Heesch's Tiling Problem

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1. INTRODUCTION. Let T be a tile in the plane. By calling T a *tile*, we mean that T is a topological disk whose boundary is a simple closed curve. But also implicit in the word "tile" is our intent to use congruent or reflected copies of T to cover the plane without gaps or overlapping; that is, we want to *tessellate* the plane with copies of T. In a minor abuse of language, one often speaks of T (as opposed to copies of it) as *tiling* or *tessellating* the plane, in the sense that T generates a tessellation. A tessellation by T may or may not be possible, so in order to learn something of T's abilities with regard to tessellating the plane, we perform the following procedure: around a centrally placed copy of T, we attempt to form a full layer, or *corona*, of congruent copies of T. We require as part of the definition that no point of T should be visible from the exterior of a corona to a Flatland creature in this plane. Also, we should form the corona without allowing gaps or overlapping, just as if we were building a tessellation. If a corona can be formed, then we attempt to surround this corona with yet another corona, and then another, and so on; if we get stuck, we go back and change a previously placed tile and try again. If T tessellates the plane, then this procedure will never end. On the other hand, if T does not tessellate the plane, and if we check all of the possible ways of forming a first corona, a second corona, and so forth, we will find that there is a maximum number of coronas that can be formed. This maximum number of layers that can be formed around a single centrally placed copy of T is called the *Heesch number* of T and is denoted by $H(T)$.

We consider a few examples before proceeding. Consider first a regular hexagon. All bees know that a regular hexagon tessellates the plane, so $H = \infty$ for a regular hexagon. Next consider a regular pentagon. It is not hard to see that not even a single corona of copies of a regular pentagon can be formed around a central copy, whence $H = 0$ for a regular pentagon. These two examples exhibit the "typical" behavior of tiles with respect to Heesch numbers: if one considers an arbitrary tile, it is usually the case either that the tile tessellates the plane ($H = \infty$) or that one cannot form even a single corona ($H = 0$). Naturally one wonders if there are tiles with Heesch number $1, 2, 3, \ldots$. This question is known as *Heesch's Tiling Problem* (see [**9**, p. 23], [**7**, p. 155], or [**8**, p. 187]):

Open Question 1 (Heesch's Tiling Problem). *Given a positive integer N, does there exist a tile T_N with $H(T_N) = N$?*

Interestingly, before Heinrich Heesch stated his problem in 1968, only one tile with Heesch number other than 0 or ∞ had been observed—a curvilinear tile called a *spandrel* (Figure 1) with Heesch number 1 was published by Lietzmann in 1928 [**10**, p. 242], [**11**]. Heesch himself identified a tile with Heesch number 1 in the same paper where he first posed his problem [**9**, p. 23]. It wasn't until 1991 that examples with finite Heesch number greater than 1 were given. First, an infinite family of U-shaped polyominoes, each having Heesch number 2, was described by Fontaine [**4**] (see Figure 2). In that same year a hexagon with notched edges (Figure 3) having Heesch number 3 that had been discovered by Robert Ammann was presented at a special session on tiling at a regional AMS meeting in Philadelphia (and was featured on the
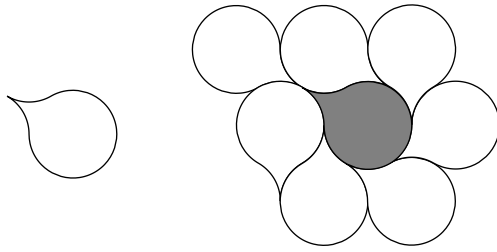
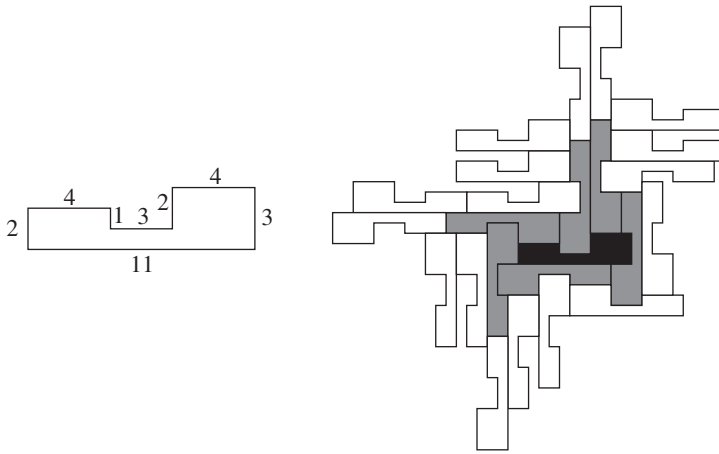**Figure 1.** Lietzmann's spandrel with one corona.
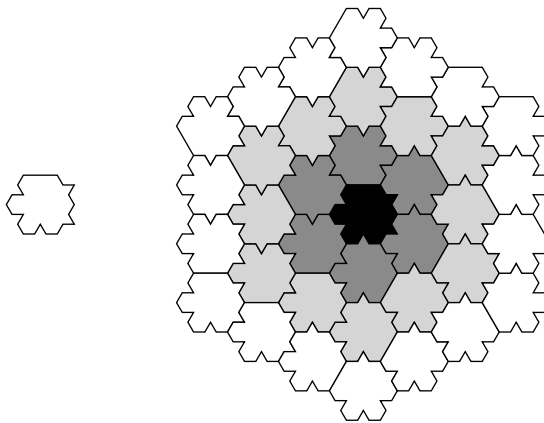

Figure 2. Fontaine's U-shaped polyomino with two coronas.

Figure 3. Ammann's notched hexagon with three coronas.

cover of the meeting program) [**17**], [**18**]. In this article we introduce a few families of tiles with Heesch numbers 3, 4, and 5, as well as other examples.

2. NEW EXAMPLES. Consider Ammann's notched hexagon more closely. First, notice that all of the bumps and nicks are centered on the edges, and all are of the

same shape. That the bumps and nicks have the same shape ensures that any bump can meet any nick; that they are centered allows two copies of the tile to meet along an edge regardless of whether or not one of the tiles has been reflected. One might wonder if a different combination of bumps and nicks could give a hexagon with Heesch number greater than 3, but this is not so. To see this, one could simply enumerate all of the possibilities and check each case individually, but there is an interesting phenomenon at play here that is worth investigating. The reason that Ammann's tile cannot tessellate the plane is that it is combinatorially unbalanced—it has more nicks than bumps. Epstein [3] gave a simple proof that no hexagon can have Heesch number greater than three if its edges are marked so that there are unequal numbers of bumps and nicks. Suppose that a fourth corona of such a notched hexagon has been formed. Such a configuration contains sixty-one hexagonal tiles and has fifty-four edges on its boundary. Say that there are more bumps than nicks on this tile. Then there are at least sixty-one unmatched bumps, bumps that must appear on the boundary of the configuration. But since there are only fifty-four edges on the boundary, there is not enough room for all of the bumps!

We can generalize Ammann's hexagon by "fusing" two or more hexagons to form a *polyhex* and marking the edges of the polyhex with bumps and nicks. To illustrate this with a new example of our own, the tile in Figure 4 is formed by gluing together three hexagons and marking a few edges with bumps and nicks. This particular example has been shown by an exhaustive computer search to have Heesch number 2.
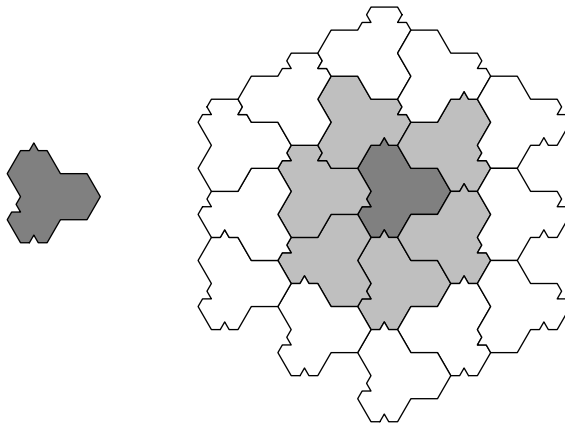

Figure 4. A combinatorially unbalanced polyhex with Heesch number 2.

The combinatorial device that constrained Ammann's original tile is still in evidence here. No polyhex with an unequal number of bumps and nicks can tile the plane, and for essentially the same reason that Ammann's original tile did not: at some stage there would necessarily be more unmatched bumps (or nicks) than boundary edges of the outer corona. However, as the size of a polyhex grows (in terms of the number of hexagons it comprises), so does the potential for achieving a high Heesch number. While we cannot accurately predict the Heesch number of an arbitrary combinatorially unbalanced polyhex, we can give an upper bound for the Heesch numbers of combinatorially unbalanced polyhexes formed from n hexagons; namely, the Heesch number is roughly $\mathcal{O}(\sqrt{n})$ [12]. This is a very interesting and important observation, for it tells us that combinatorially unbalanced polyhexes, which are geometrically and combinatorially simple (relatively speaking), have the potential to give arbitrarily large Heesch

numbers. The reader should pause and reflect on this amazing possibility: imagine a tile that allows, say, 1,000,000 coronas to be formed, but that somehow does not generate a complete tiling of the plane! One is tempted to discount this possibility, but we hope that as the reader proceeds he or she may become more open to it.

We now consider an infinite family of combinatorially unbalanced polyhexes that we have named *hexapillars*. An *n-hexapillar* is a polyhex formed by fusing *n* hexagons in a row and marking the edges with bumps and nicks as in Figure 5.
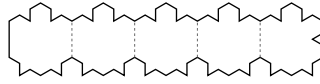

Figure 5. A 5-hexapillar.

The status of the Heesch numbers for these hexapillars is given in the following theorem [**12**, p. 25]:

Theorem 1. *The* 2- *and* 3-*hexapillars have Heesch number* 4, *while the n-hexapillars with n ≥ 4 have Heesch number* 5.

Finding the Heesch numbers of specific *n*-hexapillars for small *n* was accomplished by means of an exhaustive computer search. A proof for the general case can be found in [**12**]. It was discovered independently by Marshall that the 2-hexapillar has Heesch number 4 [**14**]. Figure 6 depicts the 5-hexapillar along with five coronas.
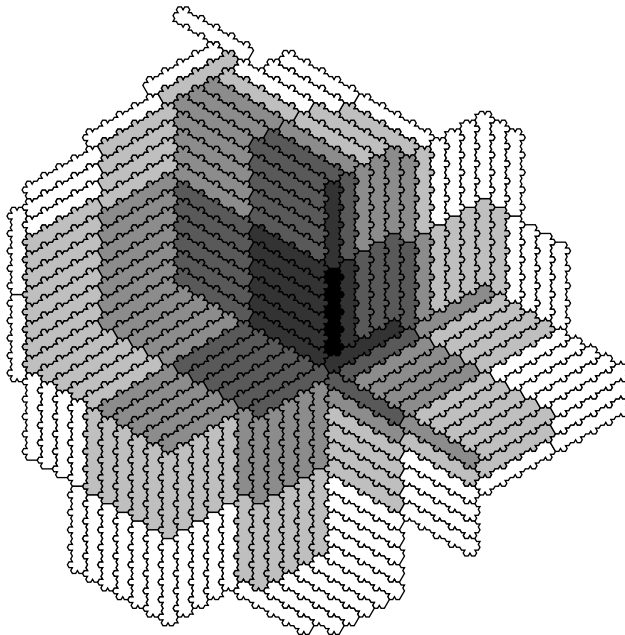

Figure 6. A 5-hexapillar surrounded by five coronas of 5-hexapillars.

We have also discovered a second infinite family of tiles in which each tile has Heesch number 3. These tiles are analogous to the hexapillars, but instead of be-

ing formed from hexagons they are constructed from squares. Tiles constructed from squares are usually called *polyominoes*. We call these particular polyominoes *polypillars* (Figure 7).
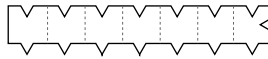

Figure 7. A 7-polypillar.

Theorem 2. *Let T_n be an n-polypillar. Then $H(T_n) = 2$ for $n = 2$ or 3, and $H(T_n) = 3$ for $n \geq 4$.*

The behavior of polypillars is similar to that of hexapillars, so Theorem 2 is established in much the same way as Theorem 1. In Figure 8 we see a 7-polypillar surrounded by three coronas.
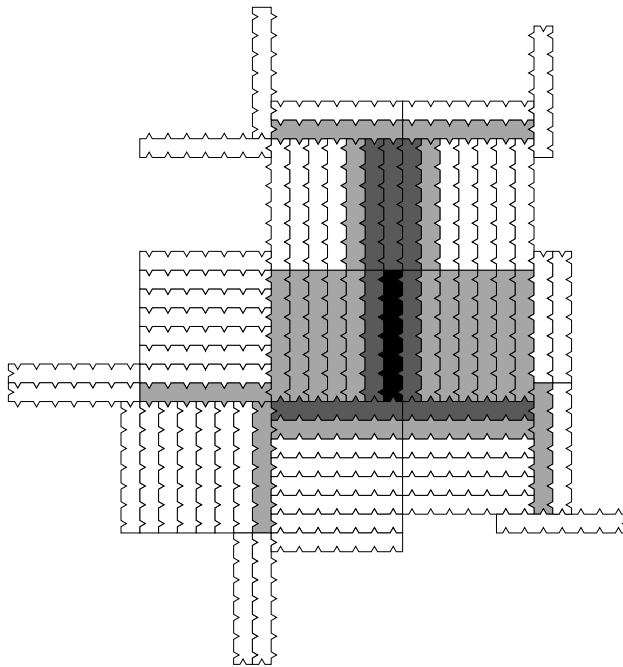

Figure 8. A 7-polypillar surrounded by three coronas.

3. THE TILING PROBLEM. Heesch's tiling problem is connected to another unsolved tiling problem that is central to the subject. Before stating this problem, we need to establish a small amount of terminology. Often one wants to use more than one kind of shape to tile the plane. The shapes with which one wants to tessellate the plane are collectively called a *protoset*. The individual tiles in a protoset are called *prototiles*. In the case that a protoset contains only one prototile, the distinction between the protoset and the prototile is blurred, and in this case both are called *monotiles*.

Open Question 2 (The Tiling Problem for Monotiles). *Does there exist an algorithm or standard procedure that can take as input an arbitrary single tile and determine in a finite number of steps whether or not that tile tessellates the plane?*

If such an algorithm exists, we say that the problem is *decidable.* The tiling problem for arbitrary protosets was shown to be undecidable by Berger in [**1**] and later by Robinson [**16**]. But for protosets of any fixed size, the problem remains open; in particular, it is still open for monotiles. To see how Heesch's tiling problem is connected to the tiling problem for monotiles, we first make the very mild restriction that two copies of the tile can meet in only finitely many ways (i.e., we consider only tiles with finitely many vertices and edges and require that tiles meet vertex-to-vertex). This restriction reduces the problem of deciding when two tiles can meet to consideration of a finite number of possibilities. Now suppose that there is a maximum finite Heesch number M that can be obtained by any single tile. An algorithm for this restricted version of the tiling problem for monotiles would then go as follows. Given a tile with only finitely many edges and vertices, start with a single centrally placed copy of the tile and begin forming coronas around it, creating all possible first coronas, then all possible second coronas, then all possible third coronas, and so forth, until one of two things happens: either we form more than M coronas, in which case the tile must tessellate the plane, or we find that it is impossible to form M coronas, in which case the tile cannot tessellate the plane. Thus we have established the following theorem:

Theorem 3. *If the tiling problem for monotiles with finitely many vertices and edges is undecidable, then there is no finite upper bound on Heesch numbers.*

On an interesting side note, the tiling problem for monotiles is connected with another famous open problem in the theory of tilings, *the Einstein problem.*[1] This problem concerns *aperiodic* monotiles. A protoset is said to be *aperiodic* if (1) its tiles tessellate the plane and (2) every possible tessellation by these tiles is nonperiodic.[2] Many aperiodic protosets have been discovered, from protosets containing many thousands of tiles [**1**] down to protosets containing only two tiles, the most famous of these being the Penrose rhombi [**15**], [**5**]. It remains to be seen whether or not an aperiodic monotile exists:

Open Question 3 (Einstein Problem). *Does there exist an aperiodic monotile?*

To see how the Einstein problem is linked to the tiling problem for monotiles, suppose that there does not exist an aperiodic monotile. Then every tile that tessellates the plane must give rise to at least one periodic tessellation of the plane. An algorithm for deciding whether or not an arbitrary tile tessellates the plane could then be constructed as follows. First, we again need to assume that copies of the tile can meet in only finitely many ways. If we then systematically form all possible finite patches of copies of the tile that cover circular disks centered at the origin of discretely incremented increasing area, one of two things can happen: (1) at some step a fundamental domain for a periodic tiling group[3] is found (there are only seventeen periodic tiling groups, or "crystallographic" groups), or (2) we find a disk that cannot be covered, implying that

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[1]The word "Einstein" in the Einstein problem is a pun: "ein" = "one" and "stein" = "tile." It is attributed to Ludwig Danzer.

[2]A tessellation of the plane is *periodic* if its symmetry group contains two nonparallel translations. There is often some confusion distinguishing between the terms "aperiodic" and "nonperiodic." The term "nonperiodic" refers to a particular tessellation, whereas the term "aperiodic" refers to a protoset.

[3]A *tiling group* is representative of an isomorphism class of symmetry groups of tessellations. A *fundamental domain* for a tiling group is a smallest configuration of tiles that when acted on by the tiling group generates a tiling of the plane.

a tessellation of the plane is not possible.[4] Thus we have proved the following theorem (originally noted by Wang in [**20**]):

Theorem 4. *If the tiling problem for monotiles is undecidable for monotiles that have finitely many vertices and edges and that meet vertex-to-vertex, then there exists an aperiodic monotile.*

The tiling problem for monotiles has implications for several other open tiling problems; for further reading on this interesting topic see [**5**].

4. MORE ON HEESCH'S PROBLEM. Heesch's tiling problem can, of course, be posed for tiles in higher dimensional Euclidean space $\mathbb{E}^n$ and in spaces with alternate geometries, such as n-dimensional hyperbolic space $\mathbb{H}^n$ and the n-dimensional sphere $\mathbb{S}^n$. Not only does the setting for Heesch's tiling problem generalize, but so does the problem itself. We briefly summarize what is known in several different contexts:

- First, we note that any planar tile can be used to form a cylindrical tile with the same Heesch number in $\mathbb{E}^3$. This means that there are space tiles with Heesch numbers 0 through 5. We have identified nontiling space tiles that potentially have higher Heesch numbers (as high as 8), but these tiles have defied exhaustive computer checks because of the vast number of combinatorial possibilities. In particular, if a rhombic dodecahedron (Figure 9) is marked with conical bumps and nicks on its faces so as to be combinatorially unbalanced, a simple counting argument reveals that the maximum possible Heesch number is 8. However, it is unknown whether or not eight coronas of such a tile can be realized.
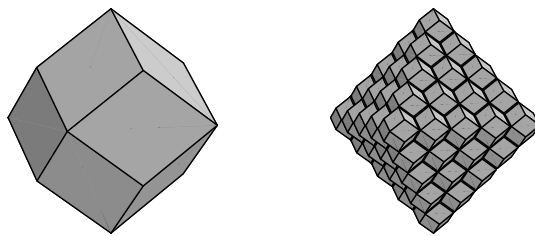
Figure 9. A portion of a tessellation (two coronas) of $\mathbb{E}^3$ by the rhombic dodecahedron.

- For any hyperbolic regular polygon each of whose edges is marked with either a bump or a nick, it is known whether or not the tile tessellates $\mathbb{H}^2$; among those that do not tessellate the hyperbolic plane, the highest Heesch number is 1 [**13**]. We have also analyzed the slightly more general class of hyperbolic regular polygons where only *some* of the edges are marked with bumps and nicks. Because of the nature of the metric on $\mathbb{H}^2$, the simple counting arguments that applied to tiles such as Ammann's tile with Heesch number 3 become much more complicated. Nevertheless, such an analysis has been performed with negative results in terms of large Heesch

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[4]One may wonder if it is possible that larger and larger circular disks can be covered, yet a tessellation of the entire plane still not be possible. Conveniently, a tessellation of the plane must exist under these conditions (although this tessellation may not contain any of the finite patches postulated). This fact is called the Extension Theorem [**8**, p. 151]. It can be also be demonstrated by an application of König's lemma in this special case.

numbers [**12**]. To date, no tile with Heesch number greater than one has been identified in $\mathbb{H}^2$.

- Tiles on the sphere have not yet been studied systematically from a Heesch number perspective. The only Heesch-number-related discovery in $\mathbb{S}^2$ known to this author is a 75° equilateral triangle with Heesch number 3 (Figure 10) [**2**].
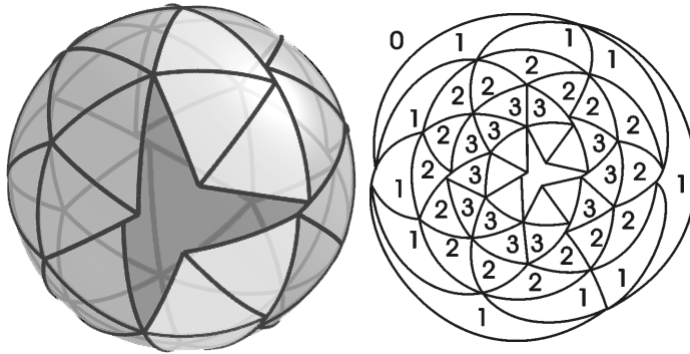

Figure 10. Dawson's Heesch number 3 tile on the sphere. The figure at right is a 360° "fisheye" view of the partial tessellation of the sphere on the left. This projection inverts the coronas so that the center tile corresponds to the exterior of the projection.

- While individual tiles in $\mathbb{E}^n$ that are marked with more bumps than nicks (or vice versa) cannot tessellate space and must have finite Heesch number, there is no uniform upper bound on the Heesch numbers of such tiles, thus allowing for the possibility that arbitrarily large Heesch numbers can be achieved using relatively simple shapes. The idea here is that we can take a nice, regular shape in $\mathbb{E}^n$ (like the cube in $\mathbb{E}^3$, for example), glue together several copies of it, and mark the faces with bumps and nicks in an unbalanced way. Then the geometry of the resulting tile is relatively tame, so we might be able to form many coronas, but not infinitely many. The more unit shapes that are fused to create the tile, the greater the upper bound on the Heesch number (based on the combinatorial imbalance alone) [**12**].

- We can consider other kinds of matching rules (other than bumps and nicks). For example, we can color the edges of the tile and require that matching edges have the same color. Or we can orient the edges and require that meeting edges have the same orientation. One can even give the edges purely combinatorial matching rules that aren't geometrically realizable. Hexagons with multiple matching rules have been studied to some extent. The author and a student have checked for their Heesch numbers all hexagons with edges marked with bumps and nicks, colors, and directions [**19**]. Our preliminary results find no tiles with Heesch number greater than three. Others have found high Heesch numbers for more general edge-matching rules.

- Heesch's tiling problem can be posed for protosets containing more than one prototile and for a starting configuration to be surrounded that comprises more than one tile. In [**8**, p. 157] the authors conjecture the existence of a function $f(n, m)$ taking positive integer values and having the following property: if $\mathcal{P}$ is a protoset of n prototiles such that every configuration containing m tiles can be surrounded at least $f(n, m)$ times, then $\mathcal{P}$ admits a tessellation of the plane. Note that in the case $n = m = 1$ this conjecture implies the existence of a maximal Heesch number (which implies the existence of a decision algorithm for the tiling problem for

monotiles). Whether or not this conjecture is true is still open; however, in light of the recent developments in finding tiles with higher and higher Heesch numbers, it might not be unreasonable to conjecture in exactly the opposite way (i.e., that such a function $f(n, m)$ does not exist).

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