Modelling of Clock Behaviour

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overheads and paper for talk available at

http://faculty.washington.edu/dbp/talks.html
Overview

• atomic clocks can keep time to unimaginable precision . . .
• . . . but some people (clock modellers!) are never satisfied and insist on focusing on unimaginable imprecisions
• will discuss
  — concepts behind clock modelling
  — ways in which current modelling practices can be improved
A Thought Experiment

• suppose we have a clock whose performance we want to evaluate
• will assume we can compare this clock to ‘perfect’ time
• at midnight we set the clock to perfect time and measure how well it does over the next 24 hours:

- clocks wanders away from perfect time over 24 hour period, ending up about 50 nanoseconds ahead of perfect time
• no obvious explanation for observed time deviations
Second Day of Thought Experiment

- let’s do this again!
- at midnight we reset the clock to perfect time and now get this:

![Graph showing time deviations over 24 hours.]

- after 24 hours, clock is now about 30 nanoseconds behind
- again, no obvious explanation for observed time deviations; however, deviations seem to have the same visual ‘bumpiness’
Thought Experiment After 100 Days

- we keep on doing this for 100 days:

- can’t predict exactly what will happen on any given day
- can make some statistical statements about the nature of the time deviations over the 24 hour period
  - average deviation after 6 hours close to 0, and 95% of curves are between about $-30$ and $30$ nanoseconds
  - average deviation after 24 hours also close to 0, but now 95% of curves are between about $-60$ and $60$ nanoseconds
Purpose of Clock Models

- clock models summarize statistical information in our data

- let’s focus again on what we observe each day at 6AM

- histogram (right-hand plot) offers some summary, but we can more with the help of a theoretical distribution
Gaussian Distribution to the Rescue!

• popular theoretical distribution is Gaussian (normal)
• bell-shaped curve determined by two parameters
  – mean, which is set by average (≈ 0) of 100 6AM deviations
  – variance, which quantifies the fact that most values occur between −30 and 30 nanoseconds
• bell-shaped curve shows Gaussian approximation:

![Graph showing Gaussian distribution with a bell-shaped curve and deviations from 6 AM measurements.]

• our simple clock model has summarized statistical information about 100 6AM measurements using just 2 parameters
Modelling Deviations at 24 Hours

- let’s now look at observations after 24 hours of elapsed time

- here are the 100 time deviations, histogram and Gaussian fit:

- mean \( \approx 0 \) still, but variance is larger, reflecting fact that 95% of the curves are now roughly between \(-60\) and \(60\) nanoseconds
Variation of Gaussian Parameters over Elapsed Time

• can repeat the above procedure for all elapsed times from 0 hours to 24 hours

• let’s plot the two Gaussian parameters versus elapsed time:

- mean is always close to 0, while variance increases approximately linearly
Summary Offered by Clock Model

- with help of Gaussian distribution, can summarize univariate statistic properties observed in this plot

![Graph showing nanoseconds vs. hours from midnight](image)

using just *one* variable (a level parameter $C$ determining rate of linear increase in variance)!!!

- impressive simplification, but only of univariate properties
Bivariate Statistical Properties

- also interested in relationship between deviations at two distinct elapsed times, e.g., 6 hours and 7 hours after midnight:

- deviations at 7 hours and 6 hours are positively correlated:
Clock Models Incorporating Multivariate Properties

• in general, interested in relationship amongst deviations at multiple elapsed times, i.e., multivariate statistical properties

• interestingly enough, with help of multivariate Gaussian distribution, can summarize information with one more parameter beyond what we need to summarize univariate properties!!!

• additional parameter $\alpha$ essentially selects a particular pattern for the increase in variance (this was linear in our experiment)

• class of models parameterized by $C$ and $\alpha$ known as ‘power law’ models
Five Canonical Power Law Models

• in practice, parameter $\alpha$ is usually set to one of five values
  * $\alpha = 0$ yields model known as white phase noise
  * $\alpha = -1$ yields flicker phase noise
  * $\alpha = -2$ yields random walk phase noise (our experiment!)
  * $\alpha = -3$ yields flicker frequency noise
  * $\alpha = -4$ yields random walk frequency noise

• variance versus elapsed time
  * is constant for $\alpha = 0$
  * increases for $\alpha = -1$ (but hard to describe exactly!)
  * increases linearly for $\alpha = -2$
  * increases for $\alpha = -3$ (again, hard to describe!)
  * increases quadratically for $\alpha = -4$
Deviations Generated by Canonical Models

- Here is one set of time deviations from each model:

- Note: Level parameter $C$ merely sets vertical scaling.
Use of Canonical Models in Theory

• when applicable, canonical models along with the multivariate Gaussian distribution offer a very simple summary of the statistical properties of time deviations (only need $C$ and $\alpha$)
• useful for comparing statistical properties of different clocks
• can use models to form an ensemble time that is better than any individual clock
Limitations of Canonical Models in Practice

• picture painted so far of clock modelling is too simplistic
• real-world complications make it harder to reap benefits
  – need different canonical models for different elapsed times
  – spacing of $\alpha$ is too coarse (models exist for all $\alpha$)
  – lack of data for decent parameter estimation
  – problem of trend and its estimation
Opportunities for Improvements

• regard $\alpha$ as parameter to be estimated rather than identified (typically preselection of $\alpha$ not accounted for, but this can be an important source of sampling variability)

• sampling variability in trend estimates usually not propagated properly

• lack of attention to sampling variability in $\alpha$/trend estimates might explain failure of ‘optimal’ procedures for forming ensemble time scales

• much has been done, but still more to do!

• use of proper statistical procedures will hopefully lead to better performance of systems depending on clocks, but work must be done to see if this is so
Final Word

- thanks to organizers for making this presentation possible!!!