Equilateral k-Isotoxal Tiles

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Abstract

In this article we introduce the notion of equilateral k-isotoxal tiles and give of examples of equilateral k-isotoxal tiles for $1 \leq k \leq 4$.

1 Introduction

A *plane tiling* $\mathcal{T}$ is a countable family of closed topological disks $\mathcal{T} = \{T_1, T_2, ...\}$ that cover the Euclidean plane $\mathbb{E}^2$ without gaps or overlaps; that is, $\mathcal{T}$ satisfies

1. $\bigcup_{i \in \mathbb{N}} T_i = \mathbb{E}^2$, and
2. $\mathrm{int}(T_i) \cap \mathrm{int}(T_j) = \emptyset$ when $i \neq j$.

The T_i are called the *tiles* of $\mathcal{T}$. For a polygonal tile it is clear what is meant when refering to the vertices and edges of a tile, but in general, the *vertices* of a tile are any finite collection of points on the boundary of the tile, and the *edges* of the tile are the arcs along the boundary of the tile between two consecutive vertices. We may also refer to the edges and vertices *of a tiling.* The intersection of any two distinct tiles in a tiling $\mathcal{T}$ is called an *edge* of the tiling, which may be a collection of disjoint arcs. The end points of the arcs comprising the edges of a tiling will be called the *vertices* of the tiling. If the vertices and edges of the tiles in a tiling coincide with the vertices and edges of the tiling, then the tiling is said to be *edge-to-edge.*

A tiling $\mathcal{T}$ is *isotoxal* if every edge of $\mathcal{T}$ is in a single transitivity class with respect to the symmetry group of $\mathcal{T}$; that is, $\mathcal{T}$ is isotoxal if for any two edges e_i and e_j of $\mathcal{T}$, there exists a symmetry of the tiling taking e_i onto e_j. One can find many simple examples of isotoxal tilings, such as the regular tilings by equilateral triangles, squares, and regular hexagons. In fact, isotoxal tilings have been classified according to topological type and incidence symbols, and there 26 such tilings [2]. The notion of isotoxal tilings can be generalized in the obvious way to k-isotoxal tilings in which there are k transitivity classes of edges.

Our focus will be on finding tiles that admit only k-isotoxal tilings. Of course, this problem is trivial if all of the edges of a tile are not congruent. For this reason, we will require our tiles to be *equilateral.* An equilateral tile that admits only k-isotoxal edge-to-edge tiles of the plane will be called *equilaterally k-isotoxal.*

The distinction between k-isotoxal tilings and equilaterally k-isotoxal tiles is analagous to the distinction between nonperiodic tilings and aperiodic protosets: a nonperiodic tiling is one whose symmetry group does not contain two nonparallel translations, while an aperiodic protoset is a set of tiles that admits only nonperiodic tilings. While nonperiodic tilings are not hard to find, aperiodic protosets are rare and difficult to find. In a similar way, k-isotoxal tilings are not difficult to generate (just consider tilings involving nonequilateral tiles), but equilaterally k-isotoxal tiles seem difficult to find.

2 Examples of equilaterally 2- and 3-isotoxal tiles

First, we note that the regular hexagon is an equilaterally 1-isotoxal tile. In Figure 1 is an equilaterally 2-isotoxal tile. This tile is a square whose sides have been marked with asymmetric bumps and nicks. We will declare that this tile has four vertices at the corners of the square. That the corners incident to the bumps and nicks are not counted as vertices is not way of cheating since it is easy round off those corners to create a smooth curve whose endpoints are the corners of the underlying square.

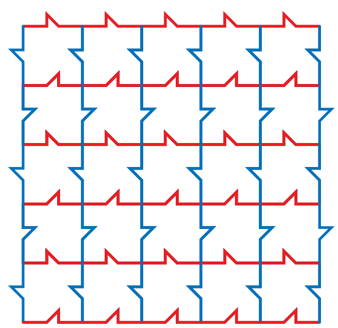

Figure 1: A patch formed by an equlaterally 2-isotoxal tile. The transitivity classes of edges are indicated by color.

Figure 2 shows an equilaterally 3-isotoxal tile.

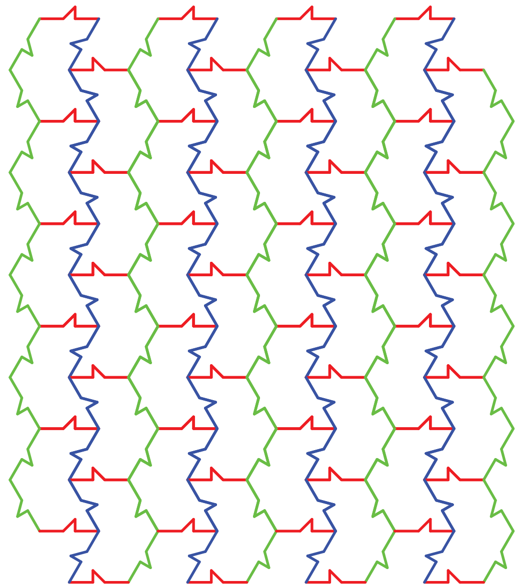

Figure 2: A patch formed by an equlaterally 3-isotoxal tile. The transitivity classes of edges are indicated by color.

Proposition 1. *The tile of Figure 1 is equilaterally 2-isotoxal and the tile of Figure 2 is equilaterally 3-isotoxal.*

Proof. The proof that the prototile of Figure 1 is equilaterally 2-isotoxal will be given here. The proof that the prototile of Figure 2 is equilaterally 3-isotoxal, being similar, is left as an exercise.

First note that due to the bumps and nicks on the prototile of Figure 1, the tile can admit only edge-to-edge tilings. It will be shown that the tiling of Figure 1 is the only tiling that its prototile admits. To keep track of the original top, bottom, left, and right sides of the prototile, color the edges of the prototile as at left in Figure 3. It is first shown that the left side of the tile must meet only the right side of the tile in any tiling formed by this prototile. That the top side meeting the right side is disallowed is argued in Figure 3, where it is seen that allowing the top side to meet the right side forces one of two configurations, neither of which can be continued. Similarly, if the bottom side meets the left side, calling on the fact that the top side may not meet the right side, one of two configurations is forced, and neither of these can be continued to a tiling of the plane (Figure 4). Thus, the left side can meet only the right side and vice versa, which allows for copies of the prototile to form infinte rows as in Figure 5. The proceeding two arguments make it clear that the top side must meet only the bottom side, so the only way to tile the plane with this prototile is to use the rows of Figure 5 in an alternating fashion, as in 1. Additionally, because the top side can meet only the bottom side and the left side can meet only the right side, the tiling admitted by this prototile is 2-isotoxal. □

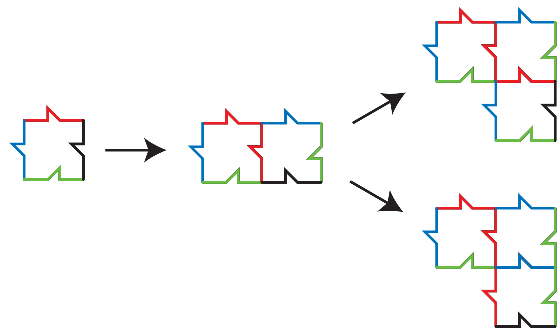

Figure 3: The top side cannot meet the right side.

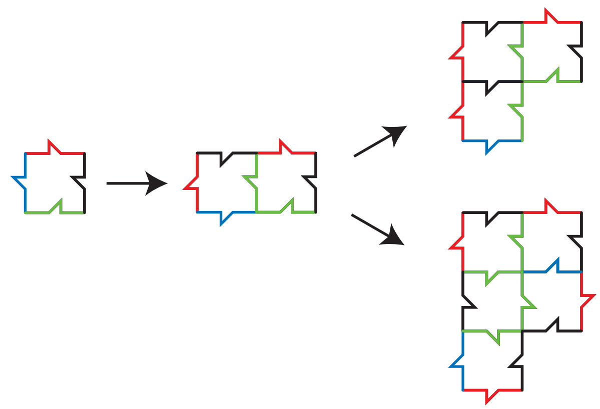

Figure 4: The bottom side cannot meet the left side.

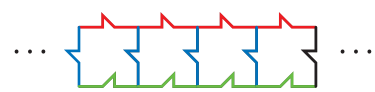

Figure 5: The left side must meet the right side to form infinite rows congruent to this.

3 Matching Rules

In our examples, we used basic polygonal shapes whose sides were marked with certain kinds of edge matching rules. In particular, we used asymmetric bumps and nicks. In full generality, the edge matching rules of a tile for which copies must meet edge-to-edge can be viewed as a formal declaration of what sides of the tile may meet with one another. However, edge matching rules with such generality often cannot be realized in terms of purely geometrically defined tiles. Indeed, some of the most common types of edge matching rules are not geometrically enforceable, such as colored edges or directed edges. *Geometrically enforceable* edge matching rules are those that can be enforced using edges of four kinds of curves: straight line segments, S-curves, C-curves, and J-curves. *S-curves* are nonstraight curvilinear segments that have 180° rotational symmetry about their midpoints. *C-curves* are nonstraight curvilinear segments that have reflective symmetry about their perpendicular bisectors. Note that if an edge of a tile is a C-curve, that edge must "point" outward or inward. *J-curves* are nonstraight curvilinear segments that possess only trivial symmetry. Like C-curves, J-curves of a tile may be thought of as pointing outward or inward.

If two planar objects are congruent by an orientation preserving isometry, then we will say that the objects are *directly congruent*, and if these two objects are congruent but not directly congruent, they are *indirectly congruent*. If T is some fixed tile whose edges are S-curves, C-curves, or J-curves, let T_1 and T_2 be tiles congruent (directly or indirectly) to T, and let e_1 be an edge of T_1 and e_2 and edge of T_2.

- If e_1 and e_2 are congruent S-curves, then T_1 and T_2 may meet along e_1 and e_2 if e_1 and e_2 are directly congruent.
- If e_1 and e_2 are congruent C-curves, then T_1 and T_2 may meet along e_1 and e_2 if e_1 and e_2 are oppositely opposed (i.e. e_1 is pointing outward on T_1 and e_2 is pointing inward on T_2, or vice versa).
- If e_1 and e_2 are congruent J-curves, T_1 and T_2 may meet along e_1 and e_2 if e_1 and e_2 are directly congruent and oppositely opposed.

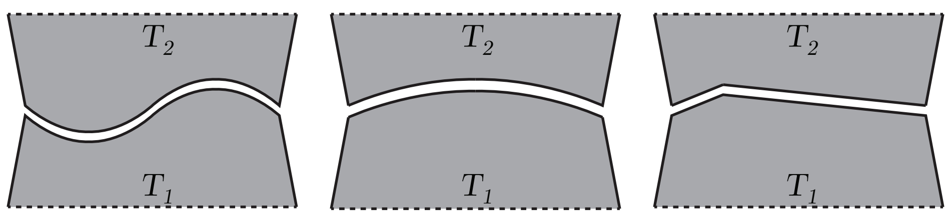

Figure 6: geometrically enforceable edge matching rules

C-curves are often represented by symmetric bumps and nicks and J-curves by asymmetric bumps and nicks. S-curves cannot be represented by any curve that has inward/outward orientation with respect to the tile, but they can be represented by directed straight edges with the rule that two edges may meet when their directions are opposite.

4 Triangles

First, we point out that any equilaterally isotoxal triangle cannot have edges that are C-curves or J-curves (which can be thought of as symmetric or asymmetric bumps or nicks). It turns out that any tile, some of whose sides are marked with bumps and nicks, must have the property that the number of nicks of a particular type must be equal to the number of bumps of the same type if that tile tessellates the plane [3]. This is a result of the Normality Lemma [2]. Since an equilateral polygon with an odd number of edges whose edges are all congruent to a C-curve or a J-curve would necessarily have an imbalance between the number of bumps and nicks, we can rule out C-curves and J-curves for polygons with an odd number of sides. Thus, an equilaterally isotoxal triangle must have either three straight sides or three congruent S-curve sides. If all three sides are straight, it is easily seen that the only tiling admitted by the equilateral triangle is 1-isotoxal. Similarly, if all three sides are directly congruent S-curves, the two tilings admitted (one using all directly congruent copies of the triangle, the other using all indirectly congruent copies of the triangle) is 1-isotoxal.

Lastly, we consider the equilateral triangle with two sides that are directly congruent S-curves and a third side that is indirectly congruent to the other two. Any tiling by such a triangle in which every tile is directly congruent must have at least two transitivity classes of edges. In Figure 7, we see a 2-isotoxal and a 3-isotoxal tiling by directly congruent copies.

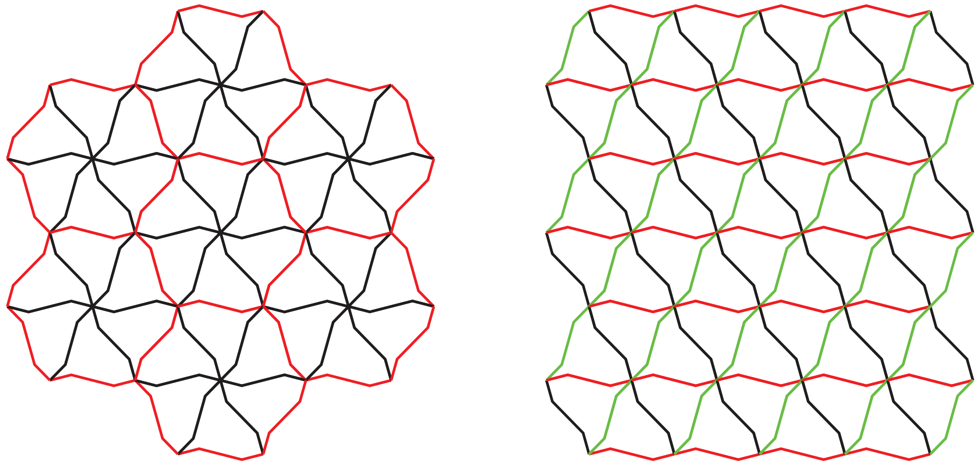

Figure 7: At left is a 2-isotoxal tiling admitted by an equilateral triangle. At right is a 3-isotoxal tiling admitted by the same triangle.

The tilings of Figure 7 show that the equilateral triangle with two sides that are directly congruent S-curves and a third side that is indirectly congruent to the other two is not equilaterally k-isotoxal for any k. If we allow for directly and indirectly copies of the prototile in a tiling, we can produce tilings that have arbitrarily large numbers of transitivity classes of edges - even infinitely many (Figure 8).

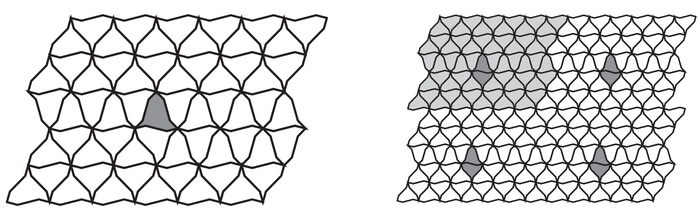

Figure 8: The tiling at left has trivial symmetry group (and so has infinitely many transitivity classes of edges. At right is a periodic tiling (fundamental region shaded) with a large (but finite) number of transitivity classes of edges. The dark gray tiles are indirectly congruent to the white tiles.

5 Pentagons

It is known that there are only two classses of convex equilateral pentagons that tile the plane [1]. Further, D. Schattschneider reports that the nonconvex equilateral pentagons that tile the plane have been classified as well [4]. Thus, pentagons provide some nice base shapes whose sides might be marked to force k-isohedrality. However, for the same reason that an equilaterally isotoxal triangle cannot have as its edges C- or J-curves, neither can any equilaterally isotoxal tile with an odd number of edges.

In Figure 9, is a type of pentagonal tiler whose angle conditions allow for the tile to be equilateral. An equilateral version of this pentagon that has sides that are all directly congruent S-curves can be made so that it is equilaterally 4-isotoxal. In Figure 10 we see an equilaterally 4-isotoxal pentagon and the unique tiling admitted by this tile. That this tile admits just one tiling with 4 transitivity classes of sides is seen by checking that in any tiling by this tile, side 0 must meet side 0 (refering to Figure 10), side 2 must meet side 2, side 3 must meet side 3, and side 1 must meet side 4. It is easily checked that any configuration in which sides meet in ways other than those just specified cannot be extended to a tiling of the entire plane. With this in mind, that the sides must meet as specified earlier allows for a unique tiling of the plane that clearly has 4 transitivity classes of edges. Lastly, we point out if the pentagon is marked with flat sides, the pentagon admits many tilings, some of which are not periodic. Thus, if the pentagon has flat sides, it is not k-isotoxal for any k.

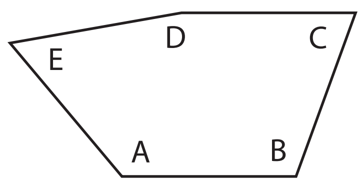

Figure 9: Any convex pentagon with $B + C = 180°$ admits a tiling of the plane.

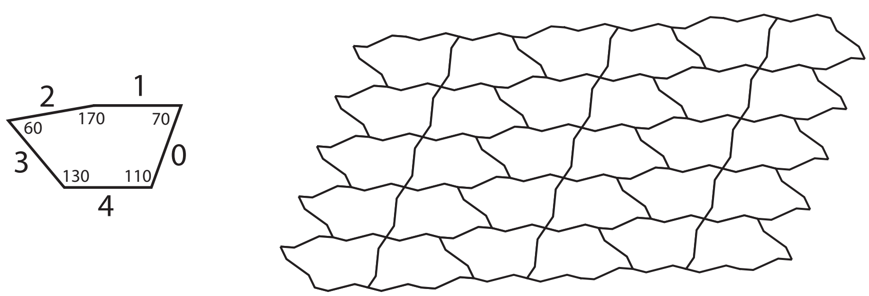

Figure 10: The equilaterally 4-isotoxal pentagon of the tiling is based on the convex pentagon at left.

We leave it as an open problem to investigate the remaining equilateral pentagons that tile the plane to see if other equilaterally k-isotoxal tiles with $1 \geq k \geq 5$ may be identified.

References

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