$, where, when $N \geq 10$ for example,

$$\mathcal{W}_1 \equiv \begin{bmatrix} h_1^\circ & h_0^\circ & h_{N-1}^\circ & h_{N-2}^\circ & h_{N-3}^\circ & \cdots & h_5^\circ & h_4^\circ & h_3^\circ & h_2^\circ \\ h_3^\circ & h_2^\circ & h_1^\circ & h_0^\circ & h_{N-1}^\circ & \cdots & h_7^\circ & h_6^\circ & h_5^\circ & h_4^\circ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ h_{N-1}^\circ & h_{N-2}^\circ & h_{N-3}^\circ & h_{N-4}^\circ & h_{N-5}^\circ & \cdots & h_3^\circ & h_2^\circ & h_1^\circ & h_0^\circ \end{bmatrix}$$

- here $h_l^\circ = h_l$ when $L \leq N$, but takes different form if $L > N$; for example, if $N = 10$ and $L = 20$, $h_l^\circ = h_l + h_{l+10}$
- can argue that $\mathcal{W}_1 \mathcal{W}_1^T = I_{N/2}$ for all L and N
- $\mathcal{W}_1$ is the top *half* of orthonormal transform matrix $\mathcal{W}$

The Scaling Filter: I

- create scaling filter $\{g_l\}$ by reversing $\{h_l\}$ and then changing sign of coefficients with even indices



- precise definition is $g_l \equiv (-1)^{l+1} h_{L-1-l}$

The Scaling Filter: II

- properties 2 and 3 (orthonormality) of $\{h_l\}$ are shared by $\{g_l\}$:

  2. unit energy:

$$\sum_{l=0}^{L-1} g_l^2 = 1$$

  3. orthogonality to even shifts: for all nonzero integers n, have

$$\sum_{l=0}^{L-1} g_l g_{l+2n} = 0$$

- squared gain function $\mathcal{G}(\cdot)$ for scaling filter satisfies

$$\mathcal{G}(f) = \mathcal{H}(f + \tfrac{1}{2}) \text{ and hence } \mathcal{H}(f) + \mathcal{G}(f) = 2$$

  is equivalent way of stating orthonormality property

Construction of Scaling Coefficients: I

- orthonormality property of $\{h_l\}$ is all that is needed to prove $\mathcal{W}_1$ is half of an orthonormal transform (never used $\sum_l h_l = 0$)
- going back and replacing h_l with g_l everywhere yields another half of an orthonormal transform
- circularly filter $\mathbf{X}$ using $\{g_l\}$ and downsample to define scaling coefficients:

$$V_{1,t} \equiv \sum_{l=0}^{L-1} g_l X_{2t+1-l \bmod N}, \quad t = 0, \ldots, \frac{N}{2} - 1$$

Construction of Scaling Coefficients: II

- have $\mathbf{V}_1 = \mathcal{V}_1 \mathbf{X}$, where $\mathcal{V}_1$ is analogous to $\mathcal{W}_1$:

$$\mathcal{V}_1 = \begin{bmatrix} g_1^\circ & g_0^\circ & g_{N-1}^\circ & g_{N-2}^\circ & g_{N-3}^\circ & \cdots & g_5^\circ & g_4^\circ & g_3^\circ & g_2^\circ \\ g_3^\circ & g_2^\circ & g_1^\circ & g_0^\circ & g_{N-1}^\circ & \cdots & g_7^\circ & g_6^\circ & g_5^\circ & g_4^\circ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ g_{N-1}^\circ & g_{N-2}^\circ & g_{N-3}^\circ & g_{N-4}^\circ & g_{N-5}^\circ & \cdots & g_3^\circ & g_2^\circ & g_1^\circ & g_0^\circ \end{bmatrix}$$

- as before, can argue that $\mathcal{V}_1 \mathcal{V}_1^T = I_{N/2}$
- in addition, each row in $\mathcal{W}_1$ is orthogonal to each row in $\mathcal{V}_1$ and hence

$$\mathcal{W} \equiv \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{V}_1 \end{bmatrix} \quad \text{is an orthonormal transform}$$

Daubechies Scaling Filters

- Daubechies (1992) constructs a family of scaling filters $\{g_l\}$ with squared gain functions given by

$$\mathcal{G}_{(\mathrm{D})}(f) \equiv 2\cos^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \sin^{2l}(\pi f)$$

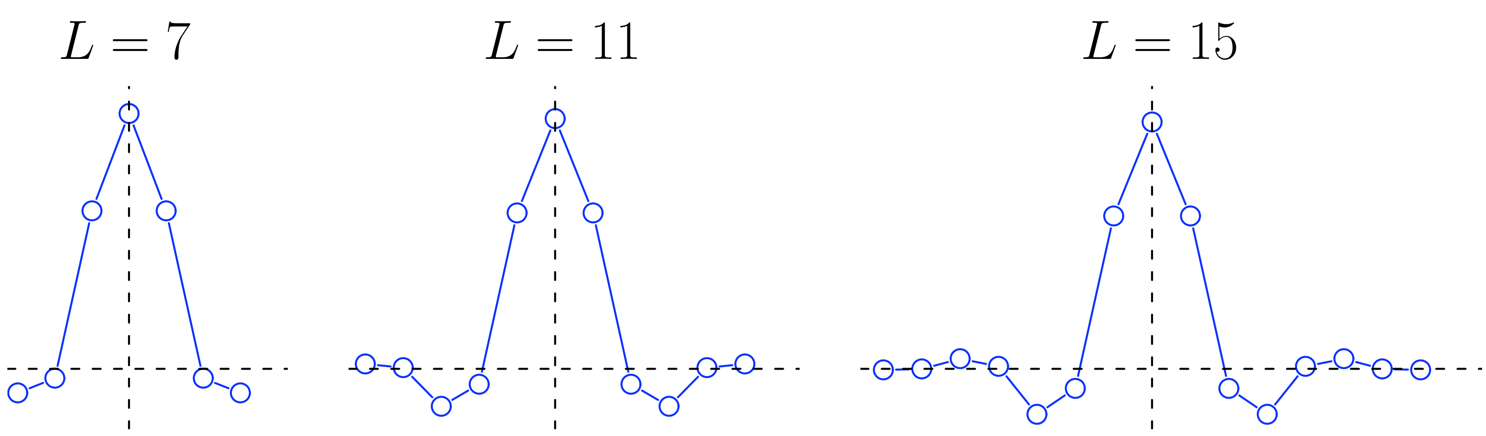
(corresponding wavelet filter given by $h_l = (-1)^l g_{L-1-l}$)

- for given L, there are multiple filters with the same $\mathcal{G}_{(\mathrm{D})}(\cdot)$, with these filters being distinguished by their phase functions $\theta(\cdot)$; i.e., their transfer functions can be written as

$$G(f) \equiv \sum_{l=0}^{L-1} g_l e^{-i2\pi f l} = \mathcal{G}_{(\mathrm{D})}^{1/2}(f) e^{i\theta(f)}$$

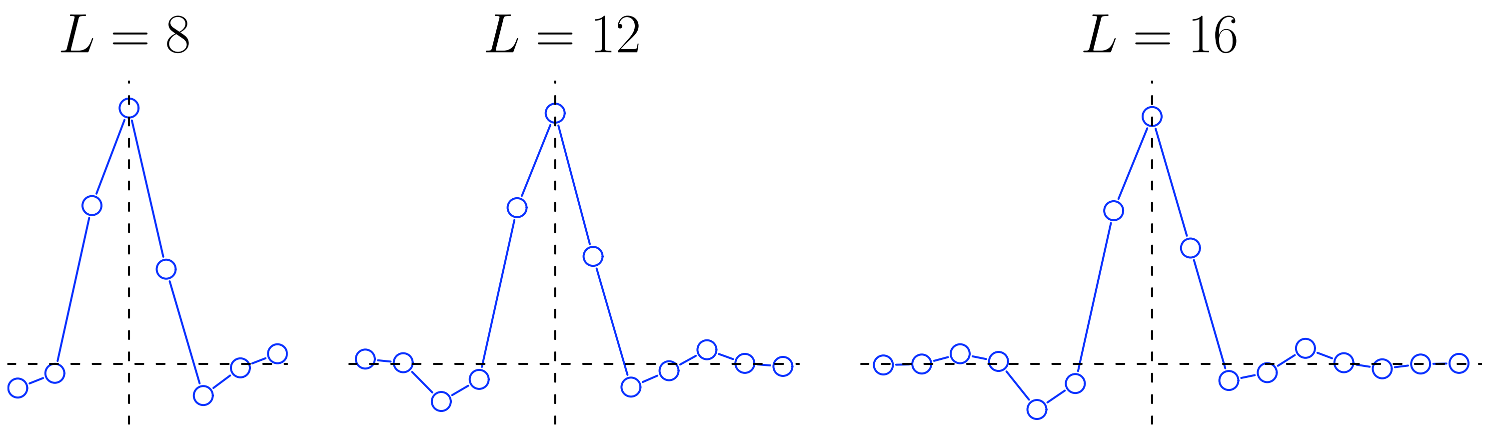
Zero-Phase Filters

- Oppenheim and Lim (1981) note that filters with zero phase (i.e., $\theta(f) = 0$ for all f) are important for eliminating distortions in filtered signals (particularly in images)
- zero-phase filters also facilitate aligning filter output with input
- conventional zero-phase filters $\{a_l\}$ must be of *odd* length, say $L = 2M + 1$, and take the form $a_{-l} = a_l$ for $l = -M, \ldots, M$
- three examples of zero-phase filters



‘Least Asymmetric’ Scaling Filters (Symlets)

- in recognition of importance of zero-phase filters, Daubechies (1992) uses spectral factorization to obtain filters of widths $L = 8, 10, 12, \ldots$ closest to having zero phase (after a reindexing)
- three members of her class of ‘least asymmetic’ scaling filters



- cannot achieve filters with *exact* zero phase under her scheme because L must be even

Zero-Phase Wavelet (Zephlet) Transform: I

- possible to construct orthonormal DWT based on filters whose squared gain functions are consistent with those of Daubechies, but with *exact* zero phase, as following theorem states
- let $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ be squared gain functions satisfying

$$\mathcal{G}(\tfrac{k}{N}) + \mathcal{G}(\tfrac{k}{N} + \tfrac{1}{2}) = 2 \text{ and } \mathcal{H}(\tfrac{k}{N}) + \mathcal{G}(\tfrac{k}{N}) = 2 \text{ for all } \tfrac{k}{N}$$

- let $\{\bar{g}_l\}$ & $\{\bar{h}_l\}$ be inverse DFTs of the sequences $\{\mathcal{G}^{1/2}(\frac{k}{N})\}$ & $\{\mathcal{H}^{1/2}(\frac{k}{N})\}$:

$$\bar{g}_l \equiv \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{G}^{1/2}(\tfrac{k}{N}) e^{i2\pi kl/N}, \quad l = 0, 1, \ldots, N-1,$$

with an analogous expression for $\bar{h}_l$

Zero-Phase Wavelet (Zephlet) Transform: II

- define the $\frac{N}{2} \times N$ matrices

$$\mathcal{D}_1 = \begin{bmatrix} \bar{h}_1 & \bar{h}_0 & \bar{h}_{N-1} & \bar{h}_{N-2} & \bar{h}_{N-3} & \cdots & \bar{h}_5 & \bar{h}_4 & \bar{h}_3 & \bar{h}_2 \\ \bar{h}_3 & \bar{h}_2 & \bar{h}_1 & \bar{h}_0 & \bar{h}_{N-1} & \cdots & \bar{h}_7 & \bar{h}_6 & \bar{h}_5 & \bar{h}_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{h}_{N-1} & \bar{h}_{N-2} & \bar{h}_{N-3} & \bar{h}_{N-4} & \bar{h}_{N-5} & \cdots & \bar{h}_3 & \bar{h}_2 & \bar{h}_1 & \bar{h}_0 \end{bmatrix}$$

and

$$\mathcal{C}_1 = \begin{bmatrix} \bar{g}_0 & \bar{g}_{N-1} & \bar{g}_{N-2} & \bar{g}_{N-3} & \bar{g}_{N-4} & \cdots & \bar{g}_4 & \bar{g}_3 & \bar{g}_2 & \bar{g}_1 \\ \bar{g}_2 & \bar{g}_1 & \bar{g}_0 & \bar{g}_{N-1} & \bar{g}_{N-2} & \cdots & \bar{g}_6 & \bar{g}_5 & \bar{g}_4 & \bar{g}_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{g}_{N-2} & \bar{g}_{N-3} & \bar{g}_{N-4} & \bar{g}_{N-5} & \bar{g}_{N-6} & \cdots & \bar{g}_2 & \bar{g}_1 & \bar{g}_0 & \bar{g}_{N-1} \end{bmatrix}$$

(note that, while $\mathcal{D}_1$ has a form analogous to $\mathcal{W}_1$ & $\mathcal{V}_1$, rows of $\mathcal{C}_1$ are circularly shifted to the left by one)

Zero-Phase Wavelet (Zephlet) Transform: III

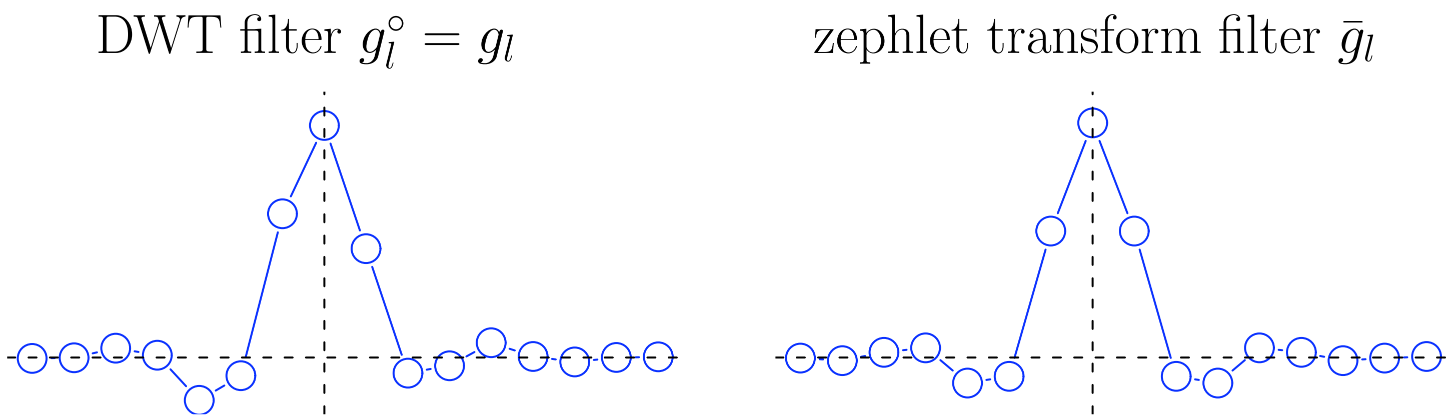
- then the $N \times N$ matrix formed by stacking $\mathcal{D}_1$ on top of $\mathcal{C}_1$ is a real-valued orthonormal matrix; i.e,

$$\mathcal{D} \equiv \begin{bmatrix} \mathcal{D}_1 \\ \mathcal{C}_1 \end{bmatrix} \text{ is such that } \mathcal{D}^T \mathcal{D} = I_N$$

- moreover, the zero-phase circular filters $\{\bar{h}_l\}$ and $\{\bar{g}_l\}$ are related by $\bar{g}_l = (-1)^l \bar{h}_l$ (note that this is in contrast to what holds for DWT filters, namely, $g_l = (-1)^{l+1} h_{L-1-l}$)
- proof of above theorem is similar in spirit to proof that $\mathcal{W}$ is orthonormal, but details differ
- algorithms for computing DWT and zephlet transform are, respectively, $\mathcal{O}(N)$ and $\mathcal{O}(N \cdot \log_2(N))$

Zero-Phase Wavelet (Zephlet) Transform: IV

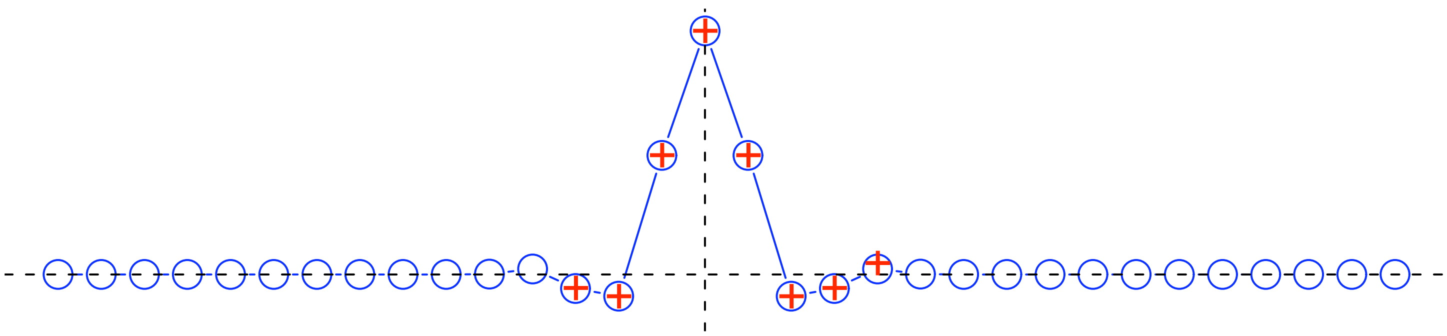
- for case $N = L = 16$, let's compare values in rows of $\mathcal{V}_1$ based on Daubechies' least asymmetric filter and corresponding $\mathcal{C}_1$ (after alignments for easier comparison)



- for any N and L, squared magnitudes of DFTs of $\{g_l^\circ\}$ & $\{\bar{g}_l\}$ at $f_k = k/N$ are exactly the same, but phase functions differ, with that for $\{\bar{g}_l\}$ given by $\theta(f_k) = 0$

Zero-Phase Wavelet (Zephlet) Transform: V

- for fixed $L \geq 8$, values in rows of zephlet transform change as N increases (DWT rows just add more 0's for all $N \geq L$)
- consider zephlet transform based on least asymmetric filter for $L = 8$ and cases $N = 8$ (pluses) and $N = 32$ (circles)



Zero-Phase Wavelet (Zephlet) Transform: VI

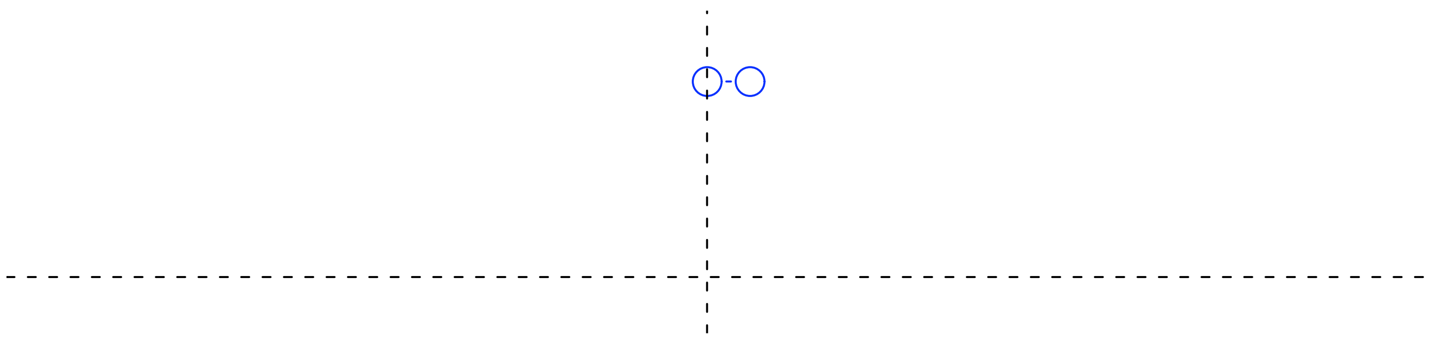
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm 1}{4})}{\sin(\pi\frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 2$:



Zero-Phase Wavelet (Zephlet) Transform: VI

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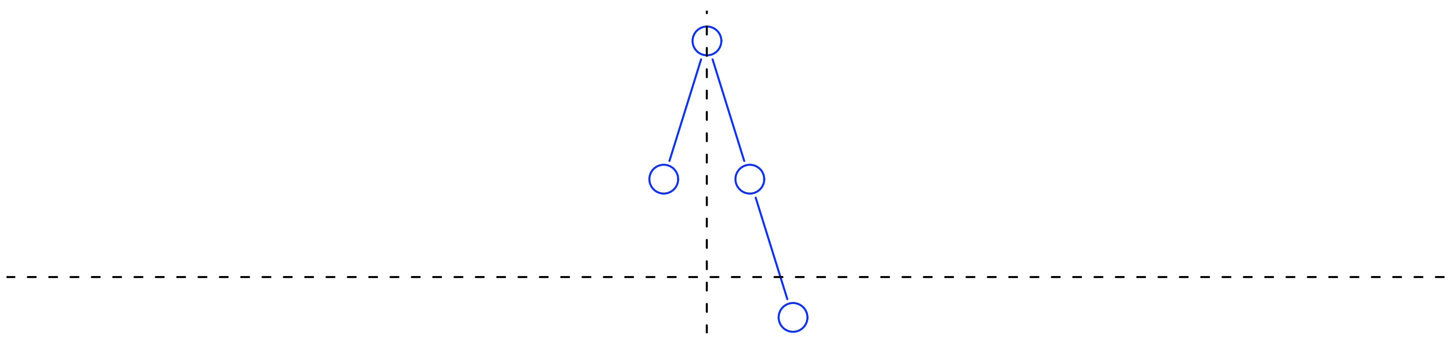
- Haar-based $\{\bar{g}_l\}$ for $N = 4$:



Zero-Phase Wavelet (Zephlet) Transform: VI

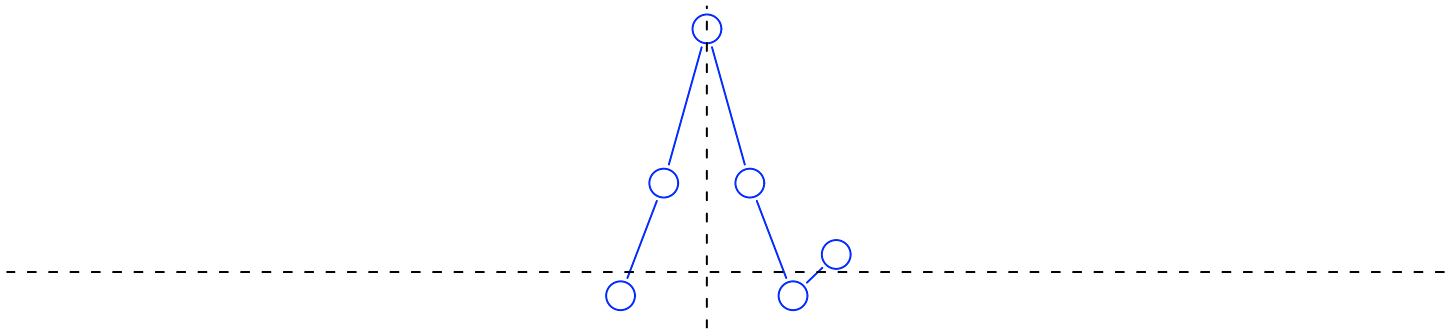
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 6$:



Zero-Phase Wavelet (Zephlet) Transform: VI

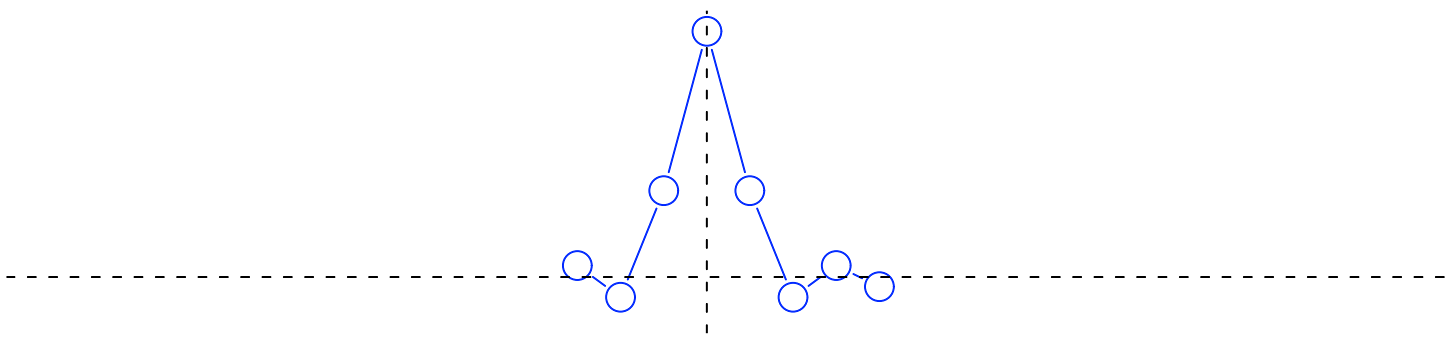
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1 - 4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 8$:



Zero-Phase Wavelet (Zephlet) Transform: VI

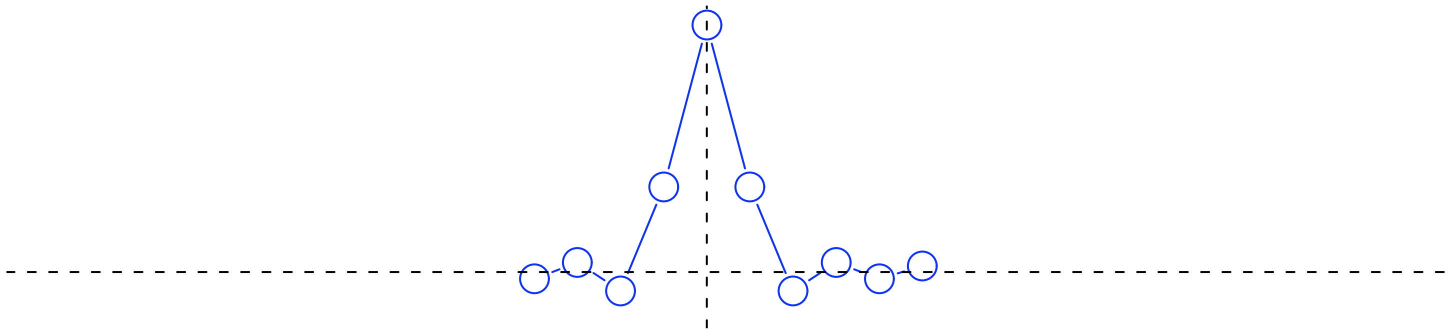
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 10$:



Zero-Phase Wavelet (Zephlet) Transform: VI

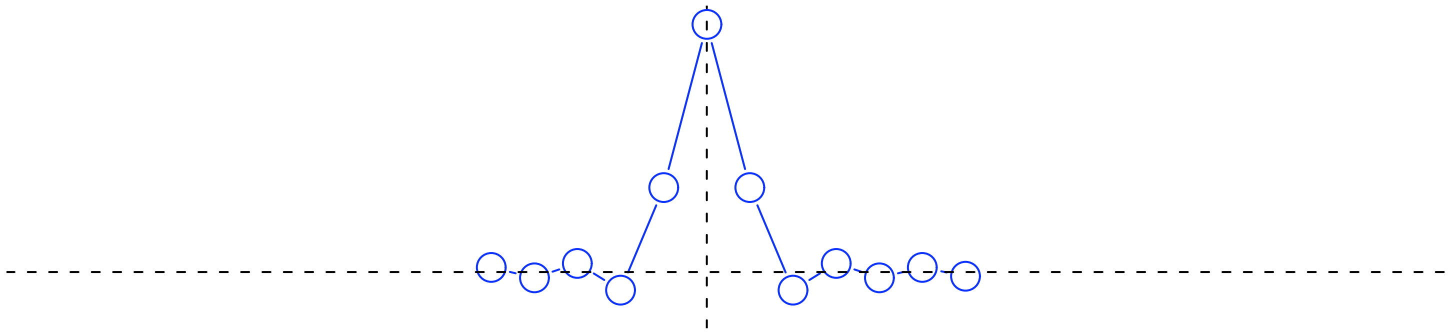
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm 1}{4})}{\sin(\pi\frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 12$:



Zero-Phase Wavelet (Zephlet) Transform: VI

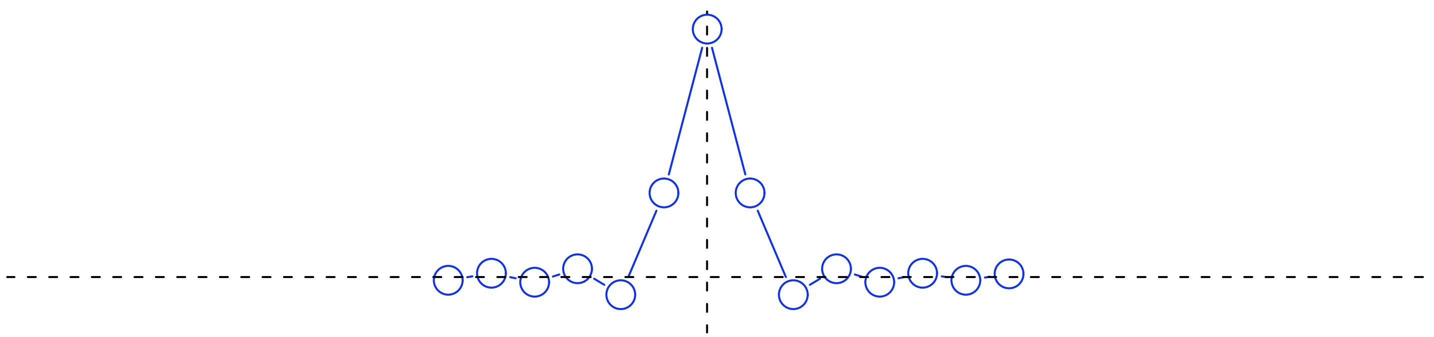
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 14$:



Zero-Phase Wavelet (Zephlet) Transform: VI

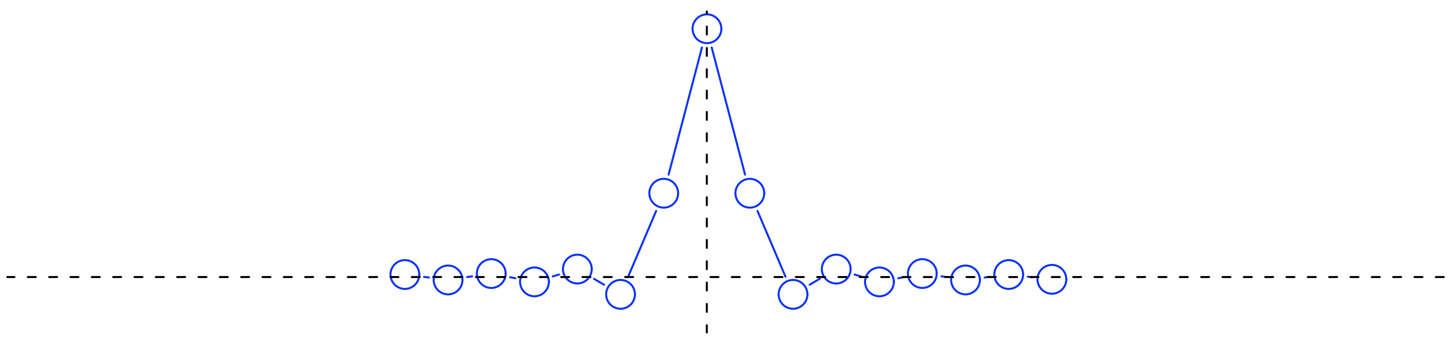
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 16$:



Zero-Phase Wavelet (Zephlet) Transform: VI

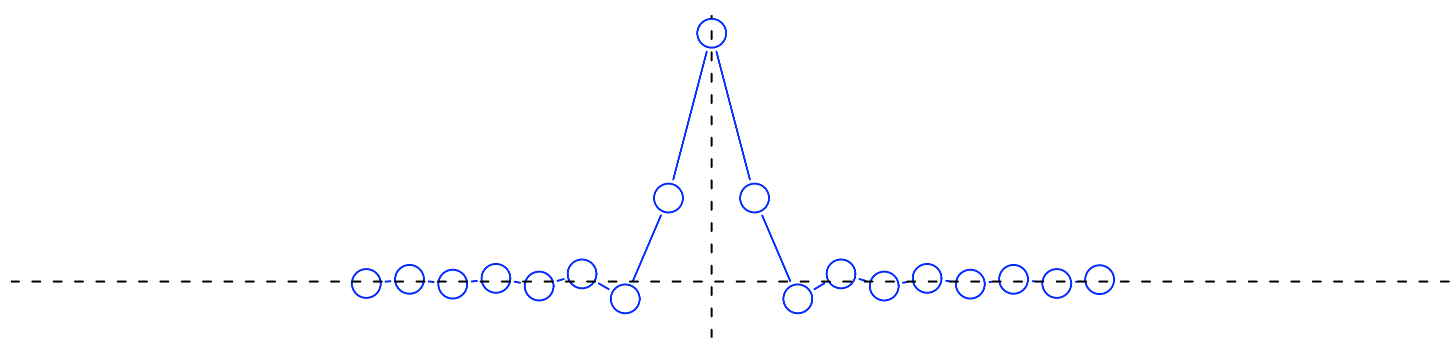
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 18$:



Zero-Phase Wavelet (Zephlet) Transform: VI

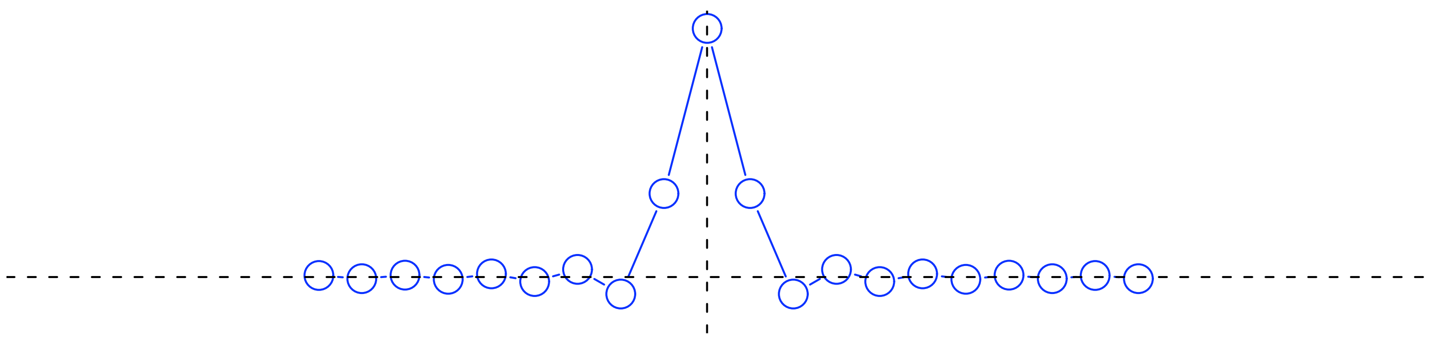
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 20$:



Zero-Phase Wavelet (Zephlet) Transform: VI

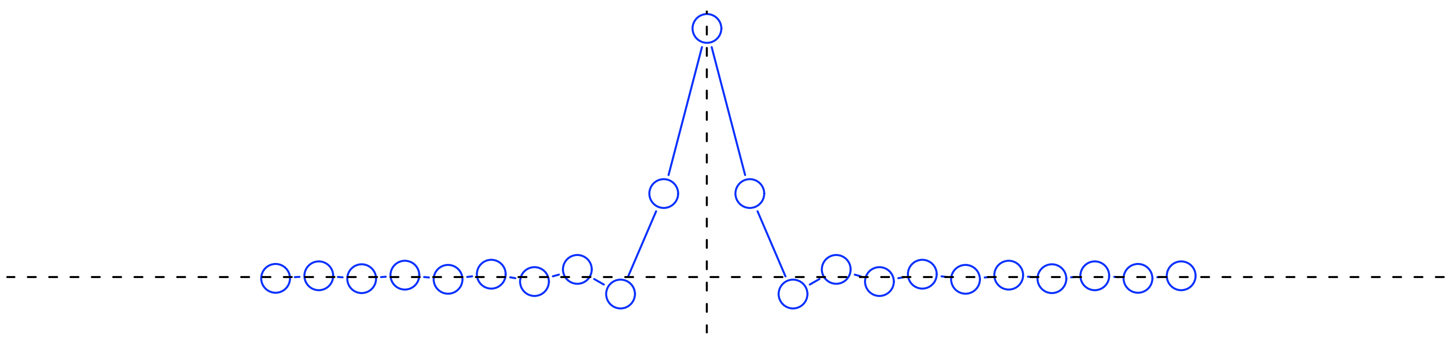
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 22$:



Zero-Phase Wavelet (Zephlet) Transform: VI

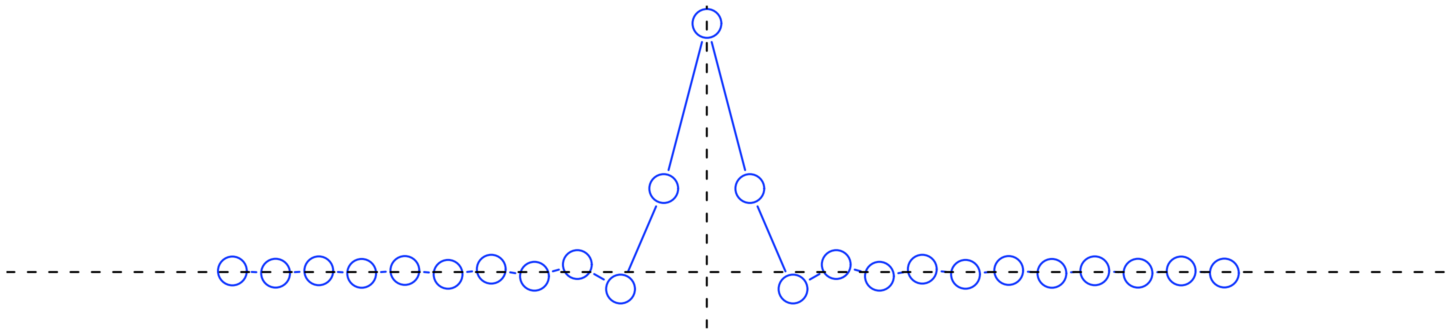
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi \tfrac{M-1}{4M}) \frac{\sin(\pi \frac{2l\pm 1}{4})}{\sin(\pi \frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 24$:



Zero-Phase Wavelet (Zephlet) Transform: VI

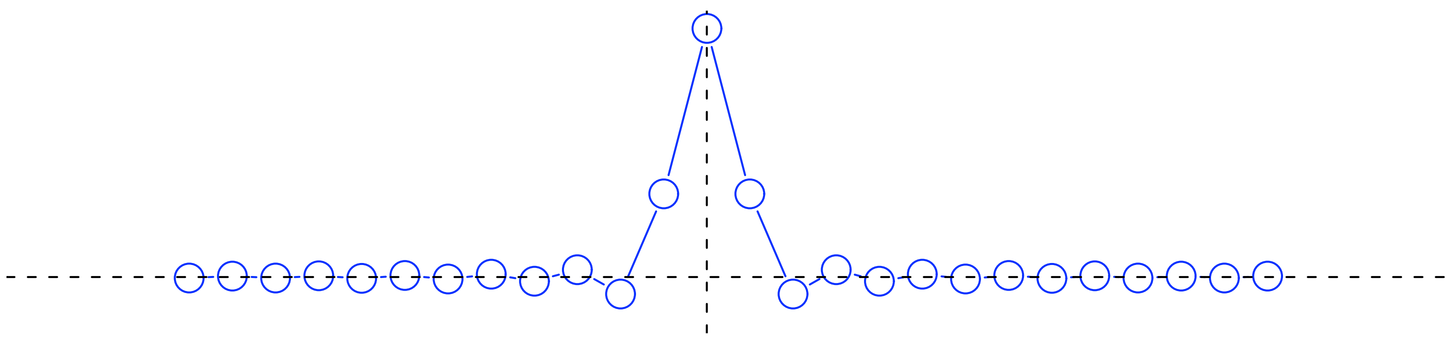
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 26$:



Zero-Phase Wavelet (Zephlet) Transform: VI

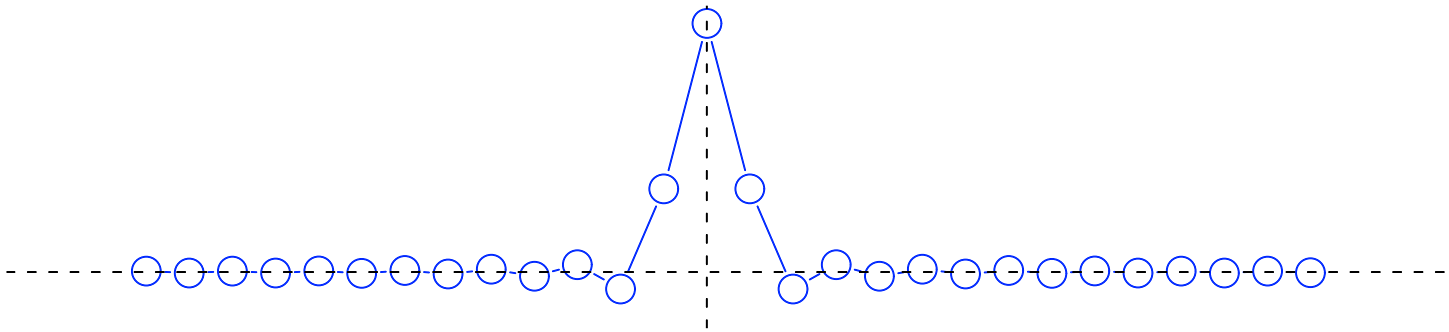
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

  for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 28$:



Zero-Phase Wavelet (Zephlet) Transform: VI

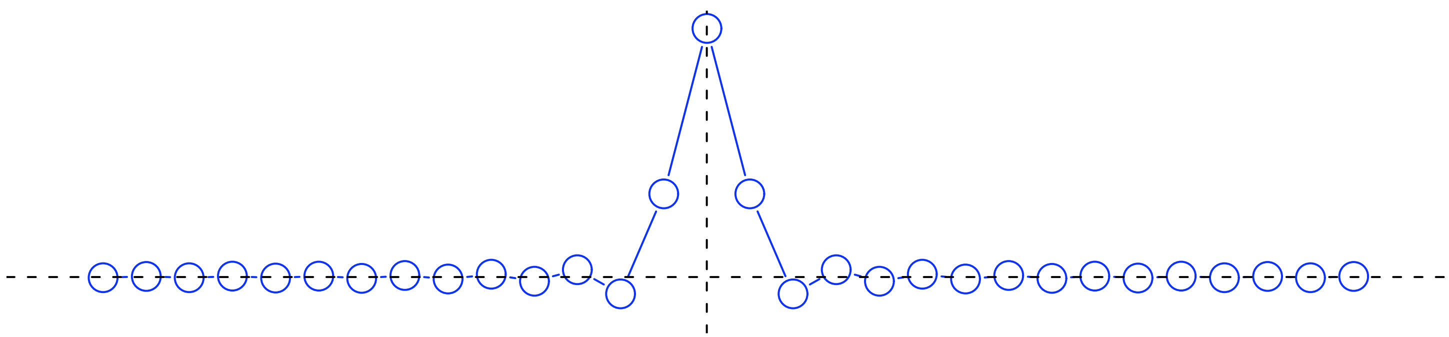
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm 1}{4})}{\sin(\pi\frac{2l\pm 1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 30$:



Zero-Phase Wavelet (Zephlet) Transform: VI

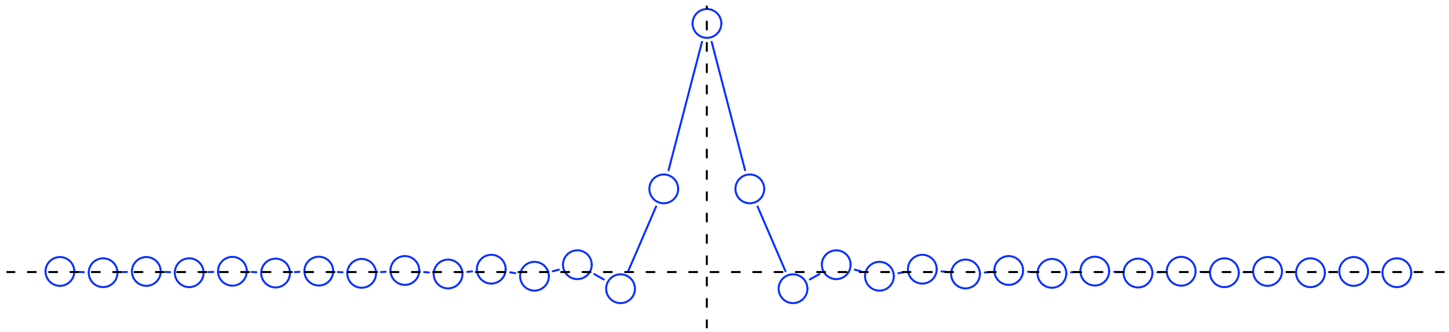
- can work out expression for elements in zephlet transform explicitly in Haar case ($L = 2$):

$$\bar{g}_l = \frac{\sqrt{2}}{N}\left[1 + (-1)^l S_{l,+} + (-1)^{l+1} S_{l,-}\right] \approx \frac{2(-1)^l\sqrt{2}}{\pi(1-4l^2)}$$

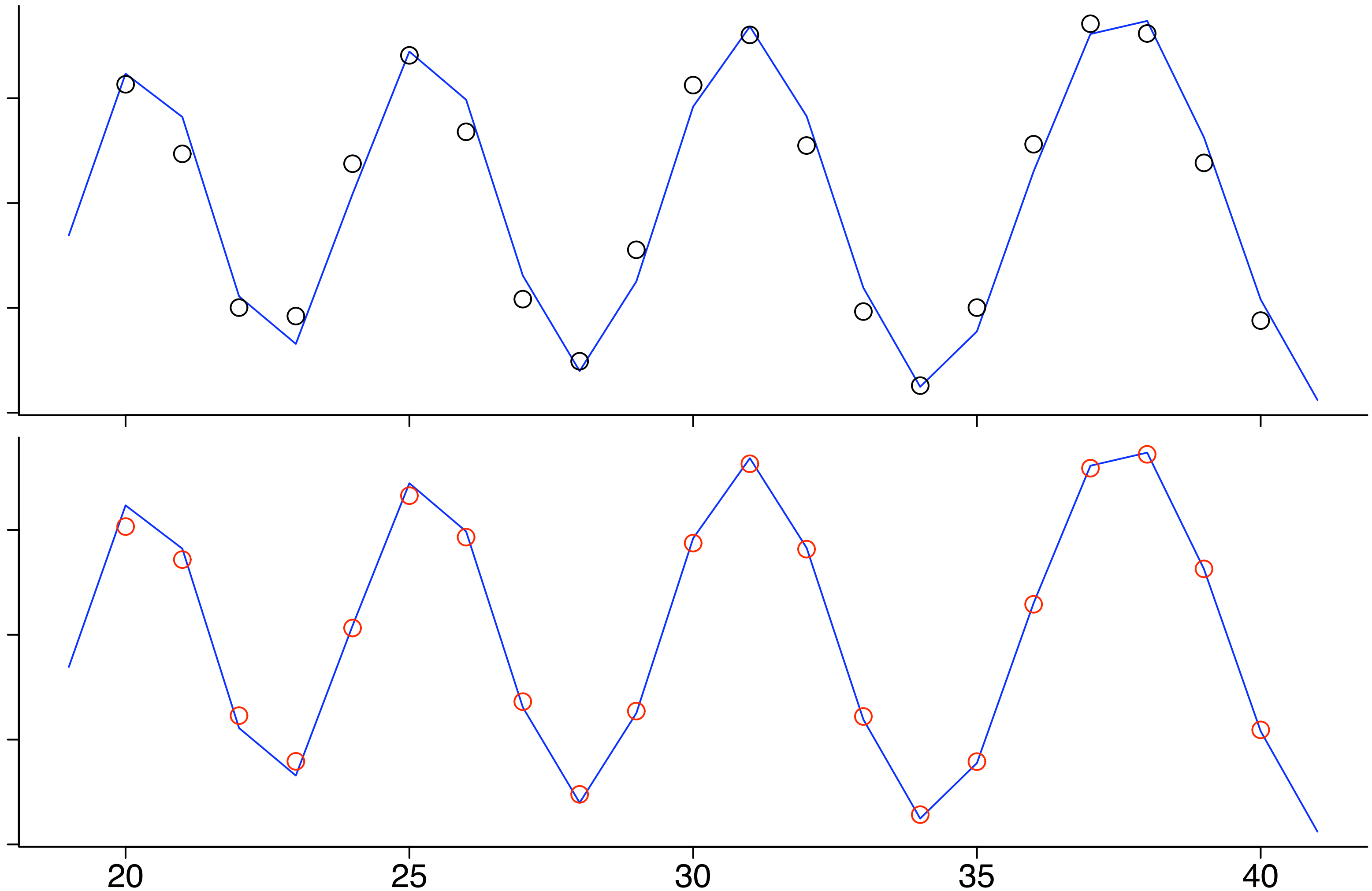
for large N, where

$$S_{l,\pm} \equiv \sin((2l \pm 1)\pi\tfrac{M-1}{4M})\frac{\sin(\pi\frac{2l\pm1}{4})}{\sin(\pi\frac{2l\pm1}{4M})}$$

- Haar-based $\{\bar{g}_l\}$ for $N = 32$:



Comparison of Outputs from LA(8) & Zephlet Scaling Filters (Input is Doppler Signal)



Concluding Remarks

- more work needed to elicit advantages/disadvantages of zephlet transform over usual DWT (in particular, for economic applications)
- can also formulate 'maximal overlap' version of zephlet transform (details in Percival, 2010)
- thanks to Ramo Gençay & conference organizers for opportunity to talk!
- research supported in part by U.S. National Science Foundation Grant No. ARC 0529955 (any opinions, findings and conclusions or recommendations expressed in this talk are those of the author and do not necessarily reflect the views of the National Science Foundation)

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