

**A Tutorial on Stochastic Models  
and Statistical Analysis  
for Frequency Stability Measurements**

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# Introduction

- time scales limited by clock noise
- can model clock noise as stochastic process  $\{X_t\}$ 
  - set of random variables (RVs) indexed by  $t$
  - $X_t$  represents clock noise at time  $t$
  - will concentrate on sampled data, for which will take  $t \in \mathbb{Z} \equiv \{\dots, -1, 0, 1, \dots\}$   
(but sometimes use  $t \in \mathbb{Z}^* \equiv \{0, 1, 2, \dots\}$ )
- Q: which stochastic processes are useful models?
- Q: how can we deduce model parameters & other characteristics from observed data?
- will cover the following in this tutorial:
  - stationary processes & closely related processes
  - fractionally differenced & related processes
  - two analysis of variance (‘power’) techniques
    - \* spectral analysis
    - \* wavelet analysis
  - parameter estimation via analysis techniques

# Stationary Processes: I

- stochastic process  $\{X_t\}$  called stationary if
  - $E\{X_t\} = \mu_X$  for all  $t$ ;  
i.e., a constant that does not depend on  $t$
  - $\text{cov}\{X_t, X_{t+\tau}\} = s_{X,\tau}$ , all possible  $t$  &  $t + \tau$ ;  
i.e., depends on lag  $\tau$ , but not  $t$
- $\{s_{X,\tau} : \tau \in \mathbb{Z}\}$  is autocovariance sequence (ACVS)
- $s_{X,0} = \text{cov}\{X_t, X_t\} = \text{var}\{X_t\}$ ;  
i.e., process variance is constant for all  $t$
- spectral density function (SDF) given by

$$S_X(f) = \sum_{\tau=-\infty}^{\infty} s_{X,\tau} e^{-i2\pi f\tau}, \quad |f| \leq 1/2$$

note:  $S_X(-f) = S_X(f)$  for real-valued processes

## Stationary Processes: II

- if  $\{X_t\}$  has SDF  $S_X(\cdot)$ , then

$$\int_{-1/2}^{1/2} S_X(f) e^{i2\pi f\tau} df = s_{X,\tau}, \quad \tau \in \mathbb{Z}$$

- setting  $\tau = 0$  yields fundamental result:

$$\int_{-1/2}^{1/2} S_X(f) df = s_{X,0} = \text{var} \{X_t\};$$

i.e., SDF decomposes  $\text{var} \{X_t\}$  across frequencies  $f$

- if  $\{a_u\}$  is a filter, then (with ‘matching condition’)

$$Y_t \equiv \sum_{u=-\infty}^{\infty} a_u X_{t-u}$$

is stationary with SDF given by

$$S_Y(f) = \mathcal{A}(f) S_X(f), \quad \text{where } \mathcal{A}(f) \equiv \left| \sum_{u=-\infty}^{\infty} a_u e^{-i2\pi f u} \right|^2$$

- if  $\{a_u\}$  narrow-band of bandwidth  $\Delta f$  about  $f$ , i.e.,

$$\mathcal{A}(f') = \begin{cases} \frac{1}{2\Delta f}, & f - \frac{\Delta f}{2} \leq |f'| \leq f + \frac{\Delta f}{2} \\ 0, & \text{otherwise,} \end{cases}$$

then have following interpretation for  $S_X(f)$ :

$$\text{var} \{Y_t\} = \int_{-1/2}^{1/2} S_Y(f') df' = \int_{-1/2}^{1/2} \mathcal{A}(f') S_X(f') df' \approx S_X(f)$$

## White Noise Process

- simplest stationary process is white noise
- $\{\epsilon_t\}$  is white noise process if
  - $E\{\epsilon_t\} = \mu_\epsilon$  for all  $t$  (usually take  $\mu_\epsilon = 0$ )
  - $\text{var}\{\epsilon_t\} = \sigma_\epsilon^2$  for all  $t$
  - $\text{cov}\{\epsilon_t, \epsilon_{t'}\} = 0$  for all  $t \neq t'$
- white noise thus stationary with ACVS

$$s_{\epsilon, \tau} = \text{cov}\{\epsilon_t, \epsilon_{t+\tau}\} = \begin{cases} \sigma_\epsilon^2, & \tau = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and SDF

$$S_\epsilon(f) = \sum_{\tau=-\infty}^{\infty} s_{X, \tau} e^{-i2\pi f\tau} = \sigma_\epsilon^2$$

## Backward Differences of White Noise

- consider first order backward difference of white noise:

$$X_t = \epsilon_t - \epsilon_{t-1} = \sum_{u=-\infty}^{\infty} a_u \epsilon_{t-u} \text{ with } a_u \equiv \begin{cases} 1, & u = 0; \\ -1, & u = 1; \\ 0, & \text{otherwise.} \end{cases}$$

- have  $S_X(f) = \mathcal{A}(f)S_\epsilon(f) = |2 \sin(\pi f)|^2 \sigma_\epsilon^2 \approx |2\pi f|^2 \sigma_\epsilon^2$   
at low frequencies (using  $\sin(x) \approx x$  for small  $x$ )
- let  $B$  be backward shift operator:  $B\epsilon_t = \epsilon_{t-1}$ ,  
 $B^2\epsilon_t = \epsilon_{t-2}$ ,  $(1 - B)\epsilon_t = \epsilon_t - \epsilon_{t-1}$ , etc.

- consider  $d$ th order backward difference of white noise:

$$\begin{aligned} X_t = (1 - B)^d \epsilon_t &= \sum_{k=0}^d \binom{d}{k} (-1)^k \epsilon_{t-k} \\ &= \sum_{k=0}^d \frac{d!}{k!(d-k)!} (-1)^k \epsilon_{t-k} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(1 - \delta)}{\Gamma(k+1)\Gamma(1 - \delta - k)} (-1)^k \epsilon_{t-k} \end{aligned}$$

with  $\delta \equiv -d$ , i.e.,  $\delta = -1, -2, \dots$

- SDF given by

$$S_X(f) = \mathcal{A}(f)S_\epsilon(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta}} \approx \frac{\sigma_\epsilon^2}{|2\pi f|^{2\delta}}$$

## Fractional Differences of White Noise

- for  $\delta$  not necessary an integer,

$$X_t = \sum_{k=0}^{\infty} \frac{\Gamma(1 - \delta)}{\Gamma(k + 1)\Gamma(1 - \delta - k)} (-1)^k \epsilon_{t-k} \equiv \sum_{k=0}^{\infty} a_k(\delta) \epsilon_{t-k}$$

makes sense as long as  $\delta < 1/2$

- $\{X_t\}$  stationary fractionally differenced (FD) process
- SDF is as before:

$$S_X(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta}} \approx \frac{\sigma_\epsilon^2}{|2\pi f|^{2\delta}}$$

- $\{X_t\}$  said to obey power law at low frequencies if

$$\lim_{f \rightarrow 0} \frac{S_X(f)}{C|f|^\alpha} = 1$$

for  $C > 0$ ; i.e.,  $S_X(f) \approx C|f|^\alpha$  at low frequencies

- FD processes obey above with  $\alpha = -2\delta$
- note: FD process reduces to white noise when  $\delta = 0$

## ACVS & PACS for FD Processes

- for  $\delta < 1/2$  &  $\delta \neq 0, -1, \dots$ , ACVS given by

$$s_{X,\tau} = \sigma_\epsilon^2 \frac{\sin(\pi\delta)\Gamma(1-2\delta)\Gamma(\tau+\delta)}{\pi\Gamma(1+\tau-\delta)};$$

when  $\delta = 0, -1, \dots$ , have  $s_{X,\tau} = 0$  for  $|\tau| > -\delta$  &

$$s_{X,\tau} = \sigma_\epsilon^2 \frac{(-1)^\tau \Gamma(1-2\delta)}{\Gamma(1+\tau-\delta)\Gamma(1-\tau-\delta)}, \quad 0 \leq |\tau| \leq -\delta$$

- for all  $\delta < 1/2$ , have

$$s_{X,0} = \text{var} \{X_t\} = \sigma_\epsilon^2 \frac{\Gamma(1-2\delta)}{\Gamma^2(1-\delta)},$$

and rest of ACVS can be computed easily via

$$s_{X,\tau} = s_{X,\tau-1} \frac{\tau + \delta - 1}{\tau - \delta}, \quad \tau \in \mathbb{Z}^+ \equiv \{1, 2, \dots\}$$

(for negative lags  $\tau$ , recall that  $s_{X,-\tau} = s_{X,\tau}$ ).

- for all  $\delta < 1/2$ , partial autocorrelation sequence (PACS) given by

$$\phi_{t,t} \equiv \frac{\delta}{t - \delta}, \quad t \in \mathbb{Z}^+$$

(useful for constructing best linear predictors)

- FD processes thus have simple and easily computed expressions for SDF, ACVS and PACS



## Simulating Stationary FD Processes

- for  $-1 \leq \delta < 1/2$ , can obtain exact simulations via ‘circulant embedding’ (Davies–Harte algorithm)
- given  $s_{X,0}, \dots, s_{X,N}$ , use discrete Fourier transform (DFT) to compute

$$S_k \equiv \sum_{\tau=0}^N s_{X,\tau} e^{-i2\pi f_k \tau} + \sum_{\tau=N+1}^{2N-1} s_{X,2N-\tau} e^{-i2\pi f_k \tau}, \quad k = 0, \dots, N$$

- given  $2N$  independent Gaussian deviates  $\varepsilon_t$  with mean zero and variance  $\sigma_\varepsilon^2$ , compute

$$\mathcal{Y}_k \equiv \begin{cases} \varepsilon_0 \sqrt{2N S_0}, & k = 0; \\ (\varepsilon_{2k-1} + i\varepsilon_{2k}) \sqrt{N S_k}, & 1 \leq k < N; \\ \varepsilon_{2N-1} \sqrt{2N S_N}, & k = N; \\ \mathcal{Y}_{2N-k}^*, & N < k \leq 2N - 1; \end{cases}$$

(asterisk denotes complex conjugate)

- use inverse DFT to construct the real-valued sequence

$$Y_t = \frac{1}{2N} \sum_{k=0}^{2N-1} \mathcal{Y}_k e^{i2\pi f_k t}, \quad t = 0, \dots, 2N - 1$$

- $Y_0, Y_1, \dots, Y_{N-1}$  is exact simulation of FD process
- implication: can represent  $X_0, X_1, \dots, X_{N-1}$  as

$$X_t = \sum_{k=0}^{2N-1} c_{t,k}(\delta) \varepsilon_k \quad \text{rather than} \quad X_t = \sum_{k=0}^{\infty} a_k(\delta) \varepsilon_{t-k}$$

## Nonstationary FD Processes: I

- suppose  $X_t^{(1)}$  is FD process with parameter  $\delta^{(s)}$  such that  $-1/2 \leq \delta^{(s)} < 1/2$
- define  $X_t, t \in \mathbb{Z}^*$ , as cumulative sum of  $X_t^{(1)}, t \in \mathbb{Z}^*$ :

$$X_t \equiv \sum_{l=0}^t X_l^{(1)}$$

(for  $l < 0$ , let  $X_t \equiv 0$ )

- since, for  $t \in \mathbb{Z}^*$ ,

$$X_t^{(1)} = X_t - X_{t-1} \quad \& \quad S_{X^{(1)}}(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta^{(s)}}},$$

filtering theory suggests using relationship

$$S_{X^{(1)}}(f) = |2 \sin(\pi f)|^2 S_X(f)$$

to *define* SDF for  $X_t$ , i.e.,

$$S_X(f) = \frac{S_{X^{(1)}}(f)}{|2 \sin(\pi f)|^2} = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta}}$$

with  $\delta \equiv \delta^{(s)} + 1$  (Yaglom, 1958)

## Nonstationary FD Processes: II

- $X_t$  has stationary 1st order backward differences
- 1 sum defines FD processes for  $1/2 \leq \delta < 3/2$
- 2 sums define FD processes for  $3/2 \leq \delta < 5/2$ , etc
- $X_t$  has stationary 2nd order backward differences, etc
- if  $X_t^{(1)}$  is white noise ( $\delta^{(s)} = 0$ ) so  $S_{X^{(1)}}(f) = \sigma_\epsilon^2$ , then  $X_t$  is random walk ( $\delta = 1$ ) with

$$S_X(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^2} \approx \frac{\sigma_\epsilon^2}{|2\pi f|^2}$$

- if  $X_t^{(2)}$  is white noise and if

$$X_t^{(1)} \equiv \sum_{l=0}^t X_l^{(2)} \quad \& \quad X_t \equiv \sum_{l=0}^t X_l^{(1)}, \quad t \in \mathbb{Z}^*,$$

then  $X_t$  is random run ( $\delta = 2$ ), and

$$S_X(f) \approx \frac{\sigma_\epsilon^2}{|2\pi f|^4}$$

## Summary of FD Processes

- $X_t$  said to be FD process if its SDF is given by

$$S_X(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta}} \approx \frac{\sigma_\epsilon^2}{|2\pi f|^{2\delta}} \text{ at low frequencies}$$

- well-defined for any real-valued  $\delta$
- FD process obeys power law at low frequencies with exponent  $\alpha = -2\delta$
- if  $\delta < 1/2$ , FD process stationary with

– ACVS given by

$$s_{X,0} = \sigma_\epsilon^2 \frac{\Gamma(1 - 2\delta)}{\Gamma^2(1 - \delta)} \quad \& \quad s_{X,\tau} = s_{X,\tau-1} \frac{\tau + \delta - 1}{\tau - \delta}, \quad \tau \in \mathbb{Z}^+$$

– PACS given by

$$\phi_{t,t} \equiv \frac{\delta}{t - \delta}, \quad t \in \mathbb{Z}^+$$

- if  $\delta \geq 1/2$ , FD process nonstationary but its  $d$ th order backward difference is stationary FD process with parameter  $\delta^{(s)}$ , where

$$d \equiv \lfloor \delta + 1/2 \rfloor \quad \text{and} \quad \delta^{(s)} \equiv \delta - d$$

(here  $\lfloor x \rfloor$  is largest integer  $\leq x$ )

## Alternatives to FD Processes: I

- fractional Brownian motion (FBM)

–  $B_H(t), 0 \leq t < \infty$ , has SDF given by

$$S_{B_H(t)}(f) = \frac{\sigma_X^2 C_H}{|f|^{2H+1}}, \quad -\infty < f < \infty,$$

where  $\sigma_X^2 > 0$ ,  $C_H > 0$  &  $0 < H < 1$

( $H$  called Hurst parameter;  $C_H$  depends on  $H$ )

– power law with  $-3 < \alpha < -1$

- discrete fractional Brownian motion (DFBM)

–  $B_t, t \in \mathbb{Z}^+$ , is DFBM if  $B_t = B_H(t)$

–  $B_t$  has SDF given by

$$S_{B_t}(f) = \sigma_X^2 C_H \sum_{j=-\infty}^{\infty} \frac{1}{|f + j|^{2H+1}}, \quad |f| \leq 1/2$$

– power law at low frequencies with  $-3 < \alpha < -1$

– reduces to random walk if  $H = 1/2$

## Alternatives to FD Processes: II

- fractional Gaussian noise (FGN)

- $X_t, t \in \mathbb{Z}^+$ , is FGN if  $X_t = B_{t+1} - B_t$

- $X_t$  has SDF given by

$$S_X(f) = 4\sigma_X^2 C_H \sin^2(\pi f) \sum_{j=-\infty}^{\infty} \frac{1}{|f+j|^{2H+1}}, \quad |f| \leq 1/2$$

- power law at low frequencies with  $-1 < \alpha < 1$

- $X_t$  is stationary, with ACVS given by

$$s_{X,\tau} = \frac{\sigma_X^2}{2} (|\tau+1|^{2H} - 2|\tau|^{2H} + |\tau-1|^{2H}), \quad \tau \in \mathbb{Z},$$

- where  $\sigma_X^2 = \text{var}\{X_t\}$

- reduces to white noise if  $H = 1/2$

- discrete pure power law (PPL) process

- SDF given by  $S_X(f) = C_S |f|^\alpha, \quad |f| \leq 1/2$

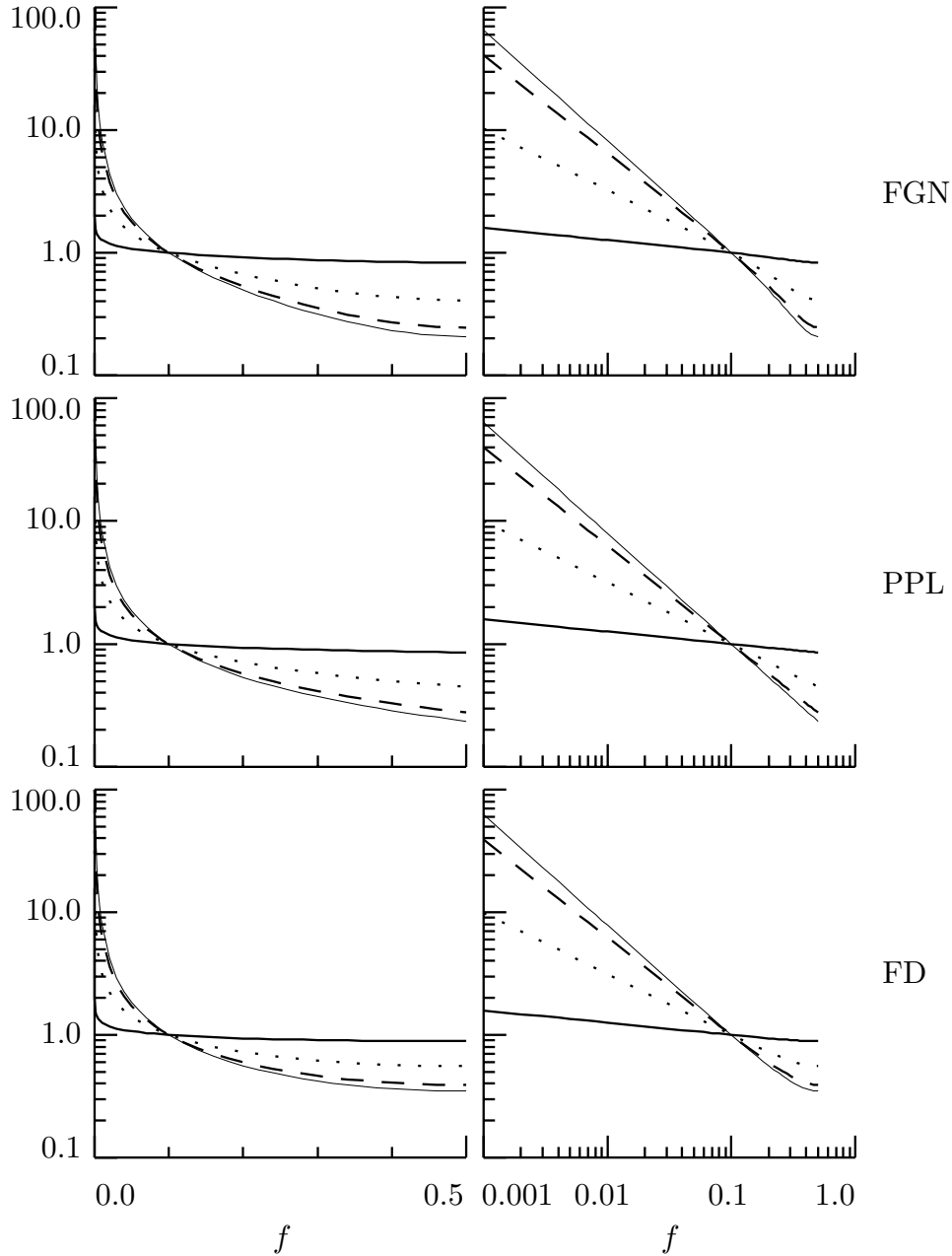
- if  $\alpha > -1$ , stationary, but ACVS takes some effort to compute

- if  $\alpha = 0$ , reduces to white noise

- $\alpha \leq -1$ , nonstationary but backward differences of certain order are stationary

## FD Processes vs. Alternatives

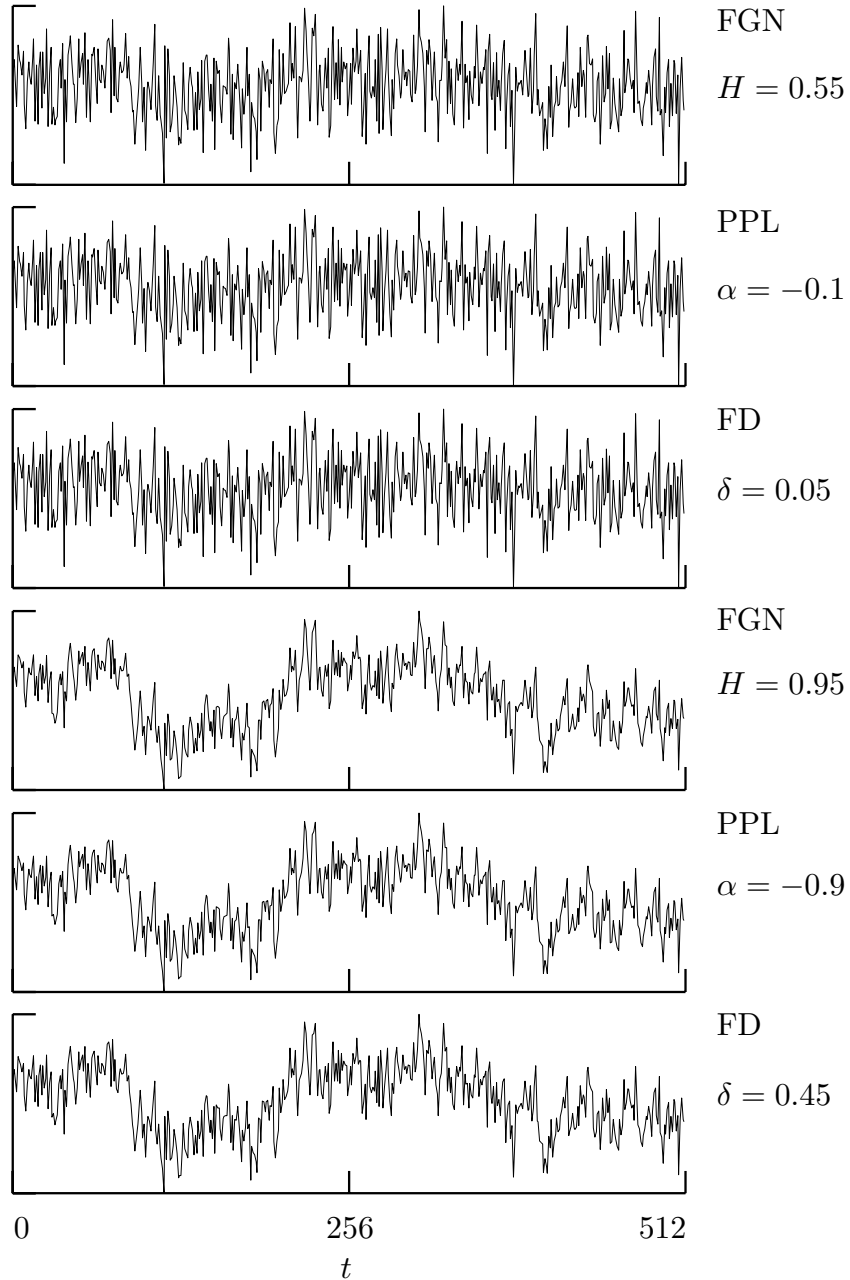
- FD processes cover full range of power laws
  - FBMs, DFBMs and FGNs cover limited range
  - PPL processes also cover full range
- differencing FD process yields another FD process; differencing alternatives yields new type of process
- FD process has simple SDF; if stationary, has simple ACVS & PACS
  - FBM has simple SDF
  - DFBM has complicated SDF
  - FGN has simple ACVS, complicated SDF & PACS
  - PPL has simple SDF, complicated ACVS & PACS
- FD, DFBM, FGN and PPL model sampled noise
  - might be problematic to change sampling rate
  - FBM models unsampled noise
- Fig. 1: comparison of SDFs for FGN, PPL & FD
- Fig. 2: comparison of realizations



**Figure 1.** SDFs for FGN, PPL and FD processes (top to bottom rows, respectively) on both linear/log and log/log axes (left- and right-hand columns, respectively). Each SDF  $S_X(\cdot)$  is normalized such that  $S_X(0.1) = 1$ . The table below gives the parameter values for the various plotted curves. (Adapted from Figure 282, Percival and Walden, 2000, copyright Cambridge University Press.)

process	thick solid	dotted	dashed	thin solid
FGN	$H = 0.55$	$H = 0.75$	$H = 0.90$	$H = 0.95$
PPL	$\alpha = -0.1$	$\alpha = -0.5$	$\alpha = -0.8$	$\alpha = -0.9$
FD	$\delta = 0.05$	$\delta = 0.25$	$\delta = 0.40$	$\delta = 0.45$





**Figure 2.** Simulated realizations of FG, PPL and FD processes. The thick (thin) solid curves in Figure 1 show the SDFs for the top (bottom) three series – these SDFs differ markedly only at high frequencies. We formed each simulated  $X_0, \dots, X_{511}$  using the circulant embedding method, which does so by transforming a realization of a portion  $\varepsilon_0, \dots, \varepsilon_{1023}$  of a white noise process. To illustrate the similarity of FG, PPL and FD processes with comparable  $H$ ,  $\alpha$  and  $\delta$ , we used the same  $\varepsilon_t$  to create all six series. Although the top (bottom) three series appear to be identical, estimates of their SDFs show high frequency differences consistent with their theoretical SDFs. (Adapted from Figure 283, Percival and Walden, 2000, copyright Cambridge University Press.)

## Extensions to FD Processes: I

- composite FD processes

$$S_X(f) = \sum_{m=1}^M \frac{\sigma_m^2}{|2 \sin(\pi f)|^{2\delta_m}};$$

i.e., linear combinations of independent FD processes

- autoregressive, fractionally integrated, moving average (ARFIMA) processes

– idea is to replace  $\epsilon_t$  in

$$X_t = \sum_{k=0}^{\infty} a_k(\delta) \epsilon_{t-k}$$

with ARMA process, say,

$$U_t = \sum_{k=1}^p \phi_k U_{t-k} + \epsilon_t - \sum_{k=1}^q \theta_k \epsilon_{t-k}$$

– yields process with SDF

$$S_X(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta}} \cdot \frac{|1 - \sum_{k=1}^q \theta_k e^{-i2\pi f k}|^2}{|1 - \sum_{k=1}^p \phi_k e^{-i2\pi f k}|^2}$$

– ARMA part can model, e.g., high-frequency structure in noise

## Extensions to FD Processes: II

- can define time-varying FD (TVFD) process via

$$X_t = \sum_{k=0}^{\infty} a_k(\delta_t) \epsilon_{t-k}$$

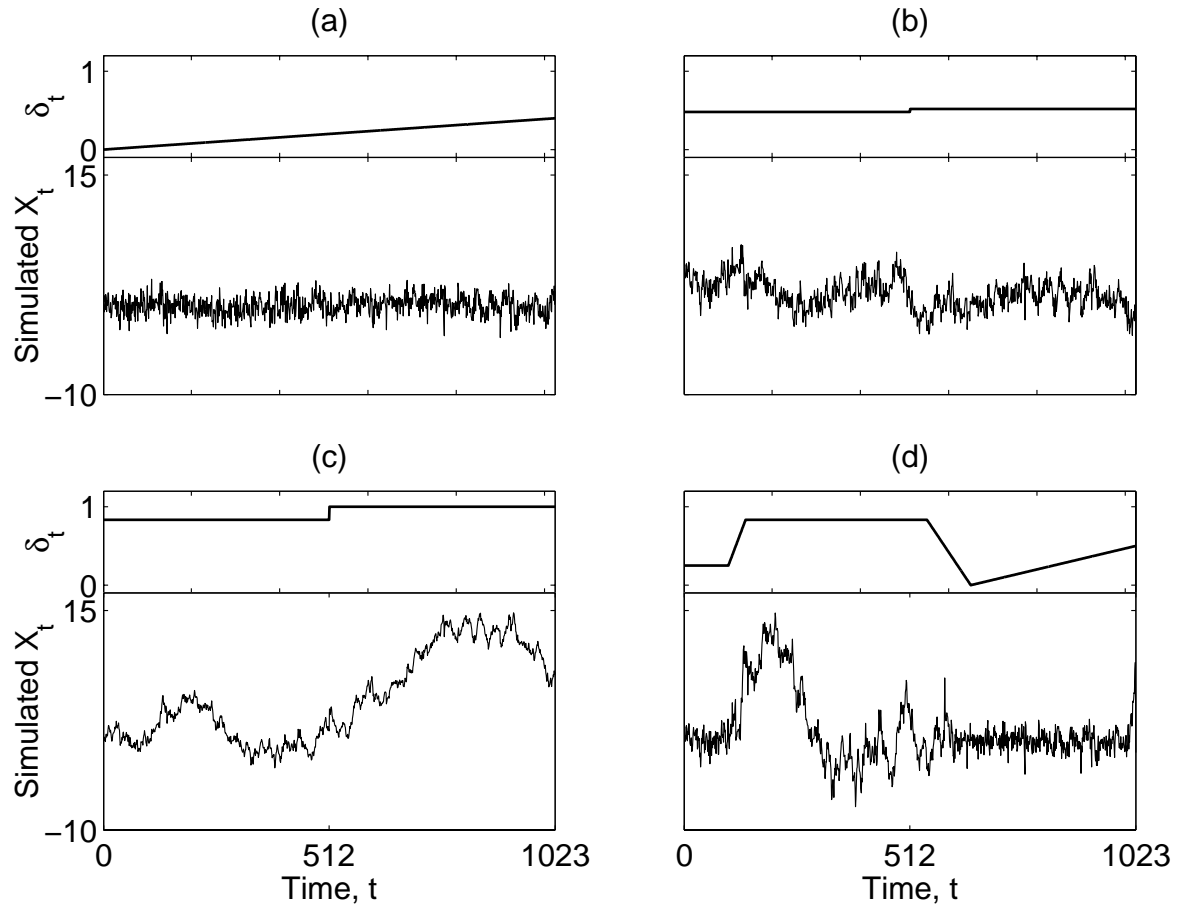
as long as  $\delta_t < 1/2$  for all  $t$

- can use representation

$$X_t = \sum_{k=0}^{2N-1} c_{t,k}(\delta_t) \epsilon_k, \quad t = 0, 1, \dots, N-1,$$

to extend definition to handle arbitrary  $\delta_t$

- Fig. 3: realizations from 4 TVFD processes
- can also make  $\sigma_\epsilon^2$  time-varying



**Figure 3.** Four examples of time series simulated from TVFD processes. The upper panel of each plot shows the sequence  $\delta_t$ ,  $t = 0, \dots, 1023$ , used to generate the simulated time series in the lower panel. All four simulated series were created using the circulant embedding method on the same set of 2048 independent deviates from a standard Gaussian distribution. (Figure 1, Percival and Constantine, 2002.)

## FD Process Parameter Estimation

- Q: given realization (clock noise) of  $X_0, \dots, X_{N-1}$  from FD process, how can we estimate  $\delta$  &  $\sigma_\epsilon^2$ ?
- *many* different estimators have been proposed!  
(area of active research)
- will concentrate on estimators based on
  - spectral analysis (frequency-based)
  - wavelet analysis (scale-based)
- advantages of spectral and wavelet analysis
  - physically interpretable
  - both are analysis of variance techniques  
(useful for more than just estimating  $\delta$  &  $\sigma_\epsilon^2$ )
  - can assess need for models more complex than simple FD process (e.g., composite FD process)
  - provide preliminary estimates for more complicated schemes (maximum likelihood estimation)

## Estimation via Spectral Analysis

- recall that SDF for FD process given by

$$S_X(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta}}$$

and thus

$$\log(S_X(f)) = \log(\sigma_\epsilon^2) - 2\delta \log(|2 \sin(\pi f)|);$$

i.e., plot of  $\log(S_X(f))$  vs.  $\log(|2 \sin(\pi f)|)$  linear with slope of  $-2\delta$

- for  $0 < f < 1/8$ , have  $\sin(\pi f) \approx \pi f$ , so

$$\log(S_X(f)) \approx \log(\sigma_\epsilon^2) - 2\delta \log(2\pi f);$$

i.e., plot of  $\log(S_X(f))$  vs.  $\log(2\pi f)$  approximately linear at low frequencies with slope of  $-2\delta = \alpha$

- basic scheme

- estimate  $S_X(f)$  via  $\hat{S}_X(f)$
- fit linear model to  $\hat{S}_X(f)$  vs.  $\log(2\pi f)$  over low frequencies
- use estimated slope  $\hat{\alpha}$  to estimate  $\delta$  via  $-\hat{\alpha}/2$
- use estimated intercept to estimate  $\sigma_\epsilon^2$

# The Periodogram: I

- basic estimator of  $S(f)$  is periodogram:

$$\hat{S}^{(p)}(f) \equiv \frac{1}{N} \left| \sum_{t=0}^{N-1} X_t e^{-i2\pi ft} \right|^2, \quad |f| \leq 1/2;$$

- represents decomposition of sample variance:

$$\int_{-1/2}^{1/2} \hat{S}^{(p)}(f) df = \frac{1}{N} \sum_{t=0}^{N-1} X_t^2$$

- for stationary processes & large  $N$ , theory says

$$\hat{S}^{(p)}(f) \stackrel{d}{=} S(f) \frac{\chi_2^2}{2}, \quad 0 < f < 1/2,$$

approximately, implying that

- $E\{\hat{S}^{(p)}(f)\} \approx E\{S(f)\chi_2^2/2\} = S(f)$
- $\text{var}\{\hat{S}^{(p)}(f)\} \approx \text{var}\{S(f)\chi_2^2/2\} = S^2(f)$

(in above ‘ $\stackrel{d}{=}$ ’ means ‘equal in distribution,’ and  $\chi_2^2$  is chi-square RV with 2 degrees of freedom)

- additionally,  $\text{cov}\{\hat{S}^{(p)}(f_j), \hat{S}^{(p)}(f_k)\} \approx 0$   
for  $f_j \equiv j/N$  &  $0 < f_j < f_k < 1/2$

## The Periodogram: II

- taking log transform yields

$$\log(\hat{S}^{(p)}(f)) \stackrel{d}{=} \log\left(S(f)\frac{\chi_2^2}{2}\right) = \log(S(f)) + \log\left(\frac{\chi_2^2}{2}\right)$$

- Bartlett & Kendall (1946):

$$E\left\{\log\left(\frac{\chi_\eta^2}{\eta}\right)\right\} = \psi(\eta) - \log(\eta) \quad \& \quad \text{var}\left\{\log\left(\frac{\chi_\eta^2}{\eta}\right)\right\} = \psi'(\eta)$$

where  $\psi(\cdot)$  &  $\psi'(\cdot)$  are di- & trigamma functions

- yields

$$\begin{aligned} E\{\log(\hat{S}^{(p)}(f))\} &= \log(S(f)) + \psi(2) - \log(2) \\ &= \log(S(f)) - \gamma \\ \text{var}\{\log(\hat{S}^{(p)}(f))\} &= \psi'(2) = \pi^2/6 \end{aligned}$$

( $\gamma \doteq 0.57721$  is Euler's constant)



## The Periodogram: III

- define  $Y^{(p)}(f_j) \equiv \log(\hat{S}^{(p)}(f_j)) + \gamma$
- can model  $Y^{(p)}(f_j)$  as

$$\begin{aligned} Y^{(p)}(f_j) &\approx \log(S(f_j)) + \epsilon(f_j) \\ &\approx \log(\sigma_\epsilon^2) - 2\delta \log(2\pi f_j) + \epsilon(f_j) \end{aligned}$$

over low frequencies indexed by  $0 < j < J$

- error  $\epsilon(f_j)$  in linear regression model such that
  - $E\{\epsilon(f_j)\} = 0$  &  $\text{var}\{\epsilon(f_j)\} = \pi^2/6$  (known!)
  - if  $\{X_t\}$  Gaussian &  $\hat{S}^{(p)}(f_j)$ 's uncorrelated, then  $\epsilon(f_j)$ 's pairwise uncorrelated
  - $\epsilon(f_j) \stackrel{d}{=} \log(\chi_2^2)$  markedly non-Gaussian
- least squares procedure yields
  - estimates  $\hat{\delta}$  and  $\hat{\sigma}_\epsilon^2$  for  $\delta$  and  $\sigma_\epsilon^2$
  - estimates of variability in  $\hat{\delta}$  and  $\hat{\sigma}_\epsilon^2$

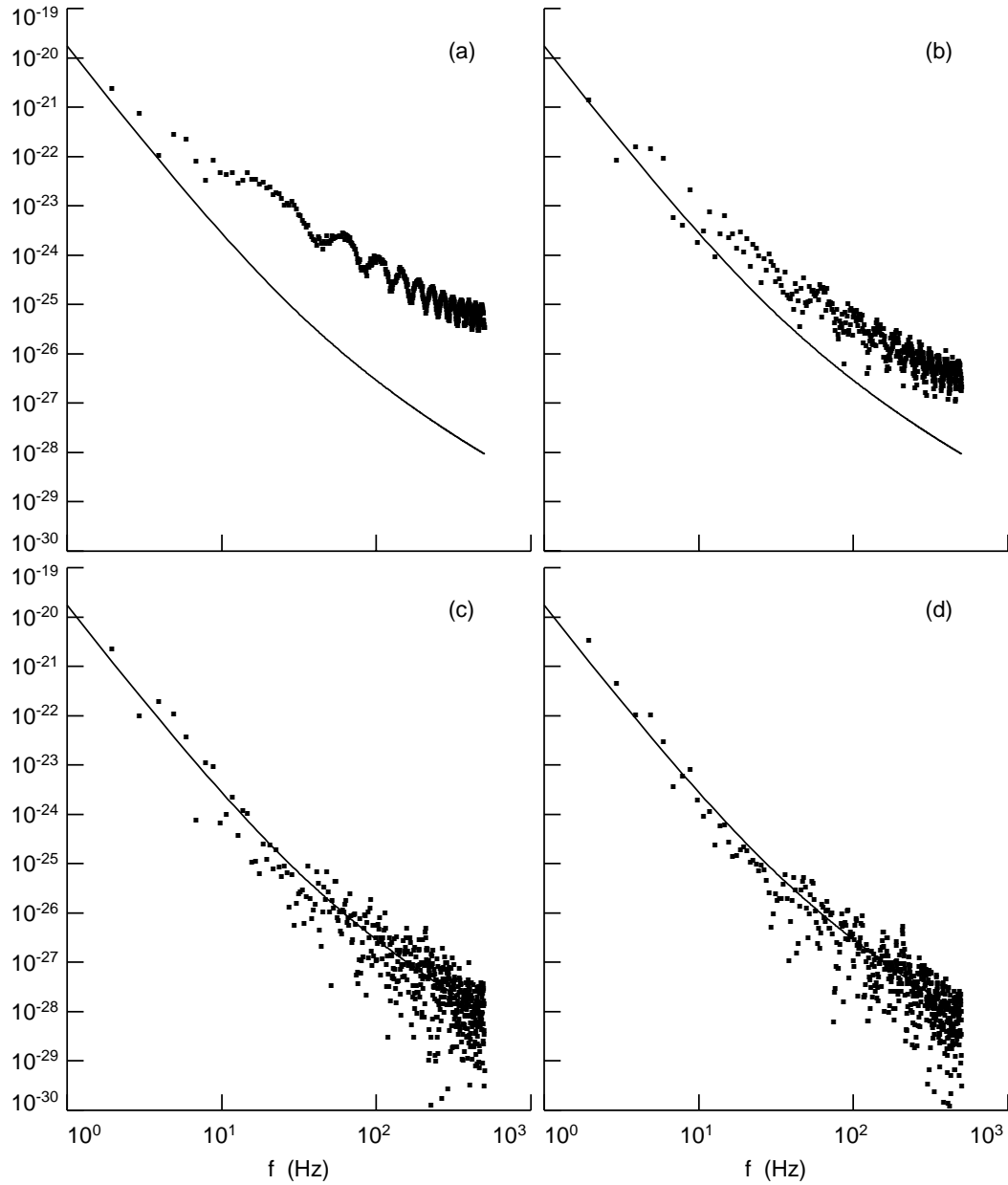
# Multitaper Spectral Estimation: I

- warnings about periodogram:
  - approximations might require  $N$  to be *very* large!
  - approximations of questionable validity for nonstationary FD processes
- Fig. 4: periodogram can suffer from ‘leakage’
- tapering is technique for alleviating leakage:

$$\hat{S}^{(d)}(f) \equiv \left| \sum_{t=0}^{N-1} a_t X_t e^{-i2\pi ft} \right|^2$$

- $\{a_t\}$  called data taper (typically bell-shaped curve)
- $\hat{S}^{(d)}(\cdot)$  called direct spectral estimator
- critique: loses ‘information’ at end of series (sample size  $N$  effectively shortened)
- Thomson (1982): multitapering recovers ‘lost info’
- use set of  $K$  orthonormal data tapers  $\{a_{n,t}\}$ :

$$\sum_{t=0}^{N-1} a_{n,t} a_{l,t} = \begin{cases} 1, & \text{if } n = l; \\ 0, & \text{if } n \neq l. \end{cases} \quad 0 \leq n, l \leq K - 1$$



**Figure 4.** Periodogram (a) and direct spectral estimates (b,c,d) of noise simulated from composite FD process with four components, namely,  $\delta = 2, 1.5, 1$  and  $0.5$  (corresponding to  $\alpha = -4, -3, -2$  and  $-1$ ). The true SDF is the solid curve in each plot, while the dots are the spectral estimates. The data tapers used in the direct spectral estimates are Slepian tapers (i.e., discrete prolate spheroidal sequences) with the resolution bandwidth  $W$  set via (b)  $NW = 1$ , (c)  $NW = 2$  and (d)  $NW = 4$ . The simulated series has length  $N = 1000$ .

## Multitaper Spectral Estimation: II

- use  $\{a_{n,t}\}$  to form  $k$ th direct spectral estimator:

$$\hat{S}_k^{(mt)}(f) \equiv \left| \sum_{t=0}^{N-1} a_{n,t} X_t e^{-i2\pi ft} \right|^2, \quad n = 0, \dots, K-1$$

- simplest form of multitaper SDF estimator:

$$\hat{S}^{(mt)}(f) \equiv \frac{1}{K} \sum_{n=0}^{K-1} \hat{S}_n^{(mt)}(f)$$

- sinusoidal tapers are one family of multitapers:

$$a_{n,t} = \left\{ \frac{2}{(N+1)} \right\}^{1/2} \sin \left\{ \frac{(n+1)\pi(t+1)}{N+1} \right\}, \quad t = 0, \dots, N-1$$

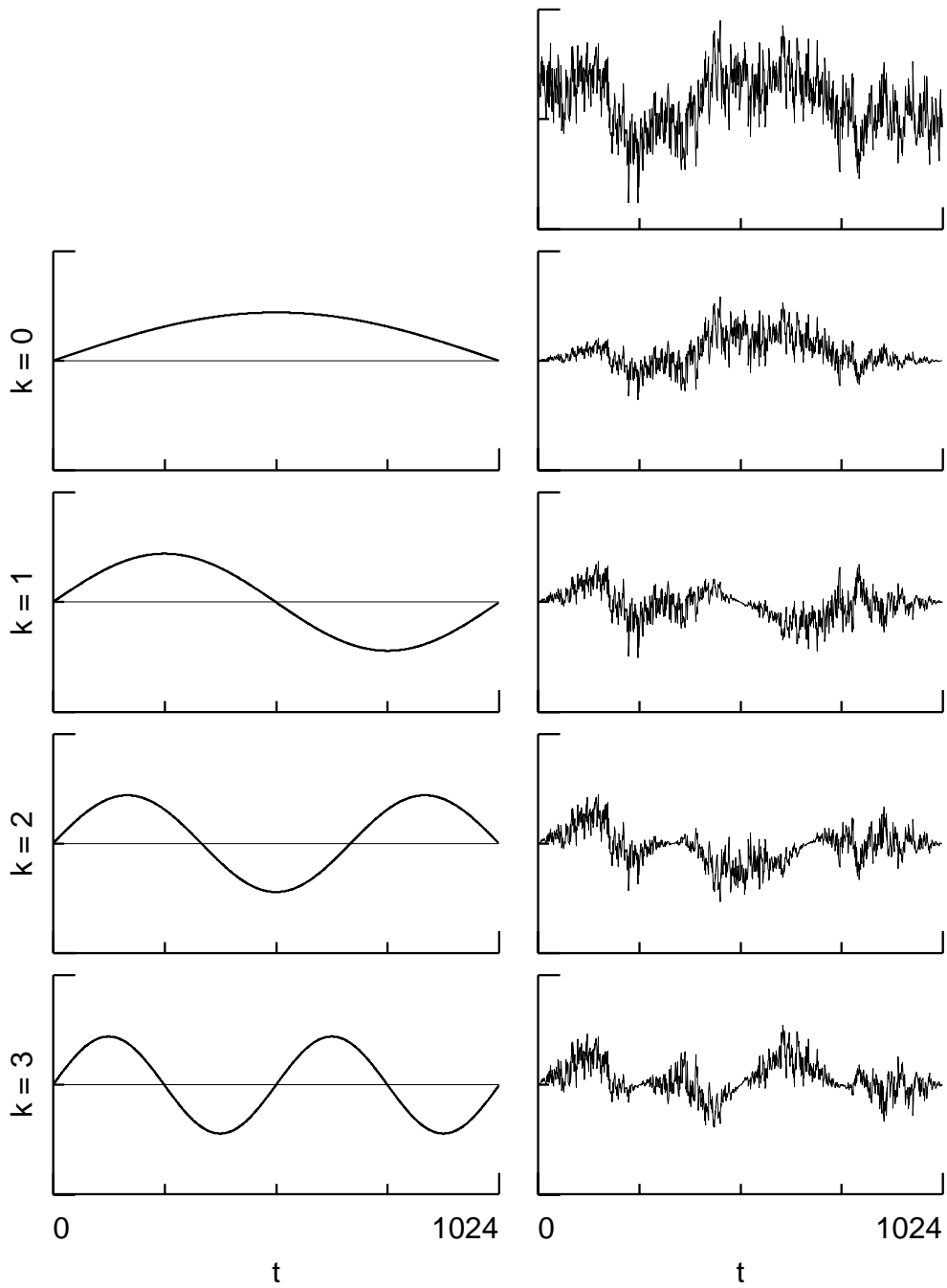
(Riedel & Sidorenko, 1995)

- Figs. 5 and 6: example of multitapering
- if  $S(\cdot)$  slowly varying around  $S(f)$  & if  $N$  large,

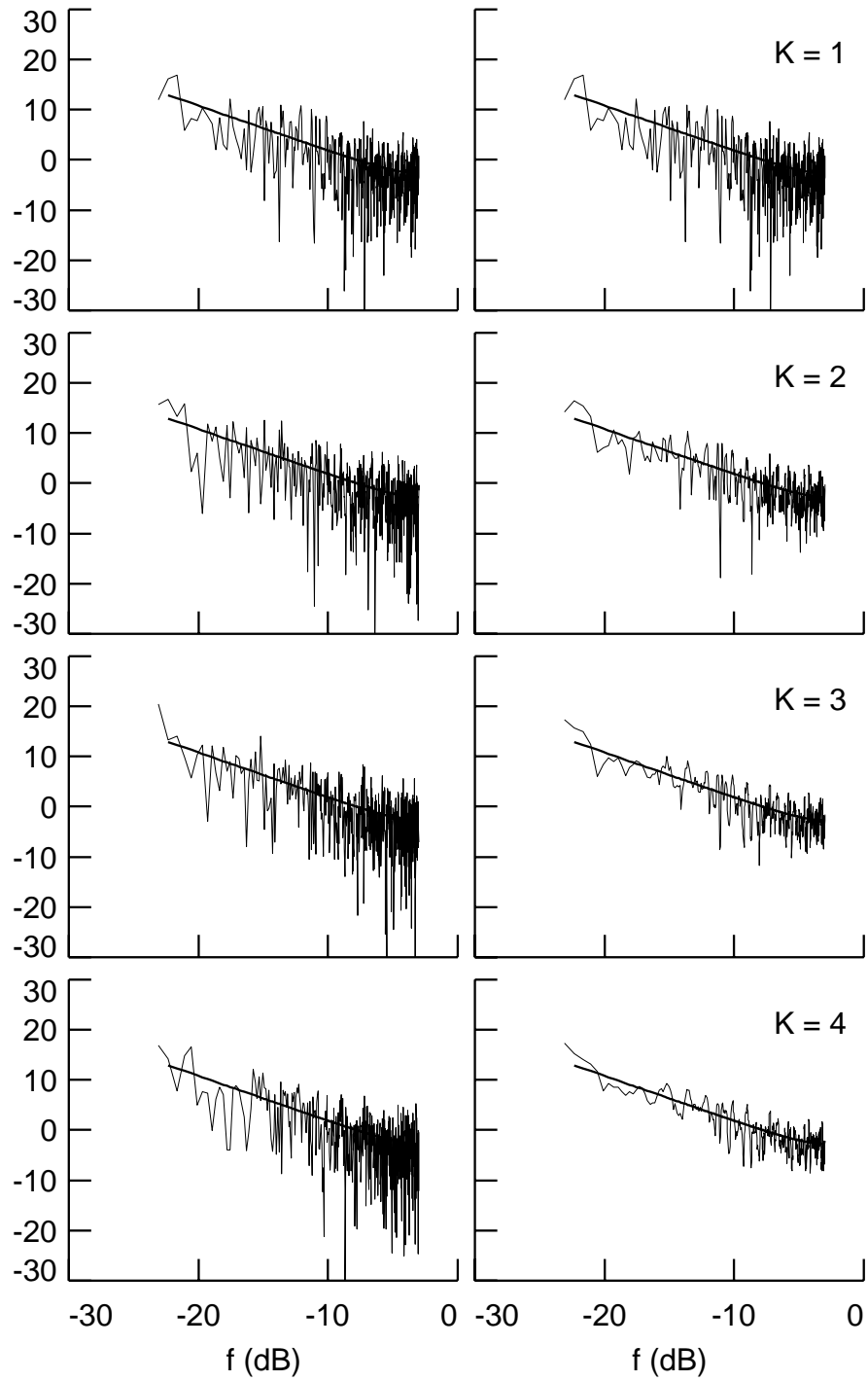
$$\hat{S}^{(mt)}(f) \stackrel{d}{=} \frac{S(f)\chi_{2K}^2}{2K}$$

approximately for  $0 < f < 1/2$ , impling

$$\text{var} \{ \hat{S}^{(mt)}(f) \} \approx \frac{S^2(f)}{4K^2} \text{var} \{ \chi_{2K}^2 \} = \frac{S^2(f)}{K}$$



**Figure 5.** Sinusoidal tapers (left-hand column) as applied to a simulated FD time series with  $\delta = 0.45$  (top plot, right-hand column), resulting in tapered series (right-hand column, second to bottom rows).



**Figure 6.** Direct spectral estimates formed using the  $k$ th sinusoidal taper,  $k = 0, \dots, 3$  (left-hand column, top to bottom row), along with multitaper estimates formed by averaging  $K = 1, \dots, 4$  of these direct spectral estimates (right-hand column, top to bottom row).

## Multitaper Spectral Estimation: III

- define  $Y^{(mt)}(f_j) \equiv \log(\hat{S}^{(mt)}(f_j)) - \psi(K) + \log(K)$
- can model  $Y^{(mt)}(f_j)$  as

$$\begin{aligned} Y^{(mt)}(f_j) &\approx \log(S(f_j)) + \eta(f_j) \\ &\approx \log(\sigma_\epsilon^2) - 2\delta \log(2\pi f_j) + \eta(f_j) \end{aligned}$$

over low frequencies indexed by  $0 < j < J$

- error  $\eta(f_j)$  in linear regression model such that
  - $E\{\eta(f_j)\} = 0$
  - $\text{var}\{\eta(f_j)\} = \psi'(K)$ , a known constant!
  - approximately Gaussian if  $K \geq 5$
  - correlated, but with simple structure:

$$\text{cov}\{\eta(f_j), \eta(f_{j+\nu})\} \approx \begin{cases} \psi'(K) \left(1 - \frac{|\nu|}{K+1}\right), & \text{if } |\nu| \leq K + 1; \\ 0, & \text{otherwise.} \end{cases}$$

- generalized least squares procedure yields
  - estimates  $\hat{\delta}$  and  $\hat{\sigma}_\epsilon^2$  for  $\delta$  and  $\sigma_\epsilon^2$
  - estimates of variability in  $\hat{\delta}$  and  $\hat{\sigma}_\epsilon^2$
- multitaper approach superior to periodogram approach

## Discrete Wavelet Transform (DWT)

- let  $\mathbf{X} = [X_0, X_1, \dots, X_{N-1}]^T$  be observed time series (for convenience, assume  $N$  integer multiple of  $2^{J_0}$ )
- let  $\mathcal{W}$  be  $N \times N$  orthonormal DWT matrix
- $\mathbf{W} = \mathcal{W}\mathbf{X}$  is vector of DWT coefficients
- orthonormality says  $\mathbf{X} = \mathcal{W}^T\mathbf{W}$ , so  $\mathbf{X} \Leftrightarrow \mathbf{W}$
- can partition  $\mathbf{W}$  as follows:

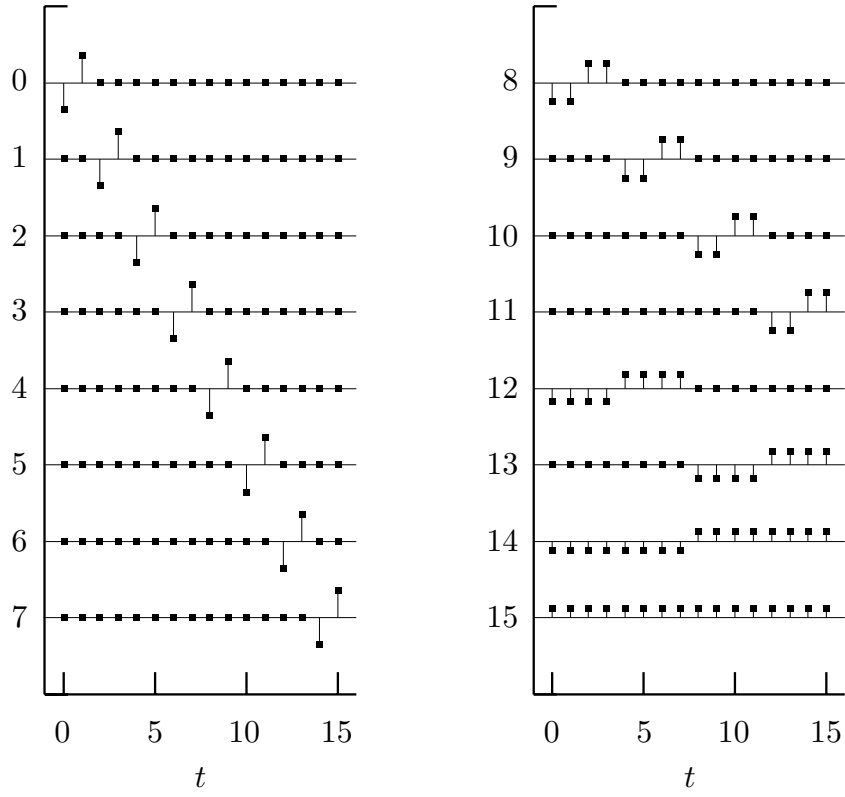
$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \vdots \\ \mathbf{W}_{J_0} \\ \mathbf{V}_{J_0} \end{bmatrix}$$

- $\mathbf{W}_j$  contains  $N_j = N/2^j$  wavelet coefficients
  - related to changes of averages at scale  $\tau_j = 2^{j-1}$  ( $\tau_j$  is  $j$ th ‘dyadic’ scale)
  - related to times spaced  $2^j$  units apart
- $\mathbf{V}_{J_0}$  contains  $N_{J_0} = N/2^{J_0}$  scaling coefficients
  - related to averages at scale  $\lambda_{J_0} = 2^{J_0}$
  - related to times spaced  $2^{J_0}$  units apart

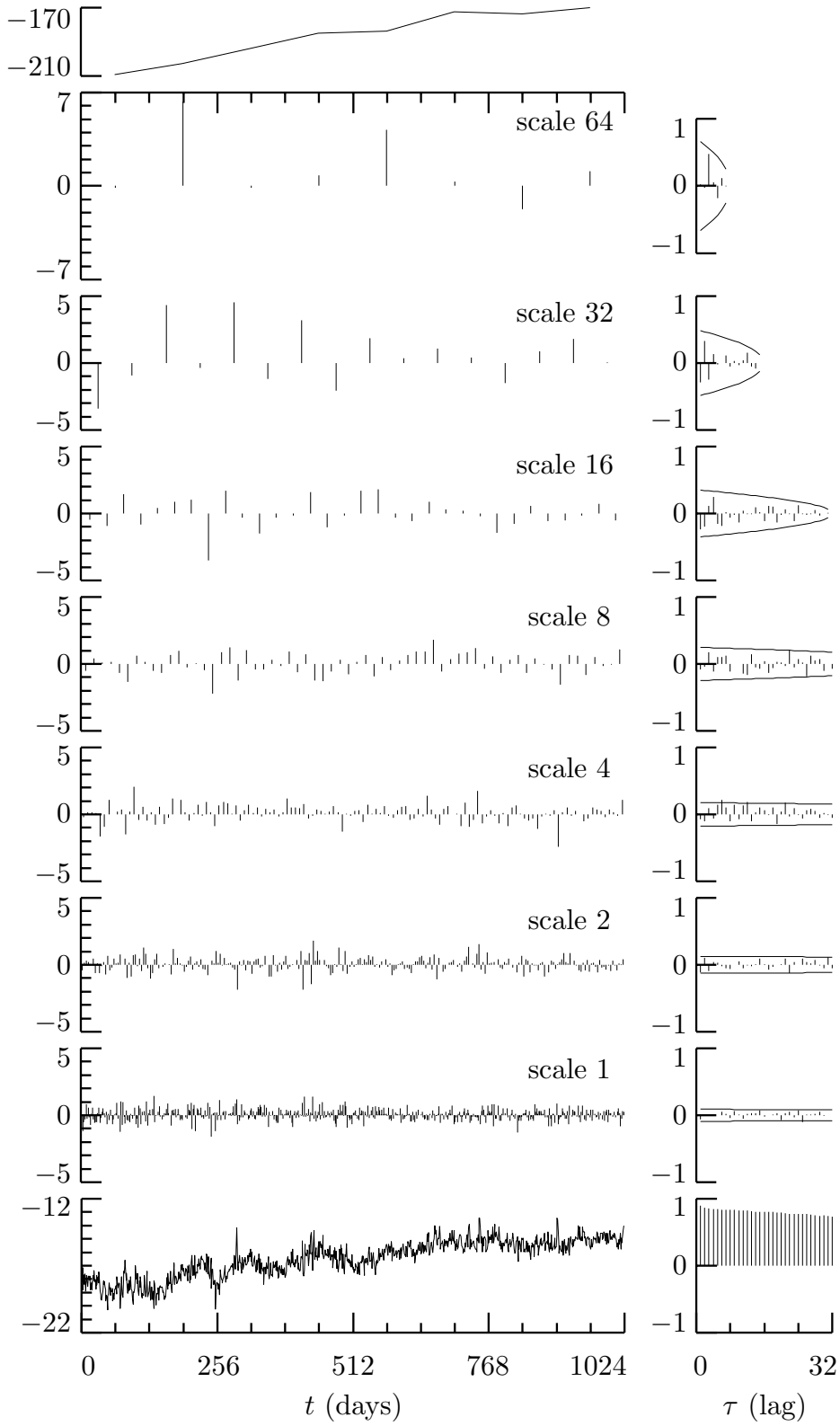


## Example: Haar DWT

- Fig. 7:  $\mathcal{W}$  for Haar DWT with  $N = 16$ 
  - first 8 rows yield  $\mathbf{W}_1 \propto \text{changes}$  on scale 1
  - next 4 rows yield  $\mathbf{W}_2 \propto \text{changes}$  on scale 2
  - next 2 rows yield  $\mathbf{W}_3 \propto \text{changes}$  on scale 4
  - next to last row yields  $\mathbf{W}_4 \propto \text{change}$  on scale 8
  - last row yields  $\mathbf{V}_4 \propto \text{average}$  on scale 16
- Fig. 8: Haar DWT coefficients for clock 571



**Figure 7.** Row vectors  $\mathcal{W}_{n\bullet}^T$  of the discrete wavelet transform matrix  $\mathcal{W}$  based on the Haar wavelet for  $N = 16$  and  $n = 0$  to  $7$  (top to bottom on left plot) and  $n = 8$  to  $15$  (right plot).



**Figure 8.** Haar DWT coefficients for clock 571 and sample autocorrelation sequences (ACSs).

## DWT in Terms of Filters

- filter  $X_0, X_1, \dots, X_{N-1}$  to obtain

$$2^{j/2}\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} h_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \dots, N-1;$$

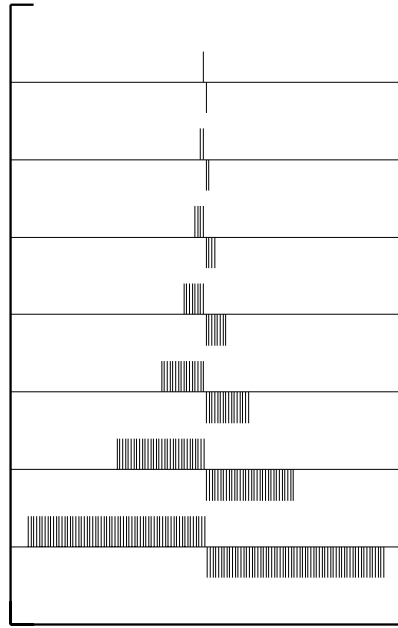
$h_{j,l}$  is  $j$ th level wavelet filter (note: circular filtering)

- subsample to obtain wavelet coefficients:

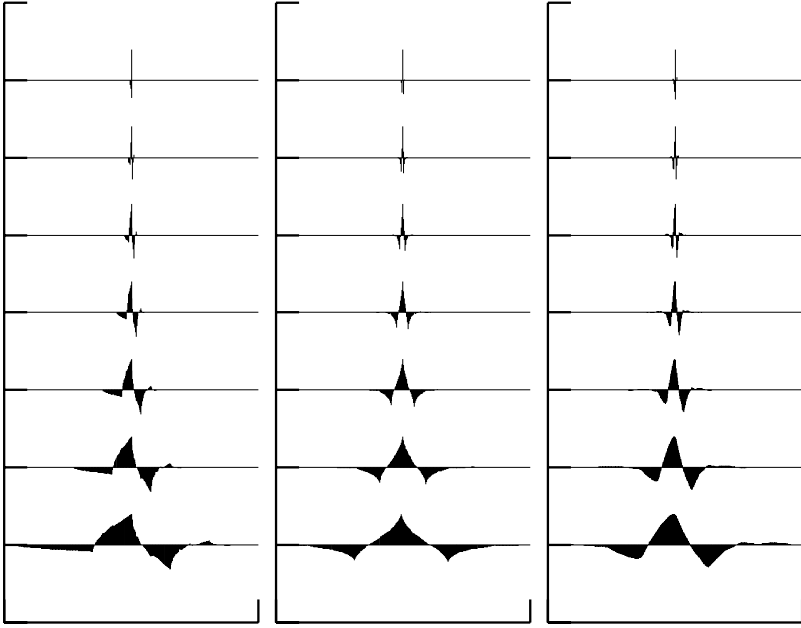
$$W_{j,t} = 2^{j/2}\widetilde{W}_{j,2^j(t+1)-1}, \quad t = 0, 1, \dots, N_j - 1,$$

where  $W_{j,t}$  is  $t$ th element of  $\mathbf{W}_j$

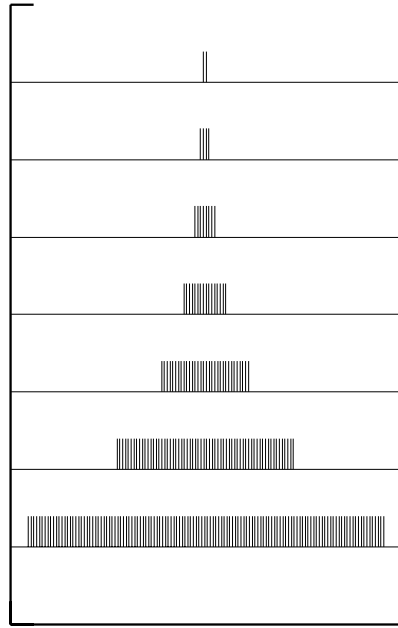
- Figs. 9 & 10: four sets of wavelet filters
- $j$ th wavelet filter is band-pass with pass-band  $[\frac{1}{2^{j+1}}, \frac{1}{2^j}]$  (i.e., scale related to *interval* of frequencies)
- similarly, scaling filters yield  $\mathbf{V}_{J_0}$
- Figs. 11 & 12: four sets of scaling filters
- $J_0$ th scaling filter is low-pass with pass-band  $[0, \frac{1}{2^{J_0+1}}]$
- as width  $L$  of 1st level filters increases,
  - band-pass & low-pass approximations improve
  - # of embedded differencing operations increases (related to # of ‘vanishing moments’)



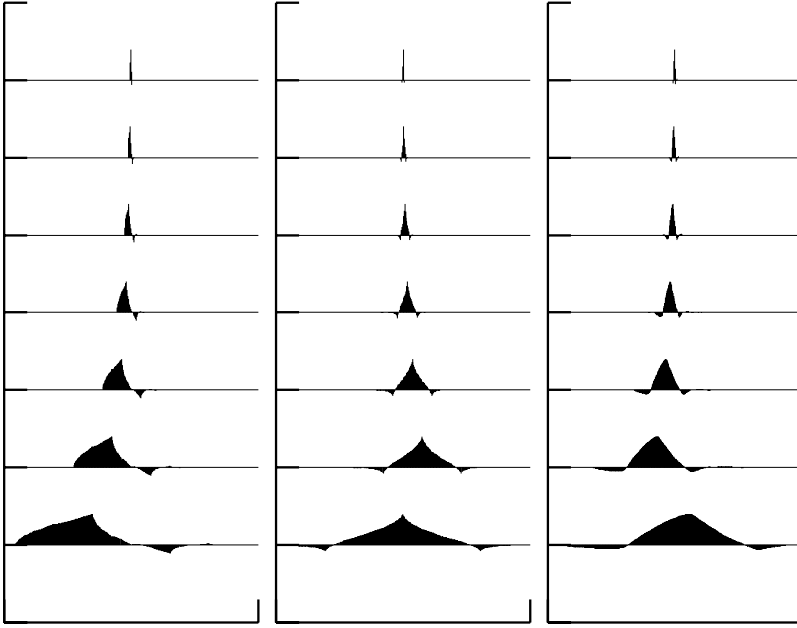
**Figure 9.** Haar wavelet filters for scales  $\tau_j = 2^{j-1}$ ,  $j = 1, 2, \dots, 7$ .



**Figure 10.** D(4), C(6) and LA(8) wavelet filters for scales  $\tau_j = 2^{j-1}$ ,  $j = 1, 2, \dots, 7$ .



**Figure 11.** Haar scaling filters for scales  $\lambda_{J_0} = 2^{J_0}$ ,  $J_0 = 1, 2, \dots, 7$ .



**Figure 12.** D(4), C(6) and LA(8) scaling filters for scales  $\lambda_{J_0} = 2^{J_0}$ ,  $J_0 = 1, 2, \dots, 7$ .



## DWT-Based Analysis of Variance

- consider ‘energy’ in time series:

$$\|\mathbf{X}\|^2 = \mathbf{X}^T \mathbf{X} = \sum_{t=0}^{N-1} X_t^2$$

- energy preserved in DWT coefficients:

$$\|\mathbf{W}\|^2 = \|\mathcal{W}\mathbf{X}\|^2 = \mathbf{X}^T \mathcal{W}^T \mathcal{W} \mathbf{X} = \mathbf{X}^T \mathbf{X} = \|\mathbf{X}\|^2$$

- since  $\mathbf{W}_1, \dots, \mathbf{W}_{J_0}, \mathbf{V}_{J_0}$  partitions  $\mathbf{W}$ , have

$$\|\mathbf{W}\|^2 = \sum_{j=1}^{J_0} \|\mathbf{W}_j\|^2 + \|\mathbf{V}_{J_0}\|^2,$$

leading to analysis of sample variance:

$$\hat{\sigma}^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} X_t^2 = \frac{1}{N} \left( \sum_{j=1}^{J_0} \|\mathbf{W}_j\|^2 + \|\mathbf{V}_{J_0}\|^2 \right)$$

- scale-based decomposition (cf. frequency-based)

## Variation: Maximal Overlap DWT

- can eliminate downsampling and use

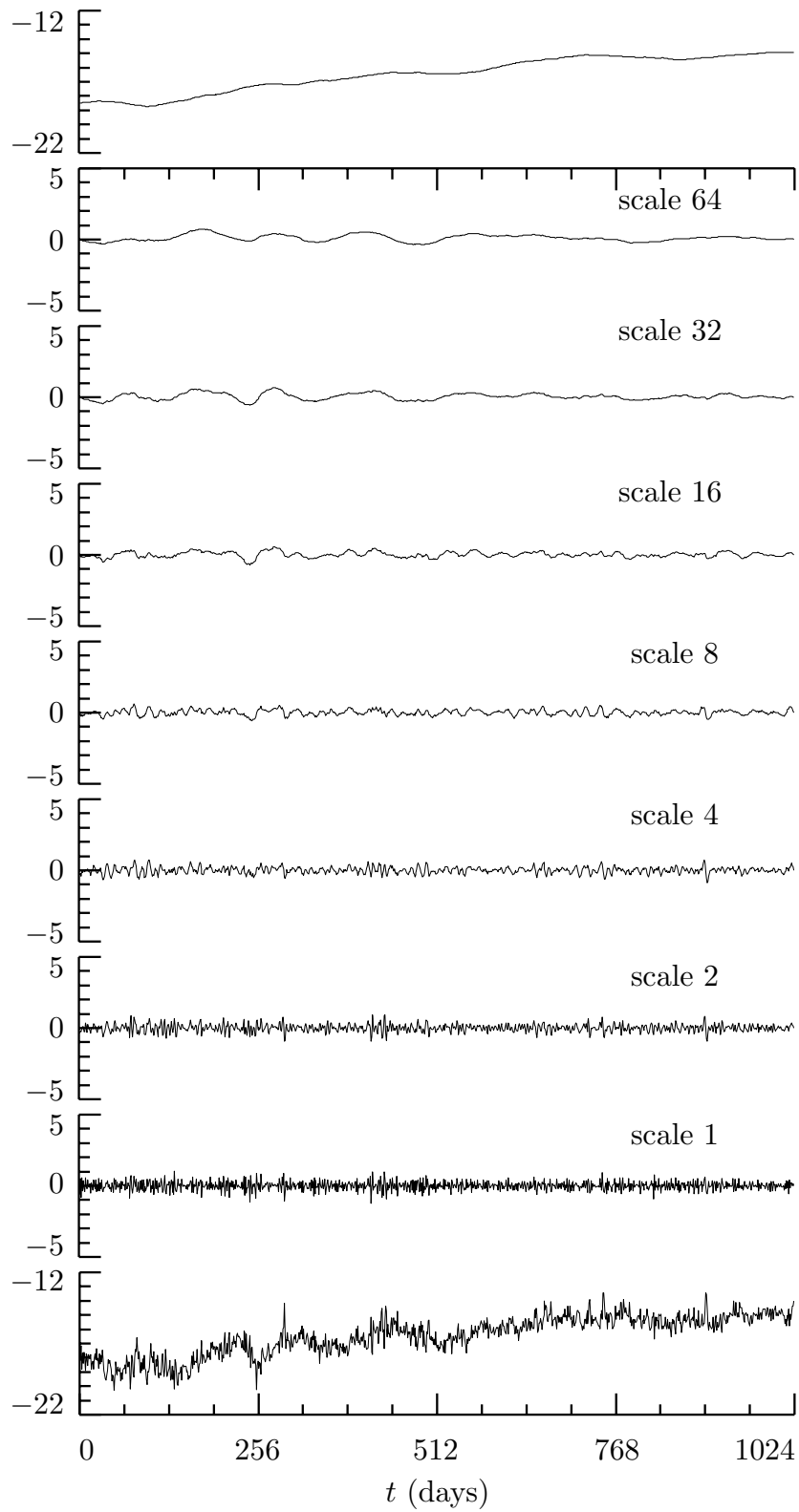
$$\widetilde{W}_{j,t} \equiv \frac{1}{2^{j/2}} \sum_{l=0}^{L_j-1} h_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \dots, N-1$$

to define MODWT coefficients  $\widetilde{\mathbf{W}}_j$  (& also  $\widetilde{\mathbf{V}}_j$ )

- unlike DWT, MODWT is not orthonormal (in fact MODWT is highly redundant)
- like DWT, can do analysis of variance because

$$\|\mathbf{X}\|^2 = \sum_{j=1}^{J_0} \|\widetilde{\mathbf{W}}_j\|^2 + \|\widetilde{\mathbf{V}}_{J_0}\|^2$$

- unlike DWT, MODWT works for all samples sizes  $N$  (i.e., power of 2 assumption is not required)
- Fig. 13: Haar MODWT coefficients for clock 571 (cf. Fig. 8 with DWT coefficients)
- can use to track time-varying FD process



**Figure 13.** Haar MODWT coefficients for clock 571.

## Definition of Wavelet Variance

- let  $X_t, t \in \mathbb{Z}$ , be a stochastic process
- run  $X_t$  through  $j$ th level wavelet filter:

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t \in \mathbb{Z}$$

- definition of time dependent wavelet variance (also called wavelet spectrum):

$$\nu_{X,t}^2(\tau_j) \equiv \text{var} \{ \overline{W}_{j,t} \},$$

assuming  $\text{var} \{ \overline{W}_{j,t} \}$  exists and is finite

- $\nu_{X,t}^2(\tau_j)$  depends on  $\tau_j$  and  $t$
- will consider time independent wavelet variance:

$$\nu_X^2(\tau_j) \equiv \text{var} \{ \overline{W}_{j,t} \}$$

(can be easily adapted to time varying situation)

- rationale for wavelet variance
  - decomposes variance on scale by scale basis
  - useful substitute/complement for SDF

## Variance Decomposition

- suppose  $X_t$  has SDF  $S_X(f)$ :

$$\int_{-1/2}^{1/2} S_X(f) df = \text{var} \{X_t\};$$

i.e., decomposes  $\text{var} \{X_t\}$  across frequencies  $f$

- involves uncountably infinite number of  $f$ 's
  - $S_X(f) \Delta f \approx$  contribution to  $\text{var} \{X_t\}$  due to  $f$ 's in interval of length  $\Delta f$  centered at  $f$
  - note:  $\text{var} \{X_t\}$  taken to be  $\infty$  for nonstationary processes with stationary backward differences
- wavelet variance analog to fundamental result:

$$\sum_{j=1}^{\infty} \nu_X^2(\tau_j) = \text{var} \{X_t\}$$

i.e., decomposes  $\text{var} \{X_t\}$  across scales  $\tau_j$

- recall DWT/MODWT and sample variance
- involves countably infinite number of  $\tau_j$ 's
- $\nu_X^2(\tau_j)$  contribution to  $\text{var} \{X_t\}$  due to scale  $\tau_j$
- $\nu_X(\tau_j)$  has same units as  $X_t$  (easier to interpret)

## Spectrum Substitute/Complement

- because  $\tilde{h}_{j,l} \approx$  bandpass over  $[1/2^{j+1}, 1/2^j]$ ,

$$\nu_X^2(\tau_j) \approx 2 \int_{1/2^{j+1}}^{1/2^j} S_X(f) df \quad (*)$$

- if  $S_X(f)$  ‘featureless’, info in  $\nu_X^2(\tau_j) \Leftrightarrow$  info in  $S_X(f)$
- $\nu_X^2(\tau_j)$  more succinct: only 1 value per octave band
- recall SDF for FD process:

$$S_X(f) = \frac{\sigma_\epsilon^2}{|2 \sin(\pi f)|^{2\delta}} \approx \frac{\sigma_\epsilon^2}{|2\pi f|^{2\delta}}$$

- (\*) implies  $\nu_X^2(\tau_j) \propto \tau_j^{2\delta-1}$  approximately
- can deduce  $\delta$  from slope of  $\log(\nu_X^2(\tau_j))$  vs.  $\log(\tau_j)$
- can estimate  $\delta$  &  $\sigma_\epsilon^2$  by applying regression analysis to log of estimates of  $\nu_X^2(\tau_j)$

## Estimation of Wavelet Variance: I

- can base estimator on MODWT of  $X_0, X_1, \dots, X_{N-1}$ :

$$\widetilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \bmod N}, \quad t = 0, 1, \dots, N-1$$

(DWT-based estimator possible, but less efficient)

- recall that

$$\overline{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t = 0, \pm 1, \pm 2, \dots$$

so  $\widetilde{W}_{j,t} = \overline{W}_{j,t}$  if mod not needed:  $L_j - 1 \leq t < N$

- if  $N - L_j \geq 0$ , unbiased estimator of  $\nu_X^2(\tau_j)$  is

$$\hat{\nu}_X^2(\tau_j) \equiv \frac{1}{N - L_j + 1} \sum_{t=L_j-1}^{N-1} \widetilde{W}_{j,t}^2 = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \overline{W}_{j,t}^2,$$

where  $M_j \equiv N - L_j + 1$

- can also construct biased estimator of  $\nu_X^2(\tau_j)$ :

$$\tilde{\nu}_X^2(\tau_j) \equiv \frac{1}{N} \sum_{t=0}^{N-1} \widetilde{W}_{j,t}^2 = \frac{1}{N} \left( \sum_{t=0}^{L_j-2} \widetilde{W}_{j,t}^2 + \sum_{t=L_j-1}^{N-1} \overline{W}_{j,t}^2 \right)$$

1st sum in parentheses influenced by circularity

## Estimation of Wavelet Variance: II

- biased estimator unbiased if  $\{X_t\}$  white noise
- biased estimator offers exact analysis of  $\hat{\sigma}^2$ ;  
unbiased estimator need not
- biased estimator can have better mean square error  
(Greenhall *et al.*, 1999; need to ‘reflect’  $X_t$ )



## Statistical Properties of $\hat{\nu}_X^2(\tau_j)$

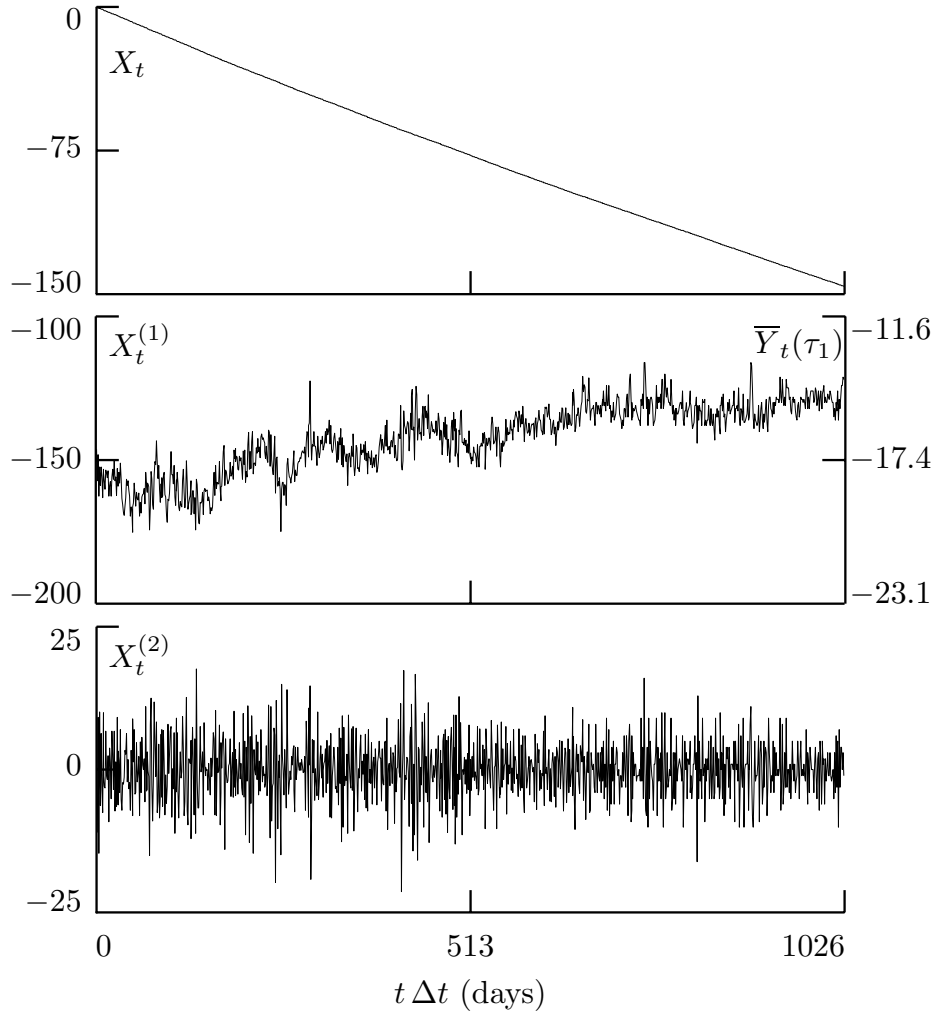
- suppose  $\{\bar{W}_{j,t}\}$  Gaussian, mean 0 & SDF  $S_j(f)$
- suppose square integrability condition holds:

$$A_j \equiv \int_{-1/2}^{1/2} S_j^2(f) df < \infty \ \& \ S_j(f) > 0$$

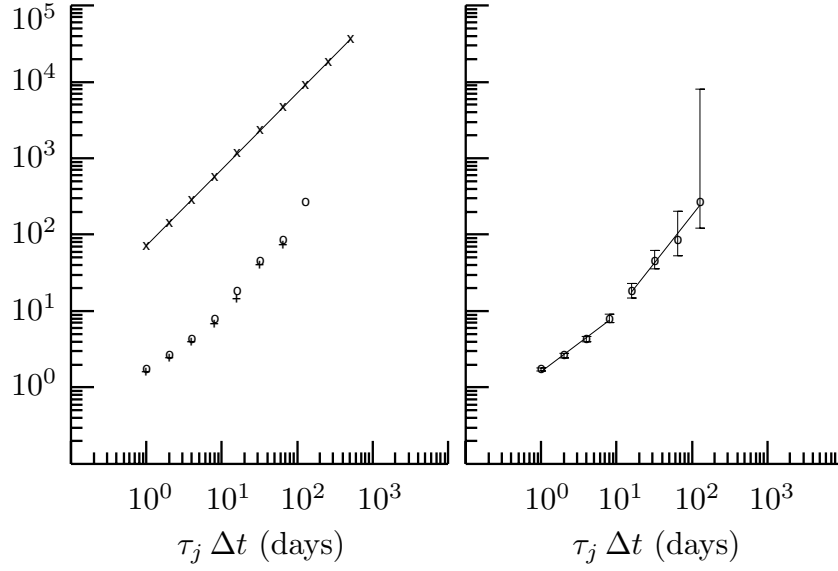
(holds for FD process if  $L$  large enough)

- can show  $\hat{\nu}_X^2(\tau_j)$  asymptotically normal with mean  $\nu_X^2(\tau_j)$  & large sample variance  $2A_j/M_j$
- can estimate  $A_j$  and use with  $\hat{\nu}_X^2(\tau_j)$  to construct confidence interval for  $\nu_X^2(\tau_j)$
- example
  - Fig. 14: clock errors  $X_t \equiv X_t^{(0)}$  along with differences  $X_t^{(i)} \equiv X_t^{(i-1)} - X_{t-1}^{(i-1)}$  for  $i = 1, 2$
  - Fig. 15:  $\hat{\nu}_X^2(\tau_j)$  for clock errors
  - Fig. 16:  $\hat{\nu}_Y^2(\tau_j)$  for  $\bar{Y}_t \propto X_t^{(1)}$
  - Haar  $\hat{\nu}_Y^2(\tau_j)$  related to Allan variance  $\sigma_Y^2(2, \tau_j)$ :

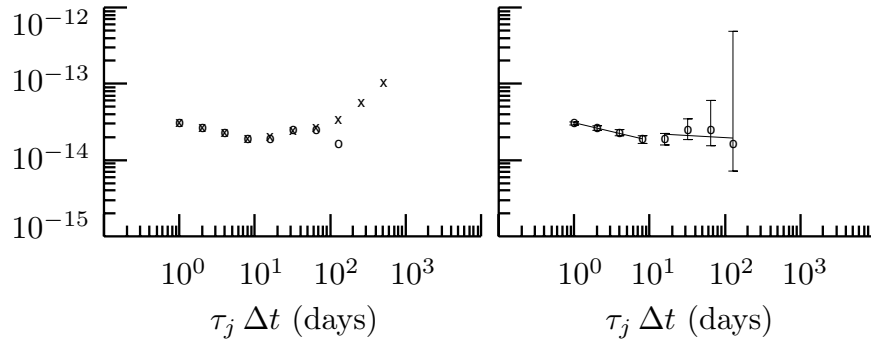
$$\nu_Y^2(\tau_j) = \frac{1}{2} \sigma_Y^2(2, \tau_j)$$



**Figure 14.** Plot of differences in time  $\{X_t\}$  as kept by clock 571 (a cesium beam atomic clock) and as kept by the time scale UTC(USNO) maintained by the US Naval Observatory, Washington, DC (top plot); its first backward difference  $\{X_t^{(1)}\}$  (middle); and its second backward difference  $\{X_t^{(2)}\}$  (bottom). In the middle plot,  $\bar{Y}_t(\tau_1)$  denotes the  $\tau_1$  average fractional frequency deviates (given in parts in  $10^{13}$ ) – these are proportional to  $X_t^{(1)}$ .



**Figure 15.** Square roots of wavelet variance estimates for atomic clock time differences  $\{X_t\}$  based upon the unbiased MODWT estimator and the following wavelet filters: Haar (x's in left-hand plot, through which a least squares line has been fit), D(4) (circles in left- and right-hand plots) and D(6) (pluses in left-hand plot). The right-hand plot also shows 95% confidence intervals for the unknown wavelet variances.



**Figure 16.** Square roots of wavelet variance estimates for atomic clock one day average fractional frequency deviates  $\{\overline{Y}_t(\tau_1)\}$  based upon the unbiased MODWT estimator and the following wavelet filters: Haar (x's in left-hand plot) and D(4) (circles in left and right-hand plots).

## Summary

- fractionally differenced processes are
  - able to cover all power laws
  - easy to work with (SDF, ACVS & PACS simply expressed)
  - extensible to composite, ARFIMA & time-varying processes
- spectral and wavelet analysis can provide
  - estimates of parameters of FD processes
  - decomposition of sample variance across
    - \* frequencies (in case of spectral analysis)
    - \* scales (in case of wavelet analysis)
  - complementary analyses
- wavelet analysis has some advantages for clock noise
  - estimates  $\delta$  &  $\sigma_\epsilon^2$  somewhat better
  - useful with time-varying noise process
  - can deal with polynomial trends (not covered here)
  - results expressed in same units as  $X_t^2$
- a big ‘thank you’ to conference organizers!