Bubbles and Trading in Incomplete Markets *

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Abstract

We show that an intrinsic property of a large class of rational bubbles is their capacity to relax the agents' debt limits. Any bubble that preserves the set of pricing kernels, or equivalently, the asset span, has effectively an identical effect on consumption and real interest rates as an appropriate relaxation of debt limits, proportional to the size of the bubble. Thus the collapse of a bubble amounts to a contraction of agents' debt limits, and conversely, a bubble can arise to supplement the credit available in the economy.

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1 Introduction

Episodes of large stock market run-ups followed by abrupt crashes, without matching movements in fundamentals, are referred to as bubbles. Formally, a (rational) bubble is defined as the price of an asset in excess of its fundamental value, computed as the discounted (at market rates) present value of dividends.

We show that a large class of rational bubbles are equivalent, from the point of view of consumption and real interest rates, to a relaxation of agents' debt limits. An equilibrium (under some fixed credit limits) with bubbles in the prices of some assets allows agents the same level of consumption they would get in a no-bubble equilibrium of an alternative economy with more *relaxed* credit limits.

We build on the insight of Kocherlakota (2008), who showed that arbitrary discounted (by the pricing kernel) positive martingales can be introduced into asset prices as bubbles, while leaving agents' consumption and the pricing kernel unchanged, as long as the debt limits of the agents are allowed to be adjusted upwards (that is, tightened) by their initial endowment of the assets multiplied with the bubble term. In other words, although at the bubbly equilibrium agents are subject to tighter debt limit, they can still enjoy the same level of consumption they would under more relaxed debt limits (and no bubbles in the asset prices). In that sense, bubbles are equivalent to a relaxation of debt limits. The modified debt limits bind in exactly the same dates and states. Kocherlakota (2008) refers to this result as the "bubble equivalence theorem", and to this technique of introducing bubbles as "bubble injections".

At the heart of the argument is that the introduction of a bubble gives consumers a windfall, proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits. Conversely, the pricking of a bubble and the resulting drop in agents' wealth can be compensated by a relaxation of debt limits.

A major limitation of Kocherlakota's (2008) result is the assumption that agents can trade in a full set of state-contingent claims to consumption next period, in addition to the existing long-lived securities. Hence one might infer that the bubble equivalence theorem is associated to knife-edge situations, and that it might not apply to incomplete markets environments or even to economies with dynamically complete markets (rather than Arrow-Debreu complete).

We prove that a version of the bubble equivalence theorem holds even when markets are incomplete, or only dynamically complete. The equivalence has two parts. The *bubble injection* direction characterizes completely the set of processes that can be injected as bubbles in asset prices through a tightening of debt limits, while preserving the real variables. Such processes are called *pricing kernel-preserving*, or *kernel-preserving*, for short. The reverse direction, or the *bubble pricking* direction, shows that a large class of bubbles (those that are kernel-preserving) can be pricked and result in identical real variables, as long as agents' debt limits are relaxed.

The kernel-preserving processes, as the name suggests, are those nonnegative processes that result in an identical set of pricing kernels if added to (bubble-free) asset prices, or conversely, if subtracted from (bubbly) asset prices. Equivalently, they are discounted martingales (under some pricing kernel) that preserve the asset span (if added to bubble-free prices, or deducted from bubbly prices). A kernel-preserving process is also a martingale when discounted by any pricing kernel associated to the initial prices or any pricing kernel associated to the prices inflated by the process. In particular, any nonnegative process which equals the value of a self-financing trading strategy will generically be a kernel-preserving process.

Our results show that the setup of Kocherlakota (2008) with Arrow-complete markets and additional, redundant long-lived assets (rather than dynamically complete markets) is not innocuous. In his framework, the pricing kernel with or without a bubble (in the long-lived assets) is the same and is uniquely pinned down by the prices of the Arrow securities. With dynamically complete markets, the injection or the pricking of a bubble can distort the asset span and the pricing kernel, and not lead to an equivalent equilibrium.

The bubble equivalence theorem has additional appeal in environments with endogenous debt limits, as in Alvarez & Jermann (2000). In these models, agents have the option to default on debt and receive a predetermined continuation utility, and the markets (competitive financial intermediaries) select the largest debt limits so that repayment is always individually rational given future bounds on debt. It turns out that both the debt limits of the bubble-free equilibrium and the tighter debt limits of the equivalent bubbly equilibrium are the endogenous bounds allowing for maximal credit expansion and preventing default. We allow for more general punishments after default than in Kocherlakota (2008). In particular, we cover the case where upon default the agents are forbidden to carry debt (Bulow & Rogoff 1989, Hellwig & Lorenzoni 2009). Therefore the "incomplete markets" in the title of the paper refers to both environments with exogenously, respectively endogenously incomplete (due to limited enforcement) markets.

It is easy to misinterpret the injection direction of the bubble equivalence theorem as a "license" to create bubbles freely. However, bubbles in positive supply assets cannot exist in economies with *high interest rates*, that is with finite present value of aggregate consumption (Santos & Woodford 1997, Werner 2014, Kocherlakota 1992), as long as agents are not prevented from reducing their share holdings, that is if their debt limits are nonpositive. Intuitively, bubbles grow on average at the rate of interest rates. With high interest rates, the bubble must become very large relative to aggregate endowment, even if this happens with small probability. But this is incompatible with the presence of optimizing, forward looking agents, who do not allow their financial wealth to exceed the present value of their future consumption. As shown by Bidian (2011, Chapter 2) and Bidian (2014a), this argument against the existence of bubbles in economies with high interest rates is extremely robust and applies to environments with asymmetric information, heterogeneous beliefs and quite general portfolio constraints.

Therefore with *high* interest rates, the tighter debt bounds needed to sterilize the wealth effects of a bubble injection in an asset in positive supply must be positive at some dates and states (even though this may happen with arbitrarily small probability). However, if an asset is in zero supply, and initially none of the agents hold any shares, a bubble injection has no wealth or allocational effects. One can inject "freely" any (nonnegative) kernel-preserving process as a bubble into the price of that asset, while preserving real interest rates and agents' consumption and debt limits.

Low interest rates arise naturally with the enforcement limitations studied in Section 4, since in equilibrium the interest rates adjust to a lower level to entice agents to repay their debt and prevent default. Hellwig & Lorenzoni (2009) (see also Werner (2014)) show that if the penalty for default is an *interdiction to borrow*, then all non-autarchic equilibria must in fact have low interest rates, and bubble injections with nonpositive debt limits are possible. Bidian (2011, Chapter 4) and Bidian (2014b) show that low interest rates can arise in equilibrium and that bubble injections with nonpositive debt limits are possible for the other common penalties for default encountered in the literature: a *permanent* or a *temporary interdiction to trade* after default. All the mentioned examples of bubble injections with nonpositive debt limits feature complete markets.

The bubble pricking direction of Theorem 3.3 shows that the intrinsic feature of kernel-preserving bubbles is to relax financial constraints. Such bubbles must be *unambiguous*, in that they do not vanish if the present value of dividends (fundamental value) is calculated using any valid pricing kernel. It follows that, for general environments with incomplete markets, an unexpected collapse of a kernel-preserving bubble would not affect agents' consumption if their debt limits are relaxed by an amount proportional to the size of the bubble. In the absence of such an increase in the availability of credit, a bubble collapse amounts to a credit crunch, and therefore can be contractionary (see, for example, Guerrieri & Lorenzoni 2011).

Therefore kernel-preserving bubbles act as devices that relax agents' debt limits. A host of recent papers point out similarly, but in very specific environments, that bubbles can arise in the presence of financial frictions, and help relax the underlying borrowing constraints (Kocherlakota 2009, Martin & Ventura 2012, Giglio & Severo 2012, Farhi & Tirole 2012). These bubbles facilitate the transfer of resources from unproductive entrepreneurs to the productive ones, by increasing the borrowing capacity of the latter. Miao & Wang (2011) make a related point, but they emphasize the multiplicity of equilibria in economies with limited enforcement, studied also in Hellwig & Lorenzoni (2009) and Bidian (2014b). In their model, bubbles are defined as the difference between the value of the firm and the value predicted using the q theory of investment. These papers analyze the production sector, shutting down (non-entrepreneurs) consumers from borrowing and lending. By contrast, we intentionally focus squarely on the consumer sector, allowing consumers to borrow and lend to each other, in a Bewley-Aiyagari environment.

2 Model

Time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, which is an increasing sequence of finite partitions $\mathcal{F}_t \subset \mathcal{F}$ on the set of states of the world Ω with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We interpret $\mathcal{F}_t$ as the information available at period t.

Let X be the set of all stochastic processes adapted to $(\mathcal{F}_t)_{t=0}^{\infty}$,[1] and denote by X_+ (respectively X_{++}) the processes $x \in X$ such that $x_t \geq 0$ P-almost surely (respectively $x_t > 0$ P-almost surely) for all $t \in \mathbb{N}$. All statements, equalities, and inequalities involving random variables are assumed to hold only "P-almost surely", and we will omit adding this qualifier. When $K, L \in \mathbb{N} \setminus \{0\}$, let $X^{K\times L}$, respectively $X_+^{K\times L}$ be the set of vector (or matrix) processes $(y^{ij})_{1\leq i\leq K, 1\leq j\leq L}$ with $y^{ij} \in X$, respectively $y^{ij} \in X_+$.

There is a single consumption good and a finite number, I, of consumers. An agent $i \in \{1, 2, \ldots, I\}$ has preferences represented by a utility $U : X_+ \to \mathbb{R}$ given by $U^i(c) = E \sum_{t=0}^{\infty} u_t^i(c_t^i)$, where c_t^i is the consumption of i, and $E(\cdot)$ is the expectation operator with respect to probability P. The per-period utility $u_t^i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing. The conditional expectation given the information available at t, $\mathcal{F}_t$, is denoted by $E_t(\cdot)$. Since there is no information at period 0, $E_0(\cdot) = E(\cdot)$. The continuation utility of agent i at t provided by a consumption stream $c \in X_+$ is $U_t^i(c) := E_t \sum_{s\geq t} u_s^i(c_s)$.

There is a finite number J of infinitely lived, disposable securities, traded at every date. The dividend and price vector processes are $d = (d^1, \ldots, d^J) \in X_+^{1\times J}$ and $p = (p^1, \ldots, p^J) \in X_+^{1\times J}$.

Consumer i has an initial endowment $\theta_{-1}^i \in \mathbb{R}_+^J$ of securities and his trading strategy is represented by a process $\theta^i \in X^{J\times 1}$. Fix some debt bounds $\phi^i \in X$ for agent i and define the budget constraint and indirect utility of an agent i from period $s \geq 0$ onward, when faced with prices $p \in X_+^{1\times J}$, debt bounds $\phi^i \in X$ and having an

[1] This is the set of sequences $x = (x_t)_{t\in\mathbb{N}}$ of random variables $x_t : \Omega \to \mathbb{R}$ such that x_t is $\mathcal{F}_t$-measurable.

initial wealth $\nu_s : \Omega \to \mathbb{R}$ which is $\mathcal{F}_s$-measurable, as

$$\begin{aligned} B_s^i(\nu_s, \phi^i, p) = \{(c^i, \theta^i) \in X_+ \times X^{J\times 1} \mid c_s^i + p_s\theta_s^i = e_s^i + \nu_s, \\ c_t^i + p_t\theta_t^i = e_t^i + (p_t + d_t)\theta_{t-1}^i, (p_t + d_t)\theta_{t-1}^i \geq \phi_t^i, \forall t > s\}, \end{aligned} \quad (2.1)$$

$$V_s^i(\nu_s, \phi^i, p) = \max_{(c^i,\theta^i)\in B_s^i(\nu_s,\phi^i,p)} U_s^i(c^i). \quad (2.2)$$

Denote by $B_s^{i,c}(\nu_s, \phi^i, p)$ the projection of $B_s^i(\nu_s, \phi^i, p)$ on consumption paths:

$$B_s^{i,c}(\nu_s, \phi^i, p) := \{c^i \in X_+ \mid \exists \theta^i \in X^{J\times 1} \text{ such that } (c, \theta^i) \in B_s^i(\nu_s, \phi^i, p)\}. \quad (2.3)$$

Definition 2.1. A vector $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ consisting of a security price process $p \in X_+^{1\times J}$, and for each agent $i \in \{1, \ldots, I\}$, consumption $c^i \in X_+$ and a trading strategy (portfolios) $\theta^i \in X^{J\times 1}$ is an equilibrium given (exogenous debt limits) $(\phi^i)_{i=1}^I$ (with $\phi^i \in X$) if the following conditions are met:

i. Consumption and portfolios of each agent i are feasible and optimal: $(c^i, \theta^i) \in B_0^i((p_0 + d_0)\theta_{-1}^i, \phi^i, p)$ and $U^i(c^i) = V_0^i\left((p_0 + d_0)\theta_{-1}^i, \phi^i, p\right)$.

ii. Markets clear: $\sum_{i=1}^I c_t^i = \sum_{i=1}^I e_t^i + d_t \cdot \sum_{i=1}^I \theta_{-1}^i, \quad \sum_{i=1}^I \theta_t^i = \sum_{i=1}^I \theta_{-1}^i, \ \forall t \in \mathbb{N}$.

Consider an equilibrium $\left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ given $(\phi^i)_{i=1}^I$. Since the utilities of the agents are strictly increasing in consumption at each date and state, prices p exclude arbitrage opportunities. Thus there cannot exist $\theta \in X^{J\times 1}$ and $t \in \mathbb{N}$ such that $p_t\theta_t \leq 0$ and $(p_{t+1} + d_{t+1})\theta_t \geq 0$, with at least one inequality being strict on a set of positive probability. Otherwise consumer i would alter his portfolio θ_t^i at t by adding to it the strategy θ_t, guaranteeing an increase in his consumption at t and $t + 1$, and a strict increase in one of the periods, with positive probability. This modified strategy still satisfies the debt constraints. The absence of arbitrage opportunities is equivalent to the existence of a process $a \in X_{++}$ such that (Santos & Woodford 1997)

$$p_t = E_t \frac{a_{t+1}}{a_t} \left(p_{t+1} + d_{t+1}\right), \forall t \geq 0. \quad (2.4)$$

We denote by $A(p)$ the set of all processes $a \in X_{++}$ satisfying equation (2.4), and we refer to them as *pricing kernels*. Equation (2.4) implies that $p_t = \frac{1}{a_t} E_t \sum_{s>t} a_s d_s +$

$\lim_{T\to\infty} \frac{1}{a_t} E_t a_T p_T$, and

$$b_t(a,p) := \frac{1}{a_t} \lim_{T\to\infty} E_t a_T p_T \tag{2.5}$$

is well defined and nonnegative, and for all $t \in \mathbb{N}$,

$$a_t b_t(a,p) = E_t a_{t+1} b_{t+1}(a,p). \tag{2.6}$$

Therefore $a \cdot b(a,p)$ is a nonnegative martingale, and $b(a,p) = 0$ if and only if $b_0(a,p) = \frac{1}{a_0} \lim_{t\to\infty} E a_t p_t = 0$. We interpret the discounted present value of dividends d under the state price density a, that is $f_t(a) := \frac{1}{a_t} E_t \sum_{s>t} a_s d_s$, as the *fundamental value* of d. Hence $b(a,p)$ represents the part of asset prices in excess of fundamental values, and will be called a *bubble* whenever $b_0(a,p) \neq 0$. Following Santos & Woodford (1997), the bubble $b(a,p)$ is *ambiguous* if $b_0(a',p) = 0$ for some $a' \in A(p)$ (while $b_0(a,p) \neq 0$). Similarly, the bubble $b(a,p)$ is *unambiguous* if $b_0(a',p) \neq 0$, for all $a' \in A(p)$. Thus a bubble is ambiguous (unambiguous) if it vanishes (does not vanish) under a different choice of a pricing kernel.

Kocherlakota (2008) assumed that in addition to trading in long-lived securities, agents can also trade in each period a full set of state-contingent claims to consumption next period. Of course this means that there is a unique pricing kernel, pegged down by the price of the Arrow securities. Hence $A(p) = \{a\}$. Given an equilibrium without bubbles in which the asset prices are p and the pricing kernel is a, and given an arbitrary process $\varepsilon \in X_+^{1\times J}$ such that $a \cdot \varepsilon$ is a martingale, he showed that there exists an equilibrium with prices $p + \varepsilon$ and with debt limits tightened in proportion to the size of ε and agents' initial endowment of assets (that is, agent i's debt limits are tightened by $\varepsilon\theta^i_{-1}$). This equilibrium is *equivalent* to the initial one, in the sense that it has *identical consumption* paths for the agents and an *identical pricing kernel* ($A(p) = A(p+\varepsilon) = \{a\}$). He dubbed this result the *"bubble equivalence theorem"*, since the process ε *injected* in the asset prices is the bubble component for the new prices $p + \varepsilon$, that is $\varepsilon = b(a, p+\varepsilon)$.

3 Bubble equivalence theorem

We extend Kocherlakota's (2008) bubble equivalence theorem to incomplete markets, or to dynamically complete (rather than Arrow-complete) markets. With incomplete markets, pricing kernels are not unique, and by not being pegged down by the price of Arrow securities, they can be distorted by the presence of a bubble. Therefore it is unclear what type of processes ε can be added to asset prices as bubbles and result in an equivalent equilibrium.

Candidate processes ε that could lead to equivalent equilibria if injected in asset prices must preserve the set of pricing kernels $A(p) = A(p+\varepsilon)$ (in addition to agents' consumption). It turns out that this is also a sufficient condition. We denote by $M(p)$ the set of all such *kernel-preserving* (or *pricing kernel preserving*) processes:

$$M(p) := \{\varepsilon \in X_+^{1\times J} \mid A(p) = A(p+\varepsilon)\}. \tag{3.1}$$

The following several equivalent characterizations of kernel-preserving processes will prove useful. The proof is given in Appendix A.

Proposition 3.1. *Fix $\varepsilon, p \in X_+^{1\times J}$. The following are equivalent:*

(i) $\varepsilon \in M(p)$.

(ii) $a \cdot \varepsilon$ *is a martingale, for any* $a \in A(p) \cup A(p+\varepsilon)$.

(iii) $\mathcal{S}_t(p) = \mathcal{S}_t(p+\varepsilon)$, *for all* $t \in \mathbb{N}$, *and* $a \cdot \varepsilon$ *is a martingale, for some* $a \in A(p) \cup A(p+\varepsilon)$, *where* $\mathcal{S}_t(p)$ *is the period* t asset span *given* p, *defined as the set of attainable payoffs at* t *under* p:

$$\mathcal{S}_t(p) := \{(p_t + d_t)\lambda \mid \lambda : \Omega \to \mathbb{R}^J \text{ and } \lambda \text{ is } \mathcal{F}_{t-1} - \text{measurable}\}. \tag{3.2}$$

(iv) ε *has the property that there exists* $\Lambda, \Gamma \in X^{J\times J}$ *such that for all* $t \geq 0$,

$$\begin{aligned} &\varepsilon_t = p_t\Lambda_t = \hat{p}_t\Gamma_t, \\ &(p_{t+1} + d_{t+1})\Lambda_t - p_{t+1}\Lambda_{t+1} = (\hat{p}_{t+1} + d_{t+1})\Gamma_t - \hat{p}_{t+1}\Gamma_{t+1} = 0, \end{aligned} \tag{3.3}$$

where $\hat{p} := p + \varepsilon$.

The equivalence $(i) \Leftrightarrow (ii)$ shows that a kernel-preserving process ε injected in (bubble-free) asset prices p will be an unambiguous bubble in the new equilibrium with prices $p + \varepsilon$.

The equivalence $(i) \Leftrightarrow (iii)$ plays a crucial technical role in establishing Theorem 3.3. It guarantees that for any feasible consumption at prices p, one can construct portfolios supporting that consumption stream at prices $p + \varepsilon$ (and conversely). As we show in Lemma A.1, if ε is a discounted martingale under all pricing kernels in $A(p)$, then each component of ε is also in the asset span for prices p, in that $\varepsilon_t^j \in S_t(p)$ (for all j), and moreover, if ε is added to prices p, it does not *enlarge* the asset span: $S_t(p + \varepsilon) \subset S_t(p)$. The fact that ε is also a discounted martingale under any pricing kernel in $A(p + \varepsilon)$ guarantees that each component of ε is also in the asset span for prices $p + \varepsilon$, thus $\varepsilon_t^j \in S_t(p + \varepsilon)$, and that ε does not lead to a *drop in rank* (does not *reduce* the asset span): $S_t(p) \subset S_t(p + \varepsilon)$.

The equivalence $(i) \Leftrightarrow (iv)$ shows that a kernel-preserving process represents also the value of two *self-financing strategies*, one at prices p, and the other at prices $p+\varepsilon$. This property can be used to construct processes in $M(p)$ (and to conclude that $M(p)$ is a large set) as follows. Start with an arbitrary $\Lambda_0 \geq 0$ with $\det(\mathbf{I}+\Lambda_0) \neq 0$, where $\mathbf{I}$ denotes the J-dimensional identity matrix, and $\det(\cdot)$ is the determinant of a matrix. Non-singularity of $\mathbf{I} + \Lambda_0$ is a mild condition (generically satisfied), equivalent with requiring that -1 is not an eigenvalue of Λ_0. Construct then iteratively, for each $t \geq 0$, $\Lambda_{t+1} \geq 0$ such that $p_{t+1}\Lambda_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t$ and $\det(\mathbf{I} + \Lambda_{t+1}) \neq 0$. Such Λ_{t+1} always exists. If ε is taken to be the value of the resulting self-financing strategy $\Lambda \in X_+^{J\times J}$ (that is, $\varepsilon_t := p_t\Lambda_t \geq 0$ for all $t \geq 0$), then ε satisfies condition (iv) of Proposition 3.1 (just take $\Gamma_t = (\mathbf{I} + \Lambda_t)^{-1}\Lambda_t, \forall t \geq 0$), and thus $\varepsilon \in M(p)$.[2]

In order to establish the bubble equivalence theorem (Theorem 3.3), we prove first that an agent's feasible consumption paths are identical at prices $p, \hat{p}$ that differ by a kernel-preserving process ε $(= \hat{p} - p)$, if his debt limits and initial wealth differ by an appropriate multiple of ε.

Proposition 3.2. *Fix a date $t \geq 0$ and an agent i. Let $p \in X_+^{1\times J}$, $\phi^i \in X$ and*

[2] It is particularly simple to construct processes ε with only one non-zero component, say the first (thus a bubble is injected only in the first asset). A self-financing strategy generating such ε must be of the form $\Lambda_t = (\Lambda_t^1, 0, \ldots, 0)$, and $\det(\mathbf{I} + \Lambda_t) \neq 0$ whenever $\Lambda_t^{11} \neq -1$, as $(\mathbf{I} + \Lambda_t)^{-1} = \mathbf{I} - \Lambda_t/(1 + \Lambda_t^{11})$.

$\varepsilon \in M(p)$. Let $\nu_t : \Omega \to \mathbb{R}$ and $\bar{\theta} : \Omega \to \mathbb{R}^J$, assumed $\mathcal{F}_t$-measurable. Set $\hat{p} := p + \varepsilon$, $\hat{\phi}^i := \phi^i + \varepsilon\bar{\theta}$, $\hat{\nu}_t := \nu_t + \varepsilon_t\bar{\theta}$. Then

(i) If $(c^i, \theta^i) \in B_t^i(\nu_t, \phi^i, p)$, there exists $\hat{\theta}^i \in X^{J\times 1}$ such that $(c^i, \hat{\theta}^i) \in B_t^i\left(\hat{\nu}_t, \hat{\phi}^i, \hat{p}\right)$.

(ii) If $(c^i, \hat{\theta}^i) \in B_t^i\left(\hat{\nu}_t, \hat{\phi}^i, \hat{p}\right)$, there exists $\theta^i \in X^{J\times 1}$ such that $(c^i, \theta^i) \in B_t^i(\nu_t, \phi^i, p)$.

Proof. (i) Let $(c^i, \theta^i) \in B_t^i(\nu_t, \phi^i, p)$. By Proposition 3.1 $(i) \Leftrightarrow (iii)$, ε preserves the asset span when added to p. Therefore there exists $\hat{\theta}^i$ such that for all $s > t$,

$$(p_s + d_s)(\theta^i_{s-1} - \bar{\theta}) = (\hat{p}_s + d_s)(\hat{\theta}^i_{s-1} - \bar{\theta}). \tag{3.4}$$

Relation (3.4) is enough to guarantee that if $(c^i, \theta^i) \in B_t^i(\nu_t, \phi^i, p)$, then $(c^i, \hat{\theta}^i) \in B_t^i\left(\hat{\nu}_t, \hat{\phi}^i, \hat{p}\right)$ (and conversely). By (3.4), for each $a \in A(p) = A(\hat{p})$ and $s \geq t$,

$$\hat{p}_s(\hat{\theta}^i_s - \bar{\theta}) = E_s\frac{a_{s+1}}{a_s}(\hat{p}_{s+1} + d_{s+1})(\hat{\theta}^i_s - \bar{\theta}) = E_s\frac{a_{s+1}}{a_s}(p_{s+1} + d_{s+1})(\theta^i_s - \bar{\theta}) = p_s(\theta^i_s - \bar{\theta}).$$

It follows that for $s \geq t+1$,

$$\begin{aligned}(\hat{p}_s + d_s)\hat{\theta}^i_{s-1} - \hat{p}_s\hat{\theta}^i_s &= (p_s + d_s)\theta^i_{s-1} - p_s\theta^i_s \quad (= c^i_s - e^i_s),\\ (\hat{p}_s + d_s)\hat{\theta}^i_{s-1} &= (p_s + d_s)\theta^i_{s-1} + \varepsilon_s\bar{\theta} \geq \phi^i_s + \varepsilon_s\bar{\theta} = \hat{\phi}^i_s,\end{aligned}$$

Moreover,

$$\hat{\nu}_t - \hat{p}_t\hat{\theta}^i_t = \nu_t + \varepsilon_t\bar{\theta} - (p_t + \varepsilon_t)\hat{\theta}^i_t = \nu_t - p_t\theta^i_t \quad (= c^i_t - e^i_t),$$

and we conclude that $(c^i, \hat{\theta}^i) \in B_t^i\left(\hat{\nu}_t, \hat{\phi}^i, \hat{p}\right)$.

(ii) The statement follows in an identical way, switching in all the formulas the roles of p and $\hat{p}$, θ^i and $\hat{\theta}^i$, ϕ^i and $\hat{\phi}^i$, ν_t and $\hat{\nu}_t$, and changing the sign in front of ε throughout. □

We can now state and prove the bubble equivalence theorem for incomplete markets.

Theorem 3.3. *Let $p, \hat{p}, \varepsilon \in X_+^{1\times J}$ such that $\hat{p} := p + \varepsilon$ and for all $i \in \{1, \ldots, I\}$, let $\phi^i, \hat{\phi}^i \in X$ such that $\hat{\phi}^i = \phi^i + \varepsilon\theta^i_{-1}$.*

(i) Let $\mathcal{E} := \left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ be a bubble-free equilibrium given $(\phi^i)_{i=1}^I$. If $\varepsilon \in M(p)$, there exist portfolios $(\hat{\theta}^i)_{i=1}^I$ such that $\hat{\mathcal{E}} := \left(\hat{p}, (c^i)_{i=1}^I, (\hat{\theta}^i)_{i=1}^I\right)$ is an equilibrium given $(\hat{\phi}^i)_{i=1}^I$, having an unambiguous bubble ε, since $\varepsilon = b(a, \hat{p})$, for all $a \in A(\hat{p})$.

(ii) Let $\hat{\mathcal{E}} := \left(\hat{p}, (c^i)_{i=1}^I, (\hat{\theta}^i)_{i=1}^I\right)$ be an equilibrium given $(\hat{\phi}^i)_{i=1}^I$, having a bubble $\varepsilon := b(a, p)$ under some $a \in A(\hat{p})$. If $\varepsilon \in M(p)$, there exist portfolios $(\theta^i)_{i=1}^I$ such that $\mathcal{E} := \left(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ is a bubble-free equilibrium given $(\phi^i)_{i=1}^I$.

Proof. (i) By setting $t := 0$, $\bar{\theta} := \theta_{-1}^i$ (initial holdings of i) and $\nu_t := (p_0 + d_0)\theta_{-1}^i$ (initial wealth of i) in Proposition 3.2, it follows that

$$B_0^{i,c}((p_0 + d_0)\theta_{-1}^i, \phi^i, p) = B_0^{i,c}((\hat{p}_0 + d_0)\theta_{-1}^i, \phi^i + \varepsilon\theta_{-1}^i, \hat{p}), \text{ where } \hat{p} = p + \varepsilon. \tag{3.5}$$

Hence c^i is optimal in $B_0^{i,c}((\hat{p}_0 + d_0)\theta_{-1}^i, \hat{\phi}, \hat{p})$. We still need to show that the portfolios $(\hat{\theta}^i)_{i=1}^I$ can be chosen so that they satisfy the market clearing condition $\sum_i \hat{\theta}^i = \sum_i \theta_{-1}^i$.

By Proposition 3.2 (i), for $i \in \{1, \ldots, I-1\}$, there exists $\hat{\theta}^i$ satisfying (3.4) with $\bar{\theta} := \theta_{-1}^i$, and such that $(c^i, \hat{\theta}^i) \in B_0^i((\hat{p}_0 + d_0)\theta_{-1}^i, \hat{\phi}, \hat{p})$. Let $\hat{\theta}^I := \sum_{i=1}^I \theta_{-1}^i - \sum_{i=1}^{I-1} \hat{\theta}^i$. Therefore $\hat{\theta}^I$ satisfies (3.4) (for $i = I$ and $\bar{\theta} = \theta_{-1}^I$). This is enough to guarantee that $(c^I, \hat{\theta}^I) \in B_0^I((\hat{p}_0 + d_0)\theta_{-1}^I, \hat{\phi}, \hat{p})$, by repeating the reasoning in the proof of Proposition 3.2 (i). Asset markets clear, as $\sum_{i=1}^I \theta_{-1}^i = \sum_{i=1}^I \hat{\theta}^i$.

Finally, for each $a \in A(\hat{p}) = A(p)$,

$$b_t(a, \hat{p}) = \lim_{s \to \infty} E_t \frac{a_s}{a_t} \hat{p}_s = \lim_{s \to \infty} E_t \frac{a_s}{a_t}(p_s + \varepsilon_s) = \varepsilon_t + \lim_{s \to \infty} E_t \frac{a_s}{a_t} p_s = \varepsilon_t.$$

(ii) Can be established in an identical manner as above, relying again on Proposition 3.2. □

Theorem 3.3 (i) represents the *bubble injection* part of the equivalence result. It provides a way to construct a bubbly equilibrium starting from a bubble-free equilibrium. When a bubble is injected in an asset in *zero supply*, and when agents' initial holdings of that asset are zero, the debt limits of the equivalent bubbly equilibrium are identical to those of the bubble-free equilibrium. Indeed, a bubble on such an asset produces no wealth effects, and therefore the bubble equivalence theorem provides an easy method to inject bubbles in such assets. The indeterminacy of equilibria (caused by bubbles) with zero-supply assets was uncovered by Kocherlakota (1992).[3]

However, bubble injections in *positive supply* assets create a wealth effect, and the tighter debt limits $\hat{\phi}^i$ needed to induce agents to save (rather than consume) this initial windfall may become positive at some dates and states. This always happens in equilibria with *high-interest rates* (making the present value of consumption finite), and it is a consequence of the non-existence of bubbles results of Santos & Woodford (1997) (for borrowing constraints) and Bidian (2014a) (for general portfolio constraints). Their results rule out unambiguous bubbles (which are the only type of bubbles involved in the bubble equivalence Theorem 3.3) without imposing the strong impatience assumption on agents used in Werner (2014).

On the other hand, with low interest rates, sufficiently small bubble injections (in positive supply assets) might be sustained with nonpositive debt limits. In economies with limited enforcement of debt contracts, low interest rates arise to induce repayment of debt, and bubble injections with nonpositive debt limits are indeed possible. This type of economies is introduced in the next section, and therefore we offer a detailed discussion of the known examples where equilibrium debt limits of bubble-free equilibria have martingale components, which can be converted into bubbles in asset prices using the injection mechanism. All those examples feature complete markets.

Theorem 3.3 (*ii*) represents the *bubble pricking* part of the equivalence result. Kernel-preserving bubbles are effectively equivalent to a *relaxation* of agents' debt limits, proportional with the size of the bubble and agents' initial endowments of assets. If those endowments are nonnegative, debt limits are relaxed when the bubble is pricked and the issue of nonpositivity of debt limits does not have a bite.

We argue, next, that the hypotheses of Theorem 3.3 cannot be relaxed further. Theorem 2 in Werner (2014) seems to suggest that processes that are a discounted martingale under one pricing kernel can be injected in asset prices, as long as they preserve the asset span. However, the equivalence (*ii*) $\Leftrightarrow$ (*iii*) in our Proposition 3.1

[3]He showed also that bubbles on zero supply assets can also have allocational effects if agents initial endowments of the asset are non-zero.

shows that such processes must be discounted martingales under *all* pricing kernels, and therefore they will result in unambiguous bubbles. One cannot dispense with the requirement that $\varepsilon \in M(p)$ (that ε is kernel-preserving, or alternatively, that it preserves the asset span), even if only the equivalence of consumption paths is desired. To illustrate that, consider a deterministic Bewley economy (Kocherlakota 1992, Huang & Werner 2000) where fiat money is the only asset. There exist two types of equilibria in this economy: equilibria with valued money (with bubbles), in which financial markets are (dynamically) complete and thus money enables trade (and risk-sharing) among agents, and an autarchic equilibrium with unvalued money (and incomplete markets). Clearly, the two types of equilibria have different consumption paths. The bubble, whether injected or pricked, alters the asset span. According to the equivalence $(i) \Leftrightarrow (iii)$ of Proposition 3.1, this bubble must alter the pricing kernel as well. Indeed, since markets are complete with valued money, there is a unique pricing kernel (and thus the bubble is unambiguous). However, at zero prices for money, there is a continuum of pricing kernels (any positive sequence is a valid kernel) and the bubble is a martingale only when discounted by exactly one of them.

The discussion shows that the setup of Kocherlakota (2008) with Arrow-complete markets and additional, redundant long-lived assets (rather than dynamically complete markets) is not innocuous. In his framework, the pricing kernel with or without a bubble (in the long-lived assets) is the same and is uniquely pinned down by the prices of the Arrow securities. Therefore the conditions of Theorem 3.3 are satisfied and a bubble injection or the pricking of a bubble always results in an equivalent equilibrium if debt limits are appropriately adjusted.

The pricking of the bubble in the Bewley example discussed above distorted the asset span, pricing kernel and consumption, but nevertheless, led to an equilibrium (autarchy). In general, pricking a bubble that is not kernel-preserving may not even result in an equilibrium. This is illustrated by the Example 4.5 in Santos & Woodford (1997). The example constructs a bubbly equilibrium in an economy with incomplete markets. The bubble is ambiguous in that it is non-zero when dividends are discounted by the risk-free rates (which is a valid pricing kernel in that economy), but vanishes if fundamental values are calculated using the representative agent's marginal utility (which is also a pricing kernel). However, that bubble cannot be pricked, as the fundamental value of the asset (calculated using the risk-free rates) violates the representative agent's Euler equations and thus it cannot be an equilibrium price.

Theorem 3.3 leaves unspecified the relation between the portfolios $\theta^i, \hat{\theta}^i$ in the equivalent equilibria $\mathcal{E}, \hat{\mathcal{E}}$. In general, these portfolios are not unique. However, they are unique if assets are *non-redundant* at prices p. Formally, assets are *non-redundant* at prices p if for all $t \geq 0$, there is no $\mathcal{F}_t$-measurable $\lambda : \Omega \to \mathbb{R}^J$ such that $\lambda \neq 0$ and $(p_{t+1} + d_{t+1})\lambda = 0$. With non-redundant assets, the dependence between $\theta^i, \hat{\theta}^i$ can be characterized explicitly as follows:

$$\hat{\theta}^i_t = (\mathbf{I} + \Lambda_t)^{-1} \left(\theta^i_t + \Lambda_t \theta^i_{-1}\right), \forall t \geq 0, \tag{3.6}$$

where $\Lambda \in X_+^{J \times J}$ is the self-financing strategy that generates ε. The existence of such strategy is guaranteed by Proposition 3.1. Its uniqueness and the non-singularity of $\mathbf{I} + \Lambda_t$ (for all $t \geq 0$) are consequences of having non-redundant assets, and are proved in Proposition A.2 in Appendix A.

Equation (3.6) makes possible an analysis of the portfolio effects of bubbles. Even though a (kernel-preserving) bubble is equivalent, from the point of view of consumption and real interest rates, to a relaxation of debt limits, such a bubble affects portfolios and trading volumes. Bejan & Bidian (2012) and Bidian (2014c) use this to show that bubbles can cause large increases in trading volume compared to equivalent bubble-free equilibria.

4 Endogenous debt limits

We allow here for the *endogenous* determination of debt constraints driven by limited commitment/imperfect enforcement as in Alvarez & Jermann (2000).

Assume that at any period t, when facing prices p (and dividends d), consumer i can choose to default on his beginning of period debt and leave the economy, receiving a continuation utility after default $\widetilde{V}^i_t(p)$ ($\mathcal{F}_t$-measurable). We allow this continuation utility to depend on exogenous variables such as endowments and dividends, but we make explicit only the functional dependence on prices, which are endogenous. Thus the *default penalty* for each agent i is described by a mapping $\widetilde{V}^i : X_+^{1 \times J} \to X$.

Alvarez & Jermann (2000), following Kehoe & Levine (1993), worked under the assumption that agents are banned from trading following default, hence for each agent i,

$$\widetilde{V}_t^i(p) := U_t^i(e^i). \tag{4.1}$$

Alternatively, Hellwig & Lorenzoni (2009), building on the work of Bulow & Rogoff (1989), assume that agents can continue to lend but not to borrow following default,

$$\widetilde{V}_t^i(p) := V_t^i(0, 0, p), \tag{4.2}$$

where the second argument in $V_t^i(0, 0, p)$ is the process in X identically equal to zero. As in Alvarez & Jermann (2000), the option to default endogenizes the debt limits to the maximum level so that repayment is always individually rational given future debt limits. This leads to the notion of debt limits that are *not-too-tight*.

Definition 4.1. Debt limits ϕ^i faced by agent i are *not-too-tight (NTT)* given prices p and penalties $\widetilde{V}^i : X_+^{1\times J} \to X$ if $V_t^i(\phi_t^i, \phi^i, p) = \widetilde{V}_t^i(p), \forall t$.

The definition captures the idea that the bounds ϕ^i have to be "tight enough" to prevent default, that is, they have to be "self-enforcing" ($V_t^i(\phi_t^i, \phi^i, p) \geq \widetilde{V}_t^i(p)$), but they should allow for maximum credit expansion (thus one should not have $V_t^i(\phi_t^i, \phi^i, p) > \widetilde{V}_t^i(p)$ on a positive probability set). One can envision the NTT debt limits as being set by competitive financial intermediaries, with agents unable to trade directly with each other. The intermediaries set debt limits such that default is prevented, but credit is not restricted unnecessarily, since competing intermediaries could relax them and increase their profits.

We extend our definition of equilibrium to allow for the endogenous determination of debt limits, in the presence of an outside option to default. A vector $\big(p, (\phi^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\big)$ is an *equilibrium with endogenous debt limits* given penalties $(\widetilde{V}^i)_{i=1}^I$ if $\big(p, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\big)$ is an equilibrium given $(\phi^i)_{i=1}^I$, and debt limits ϕ^i are NTT given $\widetilde{V}^i(p)$, for all $i \in \{1, \dots, I\}$.

Existence of equilibria with endogenous debt limits in this general environment is difficult to establish. This is due to the presence of incomplete markets, real (long-lived) securities and infinite horizon, all of which create existence problems even for the equilibria with exogenous debt limits of Definition 2.1. When markets are complete and the punishment for default is given by (4.1), the existence is established by Kehoe & Levine (1993) and Alvarez & Jermann (2000). With incomplete markets, Hernandez & Santos (1996) show that in our environment, an equilibrium with exogenous debt limits exists for a dense subset of endowment and dividend processes, if agents are impatient, have a nonnegative initial holding of securities, and if their debt is restricted by the present value of future endowments,

$$\phi_t^i = -\inf_{a \in A(p)} E_t \sum_{s \geq t} \frac{a_s}{a_t} e_s^i. \tag{4.3}$$

The debt limits in (4.3) are chosen equal to the maximum amount that an agent can borrow, if he must hold nonnegative wealth after some finite date. With complete markets, they are the NTT debt limits when the punishment for default is the confiscation of endowment.

We show next that a bubble injection ε as in Theorem 3.3 (i) preserves the NTT property of the debt limits as long as the penalties for default satisfy $\widetilde{V}^i(p+\varepsilon) = \widetilde{V}^i(p)$. The extension to endogenous debt limits of Part (ii) of Theorem 3.3 (the bubble pricking direction) is true under the same assumption on penalties for default, and is omitted for brevity.

Theorem 4.1. *Let $\left(p, (\phi^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ be an equilibrium with endogenous debt limits given $(\widetilde{V}^i)_{i=1}^I$. Choose $\varepsilon \in M(p)$ and let $\hat{p} := p + \varepsilon$, $\hat{\phi}^i := \phi^i + \varepsilon\theta_{-1}^i$ for all $i \in \{1, \ldots, I\}$. If $\widetilde{V}^i(p+\varepsilon) = \widetilde{V}^i(p)$ for all $i \in \{1, \ldots, I\}$, then there exist portfolios $(\hat{\theta}^i)_{i=1}^I$ such that $\left(\hat{p}, (\hat{\phi}^i)_{i=1}^I, (c^i)_{i=1}^I, (\hat{\theta}^i)_{i=1}^I,\right)$ is an equilibrium with endogenous debt limits given $(\widetilde{V}^i)_{i=1}^I$.*

Proof. Setting $\bar{\theta} := \theta_{-1}^i$ and $\nu_t := \phi_t^i$ in Proposition 3.2, it follows that $B_t^{i,c}(\phi_t^i, \phi^i, p) = B_t^{i,c}(\phi_t^i + \varepsilon_t\theta_{-1}^i, \phi^i + \varepsilon\theta_{-1}^i, p + \varepsilon)$, and therefore

$$V_t^i(\phi_t^i, \phi^i, p) = V_t^i(\phi_t^i + \varepsilon_t\theta_{-1}^i, \phi^i + \varepsilon\theta_{-1}^i, p + \varepsilon). \tag{4.4}$$

By (4.4), $\hat{\phi}^i$ are not-too-tight at prices $\hat{p}$, since

$$\widetilde{V}_t^i(\hat{p}) = \widetilde{V}_t^i(p) = V_t^i(\phi_t^i, \phi^i, p) = V_t^i(\phi_t^i + \varepsilon_t\theta_{-1}^i, \phi^i + \varepsilon\theta_{-1}^i, p + \varepsilon) = V_t^i(\hat{\phi}_t^i, \hat{\phi}^i, \hat{p}).$$

Theorem 3.3 (i) shows that there exist portfolios $(\hat{\theta}^i)_{i=1}^I$ such that $\left(\hat{p}, (c^i)_{i=1}^I, (\hat{\theta}^i)_{i=1}^I,\right)$ is an equilibrium given $(\hat{\phi}^i)_{i=1}^I$, and therefore the conclusion follows. $\square$

The requirement $\widetilde{V}^i(p+\varepsilon) = \widetilde{V}^i(p)$ that default penalties are not affected by a bubble injection is necessary and sufficient to ensure that the equivalence result in Theorem 3.3 extends to equilibria with endogenous debt limits. Indeed, agents' continuation utilities when starting with maximal (binding) amounts of debt are identical in the bubbly and bubble-free equilibrium, and therefore the penalties for default with and without a bubble have to coincide, by the definition of NTT debt limits. This condition holds when the continuation utilities after default are of the form (4.1), or more generally when $\widetilde{V}^i$ does not depend on prices. These are the only types of penalties considered in Kocherlakota (2008). It holds also for the interdiction to borrow after default (4.2). In fact, it holds for a more general class of penalties, where after default, an agent i is subjected to some exogenous debt limits $\widetilde{\phi}^i$ (equal to zero for an interdiction to borrow). Indeed, by setting $\bar{\theta} := 0$ and $\nu_t := \phi_t^i$ in Proposition 3.2, it follows that $B_t^{i,c}(\phi_t^i, \phi^i, p) = B_t^{i,c}(\phi_t^i, \phi^i, \hat{p})$, and therefore

$$V_t^i(\phi_t^i, \phi^i, p) = V_t^i(\phi_t^i, \phi^i, p+\varepsilon). \tag{4.5}$$

As a consequence of (4.5),

$$\widetilde{V}_t^i(p+\varepsilon) = V_t^i(\widetilde{\phi}_t^i, \widetilde{\phi}^i, p+\varepsilon) = V_t^i(\widetilde{\phi}_t^i, \widetilde{\phi}^i, p) = \widetilde{V}_t^i(p). \tag{4.6}$$

The bubble injection ε can only result in nonpositive debt limits $\hat{\phi}^i$ if the initial debt limits ϕ^i have (discounted) martingale components in them, as one needs to have $\phi^i \leq -\varepsilon\theta_{-1}^i$. Such an example is provided in Hellwig & Lorenzoni (2009) (see also Werner (2014)), for the interdiction to borrow (4.2). Bidian & Bejan (2012) show that debt limits have martingale components also under the class of penalties discussed above which generalize the interdiction to borrow. Thus, after default, agents are punished by being subjected to some fixed penalty debt limits $(\widetilde{\phi}_i)_{i=1}^I$, not necessarily zero (but potentially arbitrarily close to zero). Bidian (2011, Chapter 4) and Bidian (2014b) show that bubble injections (with nonpositive debt limits) are also possible for the interdiction to trade (4.1). That is also the case for a temporary (rather than permanent) interdiction to trade upon default, where the duration of exclusion can be deterministic or stochastic (Bidian 2014b). Interestingly, with a permanent or temporary interdiction to trade, only some of the non-autarchic equilibria (rather than all) allow for bubble injections with nonpositive debt limits. All the examples mentioned feature complete markets.

5 Conclusion

We consider the (large) class of nonnegative processes that preserve the set of pricing kernels when added to asset prices ("kernel-preserving" processes). Any such process can be injected as a rational bubble in asset prices, leading to an equilibrium with identical allocations and pricing kernels, but with debt limits tightened proportionally to the size of the bubble. Moreover, with enforcement limitations, if the debt bounds are endogenized as in Alvarez & Jermann (2000) to prevent default but to allow for maximal credit expansion, the modified debt limits in the equilibrium with bubbles still arise endogenously from the existing enforcement limitations.

Conversely, a kernel-preserving bubble is equivalent from the point of view of consumption and real interest rates to a relaxation of debt limits. Nevertheless, such a bubble is non-neutral from the point of view of returns and trading volumes. Bejan & Bidian (2012) and Bidian (2014c) show that a bubble can cause large increases in trading volume compared to the equivalent bubble-free equilibrium. High trading volumes typically accompany bubble episodes (Cochrane 2002).

A Kernel-preserving processes

We characterize the set $M(p)$ of kernel-preserving processes.

Lemma A.1. *Let $p \in X_+^{1\times J}$ such that $A(p) \neq \emptyset$. Let $\varepsilon \in X^{1\times J}$ such that $p + \varepsilon \in X_+^{1\times J}$. The following are equivalent:*

(i) There exists $\Lambda \in X^{J\times J}$ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for all $t \geq 1$ and there exists $a \in A(p)$ such that $a \cdot \varepsilon$ is a martingale.

(ii) $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$ *for all* $t \geq 1$ *and there exists* $a \in A(p)$ *such that* $a \cdot \varepsilon$ *is a martingale.*

(iii) There exists $\Lambda \in X^{J\times J}$ *such that* $\varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t$, $\varepsilon_t = p_t\Lambda_t$, *for all* $t \geq 0$.

(iv) $A(p) \subset A(p+\varepsilon)$

(v) For each $a \in A(p)$, $a \cdot \varepsilon$ *is a martingale.*

Proof. $(i) \Leftrightarrow (ii)$ For the implication $(i) \Rightarrow (ii)$, let $(p_t + d_t)\lambda \in \mathcal{S}_t(p+\varepsilon)$, with $\lambda : \Omega \to \mathbb{R}^J$, $\mathcal{F}_{t-1}$-measurable. Then

$$(p_t + \varepsilon + d_t)\lambda = (p_t + d_t)(\mathbf{I} + \Lambda_{t-1})\lambda \in \mathcal{S}_t(p).$$

Conversely ($(ii) \Rightarrow (i)$), for any $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_{t-1}$-measurable, there exists $\lambda'_{t-1} : \Omega \to \mathbb{R}^J$, $\mathcal{F}_{t-1}$-measurable, such that $(p_t + d_t + \varepsilon_t)\lambda_{t-1} = (p_t + d_t)\lambda'_{t-1}$. It follows that $\varepsilon_t\lambda_{t-1} = (p_t + d_t)(\lambda'_{t-1} - \lambda_{t-1})$, and since λ_{t-1} was arbitrary, we conclude that each of the J components of ε_t belongs to $\mathcal{S}_t(p)$. Thus $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for some $\mathcal{F}_{t-1}$-measurable $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$.

$(i) \Rightarrow (iii)$ The conclusion is immediate, since for all $t \geq 0$,

$$\varepsilon_t = E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1} = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})\Lambda_t = p_t\Lambda_t.$$

$(iii) \Rightarrow (iv)$ Let $a \in A(p)$. The conclusion follows, since

$$E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1}) = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})(\mathbf{I} + \Lambda_t) = p_t(\mathbf{I} + \Lambda_t) = p_t + \varepsilon_t.$$

$(iv) \Rightarrow (v)$ Let $a \in A(p)$. Thus $a \in A(p+\varepsilon)$. The conclusion follows, since

$$p_t + E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1} \overset{a\in A(p)}{=} E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1}) \overset{a\in A(p+\varepsilon)}{=} p_t + \varepsilon_t, \forall t \geq 0.$$

$(v) \Rightarrow (i)$ Fix a date $t \geq 0$ and a state $\omega \in \Omega$, and denote by $\mathcal{F}_t(\omega)$ the atom of the partition $\mathcal{F}_t$ containing ω (that is, the date t "node" containing ω, in the event-tree language). Assume that $\mathcal{F}_{t+1}$ has S atoms belonging to $\mathcal{F}_t(\omega)$ (i.e. there are S branches leaving the chosen node). All the $\mathcal{F}_{t+1}$-measurable random variables, with the domain restricted to $\mathcal{F}_t(\omega)$ (that is, given that $\mathcal{F}_t(\omega)$ occurred), which will be the case for the rest of the proof, can be viewed simply as elements of $\mathbb{R}^S$. For any two such random variables v_1, v_2, the conditional expectation $E_t(v_1 \cdot v_2)$ introduces an inner product on $\mathbb{R}^S$, denoted by $\langle v_1, v_2 \rangle$. To simplify notation, let $A := \{x \in \mathbb{R}^S_{++} \mid p_t = \langle x, p_{t+1} + d_{t+1} \rangle\}$. Thus A is the set of all a_{t+1}/a_t with $a \in A(p)$. Similarly, let $\hat{A} := \{x \in \mathbb{R}^S \mid p_t = \langle x, p_{t+1} + d_{t+1} \rangle\}$ be the set off all a_{t+1}/a_t satisfying $p_t = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})$ (strict positivity is not imposed). Thus $A = \hat{A} \cap \mathbb{R}^S_{++}$. Let $Z := \mathcal{S}_{t+1}(p)$ be the asset span at $t+1$, and hence a subspace of $\mathbb{R}^S$. In other words, $Z := \{\sum_{j=1}^J \lambda_j (p^j_{t+1} + d^j_{t+1}) \mid \lambda_1, \ldots, \lambda_J \in \mathbb{R}\}$.

We show next that $\hat{A} = \operatorname{aff}(A)$, where $\operatorname{aff}(A)$ is the affine hull of A, defined as $\operatorname{aff}(A) := a + \operatorname{span}(A - A)$, for any $a \in A$ (and $\operatorname{span}(A - A)$ is the linear span of $A - A$). Let $Z^\perp \subset \mathbb{R}^S$ be the subspace of all vectors orthogonal to Z, $Z^\perp := \{z' \in \mathbb{R}^S \ : \ \langle z', z \rangle = 0, \forall z \in Z\}$. For each $a \in \hat{A}$, the definition of $\hat{A}$ implies that $\hat{A} = \{a' \in \mathbb{R}^S \mid \langle a' - a, z \rangle = 0, \forall z \in Z\}$. Therefore for each $a \in \hat{A}$, $\hat{A} = a + Z^\perp$. Thus $\hat{A}$ is an affine subspace parallel to $Z^\perp$. Since $A = \hat{A} \cap \mathbb{R}^S_{++}$, it follows that any element in A is also in the core (algebraic interior) of A relative to $\hat{A}$.[4] The convexity of A implies that A is of full dimension in $\hat{A}$, that is $\operatorname{aff}(A) = \hat{A}$ (Holmes 1975, Theorem 2.C), and therefore $\operatorname{span}(A - A) = Z^\perp$.

Choose a $j \in \{1, \ldots, J\}$. For all $a \in A$, $\varepsilon^j_t = \langle a, \varepsilon^j_{t+1} \rangle$. Hence for all $a, a' \in A$, $\langle a' - a, \varepsilon^j_{t+1} \rangle = 0$. It follows that $\varepsilon^j_{t+1} \perp A - A$, and therefore $\varepsilon^j_{t+1} \perp \operatorname{span}(A - A)$. Since $\operatorname{span}(A - A) = Z^\perp$, $\varepsilon^j_{t+1} \in (Z^\perp)^\perp = Z$. As a consequence, for each $j \in \{1, \ldots, J\}$, there exists $\lambda^j_t = (\lambda^j_{1,t}, \ldots, \lambda^j_{J,t})' \in \mathbb{R}^J$ such that $\varepsilon^j_{t+1} = (p_{t+1} + d_{t+1}) \lambda^j_t = \sum_{k=1}^J (p^k_{t+1} + d^k_{t+1}) \lambda^j_{k,t}$. By setting $\Lambda_t = (\lambda^1_t, \ldots, \lambda^J_t)$, it follows that $\varepsilon_{t+1} = (p_{t+1} + d_{t+1}) \Lambda_t$. This concludes the proof, since $t \geq 0$ and $\omega \in \Omega$ were chosen arbitrarily. $\square$

Proof of Proposition 3.1

Proof. $(i) \Leftrightarrow (ii)$ Follows from the equivalence $(iv) \Leftrightarrow (v)$ in Lemma A.1. $A(p) \subset A(p + \varepsilon)$ is equivalent to $a \cdot \varepsilon$ being a martingale for any $a \in A(p)$. Similarly,

[4] An element $\bar{a} \in A$ belongs to the core of A relative to $\hat{A}$ if for each $a \in \hat{A}$, there exists $\gamma_a > 0$ such that $\bar{a} + \gamma(a - \bar{a}) \in A$, for all $\gamma \in [0, \gamma_a]$. It is thus possible to move linearly from $\bar{a}$ towards any point in $\hat{A}$ while staying in A.

$A(p+\varepsilon) \subset A(p+\varepsilon+(-\varepsilon))$ is equivalent to $a \cdot (-\varepsilon)$ (hence $a \cdot \varepsilon$) being a martingale for all $a \in A(p+\varepsilon)$.

$(i) \Rightarrow (iii)$ $A(p) \subset A(p+\varepsilon)$ implies $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$ and $a \cdot \varepsilon$ is a martingale, for some $a \in A(p)$ (Lemma (A.1) $(iv) \Rightarrow (ii)$). Similarly, $A(p+\varepsilon) \subset A(p+\varepsilon+(-\varepsilon))$ gives $\mathcal{S}_t(p) = S_t(p+\varepsilon+(-\varepsilon)) \subset \mathcal{S}_t(p+\varepsilon)$.

$(iii) \Rightarrow (i)$ Choose $a \in A(p) \cup A(p+\varepsilon)$ such that $a \cdot \varepsilon$ is a martingale. This implies that if $a \in A(p)$, then $a \in A(p+\varepsilon)$, and similarly, if $a \in A(p+\varepsilon)$, then $a \in A(p)$. Thus $a \in A(p) \cap A(p+\varepsilon)$. Since $\mathcal{S}_t(p+\varepsilon) \subset \mathcal{S}_t(p)$, it follows that $A(p) \subset A(p+\varepsilon)$, by Lemma A.1 $(ii) \Rightarrow (iv)$. Similarly, $A(p+\varepsilon) \subset A(p+\varepsilon+(-\varepsilon))$ since $\mathcal{S}_t(p+\varepsilon+(-\varepsilon)) \subset \mathcal{S}_t(p+\varepsilon)$ (Lemma A.1 $(ii) \Rightarrow (iv)$).

$(iii) \Rightarrow (iv)$ This is an immediate consequence of the equivalence $(iii) \Leftrightarrow (ii)$ proved above and Lemma A.1 $(ii) \Rightarrow (iii)$ applied successively for the pairs (p, ε) and $(\hat{p}, -\varepsilon)$.

$(iv) \Rightarrow (i)$ By Lemma (A.1) $(iii) \Rightarrow (iv)$, it follows that $A(p) \subset A(p+\varepsilon)$. Notice that $-\varepsilon_{t+1} = (\hat{p}_{t+1} + d_{t+1})(-\Gamma_t)$ and $-\varepsilon_t = \hat{p}_t(-\Gamma_t)$, and therefore by Lemma (A.1) $(iii) \Rightarrow (iv)$ (applied to $\hat{p}, -\varepsilon, -\Gamma$ rather than p, ε, Λ), $A(p+\varepsilon) = A(\hat{p}) \subset A(\hat{p}-\varepsilon) = A(p)$. □

Proposition A.2. *Let $\varepsilon \in M(p)$ and*

$$\Lambda(\varepsilon, p) := \left\{ \Lambda \in X^{J \times J} \mid \forall t \geq 0, \varepsilon_t = p_t \Lambda_t, (p_{t+1} + d_{t+1})\Lambda_t = p_{t+1}\Lambda_{t+1} \right\}.$$

If assets are non-redundant at prices p, then the set $\Lambda(\varepsilon, p)$ is a singleton, $\Lambda(\varepsilon, p) := \{\Lambda\}$, and $\det(\mathbf{I} + \Lambda_t) \neq 0, \forall t \geq 0$. Moreover, the portfolios $\theta^i, \hat{\theta}^i$ in the equivalent equilibria $\mathcal{E}, \hat{\mathcal{E}}$ of Theorem 3.3 satisfy, for all $t \geq 0$,

$$\hat{\theta}^i_t = (\mathbf{I} + \Lambda_t)^{-1} \left(\theta^i_t + \Lambda_t \theta^i_{-1} \right).$$

Proof. If $\Lambda, \Lambda' \in \Lambda(\varepsilon, p)$, then for all $t \geq 0$, $(p_{t+1} + d_{t+1})(\Lambda_t - \Lambda'_t) = 0$. Since assets are non-redundant at prices p, $\Lambda_t = \Lambda'_t$ for all $t \geq 0$, and thus $\Lambda(\varepsilon, p)$ is a singleton, $\Lambda(\varepsilon, p) := \{\Lambda\}$. By Proposition 3.1 $(i) \Rightarrow (iv)$, there exist Γ such that $\varepsilon_{t+1} = (p_{t+1} + d_{t+1})\Lambda_t = (p_{t+1} + d_{t+1})(\mathbf{I} + \Lambda_t)\Gamma_t$. Using again non-redundancy of assets at prices p, we obtain $(\mathbf{I}+\Lambda)\,\Gamma=\Lambda$. It follows that

$$\mathbf{I}=\mathbf{I}+\Lambda-\Lambda=\mathbf{I}+\Lambda-(\mathbf{I}+\Lambda)\Gamma=(\mathbf{I}+\Lambda)(\mathbf{I}-\Gamma),$$

and therefore $\mathbf{I}+\Lambda$ is non-singular.

If $(c^i,\theta^i),(c^i,\tilde{\theta}^i)\in B_0^i((p_0+d_0)\theta^i_{-1},\phi^i,p)$ are optimal for i, then $(p_t+d_t)\theta^i_{t-1}=(p_t+d_t)\tilde{\theta}^i_{t-1}$ for all $t\geq 0$. Therefore $\theta^i=\tilde{\theta}^i$, as assets are non-redundant. Indeed, if for example $(p_t+d_t)\theta^i_{t-1}>(p_t+d_t)\tilde{\theta}^i_{t-1}$, as agents' utilities are strictly increasing, $V_t^i((p_t+d_t)\theta^i_{t-1},\phi^i,p)>V_t^i((p_t+d_t)\tilde{\theta}^i_{t-1},\phi^i,p)$, contradicting the optimality of $(c^i,\tilde{\theta}^i)$.

Therefore the portfolios $\theta^i,\hat{\theta}^i$ in the equilibria $\mathcal{E},\hat{\mathcal{E}}$ of Theorem 3.3 are uniquely determined. They satisfy (see (3.4) and the proof of Theorem 3.3)

$$(p_t+d_t)(\theta^i_{t-1}-\theta^i_{-1})=(\hat{p}_t+d_t)(\hat{\theta}^i_{t-1}-\theta^i_{-1}),\forall t\geq 0, \tag{A.1}$$

where $\hat{p}=p+\varepsilon$, and $\varepsilon\in M(p)$. It follows that for all $t\geq 1$, $(p_t+d_t)\Lambda_{t-1}=p_t\Lambda_t=\varepsilon_t$. Using (A.1),

$$(p_t+d_t)(\theta^i_{t-1}-\theta^i_{-1}-(\mathbf{I}+\Lambda_{t-1})(\hat{\theta}^i_{t-1}-\theta^i_{-1}))=0,$$

and therefore $\theta^i_{t-1}-\theta^i_{-1}=(\mathbf{I}+\Lambda_{t-1})(\hat{\theta}^i_{t-1}-\theta^i_{-1})$. Solving for $\hat{\theta}^i_{t-1}$ leads to (3.6).

□

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