Estimating the Frequency Response of a Transfer Function

Xu Chen
Department of Mechanical Engineering
University of Washington
chx AT uw.edu
September 29, 2019

1 Main results

Let \( u \) and \( y \) be the input and the output of a linear time-invariant (LTI) transfer function \( G_{yu} \). The frequency response of \( G_{yu} \) is the quotient of \( \Phi_{yu} \), the cross power spectral density of \( u \) and \( y \), and \( \Phi_{uu} \), the power spectral density of \( u \):

\[
G_{yu} = \frac{\Phi_{yu}}{\Phi_{uu}}
\]

It is also true that

\[
G_{yu} = \frac{\Phi_{yy}}{\Phi_{yu}}
\]

2 The underlying theory

2.1 Definitions

- \( X_{xx}(l) \): auto covariance of a stationary random process \( x \), defined by

\[
X_{xx}(l) = E [(x(k) - E[x])(x(k+l) - E[x])]
\]

where \( E[] \) is the operation of computing the mean.

- \( X_{xy}(l) \): cross covariance between two stationary random processes \( x \) and \( y \), defined by

\[
X_{xy}(l) = E [(x(k) - E[x])(y(k+l) - E[y])]
\]

Under mild conditions (called ergodic) that are commonly satisfied in practice, auto and cross covariances can be computed by the ensemble averages, namely

\[
X_{xx}(l) = E [(x(k) - E[x])(x(k+l) - E[x])] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{j=-N}^{N} (x(j) - E[x])(x(j+l) - E[x])
\]

- \( \Phi_{xx}(\omega) \): power spectral density is the Fourier transform of auto covariance, defined by

\[
\Phi_{xx}(\omega) = \sum_{l=-\infty}^{\infty} X_{xx}(l) e^{-j\omega l}
\]

Remark: Given the time sequence of \( x \) and \( y \), there are existing functions to calculate the power spectral densities in MATLAB.
2.2 Derivations

Consider passing a stationary random process \( u(k) \) through an LTI transfer function \( G(z) \). The resulting output is defined by the convolution:

\[
g(k) = g(k) * u(k) = \sum_{i=-\infty}^{\infty} g(i) u(k-i)
\]

where \( g(k) \) is the impulse response of \( G(z) \).

- if \( u \) is zero mean and ergodic, then

\[
X_{uy}(l) = u(k) \sum_{i=-\infty}^{\infty} u(k+l-i)g(i) = \sum_{i=-\infty}^{\infty} u(k)u(k+l-i)g(i) = \sum_{i=-\infty}^{\infty} X_{uu}(l-i)g(i) = g(l) * X_{uu}(l)
\]

similarly

\[
X_{yu}(l) = \sum_{i=-\infty}^{\infty} X_{yu}(l-i)g(i) = g(l) * X_{yu}(l)
\]

- in pictures we have

\[
\begin{align*}
X_{uu}(l) & \xrightarrow{G(z)} X_{uy}(l) \quad X_{yu}(l) & \xrightarrow{G(z)} X_{yy}(l)
\end{align*}
\]

- for a general LTI system

\[
u(k) \xrightarrow{G(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}} y(k)
\]

convolution in time domain is multiplication in frequency domain:

\[
Y(z) = G(z)U(z) \iff Y(e^{j\omega}) = G(e^{j\omega})U(e^{j\omega})
\]

- hence for the auto/cross covariances:

\[
\begin{align*}
X_{uu}(l) & \xrightarrow{G(z)} X_{uy}(l) \quad X_{yu}(l) & \xrightarrow{G(z)} X_{yy}(l)
\end{align*}
\]

we have

\[
\Phi_{yy}(\omega) = G(e^{j\omega})\Phi_{yu}(\omega)
\]

\[
\Phi_{yu}(\omega) = G(e^{j\omega})\Phi_{uu}(\omega)
\]