# Generalizing the Non-Cooperative Implementation of the Core to Predict Coalition Formation*

**Camelia Bejan**
Department of Economics
Rice University MS-22
6100 Main Street
Houston, TX 77005
E-mail: camelia@rice.edu

**Juan Camilo Gómez**
Business Administration Program
University of Washington - Bothell
18115 Campus Way NE
Bothell, WA 98011
E-mail: jcgomez@u.washington.edu

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**Abstract**

We propose two non-cooperative mechanisms that deliver aspiration core allocations (Cross 1967, Bennett 1983) together with their supporting coalitions. The equilibrium concepts used are strong and subgame perfect Nash equilibria. Overlapping coalition configurations are allowed and our bargaining procedures interpret this fact in different ways. The first mechanism allows players to participate simultaneously in more than one coalition, while the second assigns probabilities to the formation of different (and potentially overlapping) coalitions. Both procedures generalize previous core implementation results.

**Keywords:** Coalition formation, non-cooperative implementation, aspiration core
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# 1 Introduction

This work brings together two distinct branches of the literature. The first refers to the non-cooperative implementation of the core, while the second deals with coalition formation. Which coalitions arise is a non-issue for balanced games: it can be assumed that the grand coalition always forms, since it generates the highest total surplus for the players of the game. However, if balancedness (or super-additivity) is not satisfied, it is desirable to endogenously determine not only payoffs but also coalitional outcomes. We adapt ideas from well-known core implementation procedures to a setting in which the grand coalition is not assumed to form. Our mechanisms deliver, as equilibrium outcomes, the aspiration core[1] vectors (Cross 1967, Bennett 1983) and their supporting coalitions.

Aspirations can be interpreted as price vectors resulting from demands placed by the players for participating in any formed coalition. An aspiration cannot be improved upon by any coalition but, compared to a core vector, has a much weaker feasibility requirement: each player's payoff must be low enough so at least one coalition is able to support it. In particular, aspiration payoffs may not be feasible for the grand coalition, allowing the possibility for other coalitions to be formed.

[1]The aspiration core is also known as the balanced aspiration set.

The aspiration core is defined as the set of aspirations of minimal sum and it is a non-empty core extension. Our bargaining procedures are related to previous core implementation results (e.g., (Perry and Reny 1994, Serrano 1995, Serrano and Vohra 1997, Pérez-Castrillo 1994, Moldovanu and Winter 1994, Okada and Winter 2002)) but with an added coalition formation dimension to the problem. Still, if the game is balanced, the grand coalition forms and equilibrium payoffs coincide with the core.

Coalitions supporting a given aspiration are not necessarily disjoint. The literature on coalition formation has generally discarded this type of outcome as lacking predicting power. Greenberg (1994), for example, noted: "Clearly, the set of coalitions that support an aspiration is not a partition. Thus, despite its appeal, an aspiration fails to predict which coalition will form, and moreover, it ignores the possibility that players who are left out will decide to lower their reservation price." Both of our aspiration core implementations deal with this criticism, albeit in different ways.

In our first bargaining game aspiration core payoffs are attained via fractional coalition participation. While belonging to a single coalition is an appropriate assumption to model some situations (one person cannot belong to the Democrat and the Republican party simultaneously), there are other instances in which it is reasonable to assume that an agent would divide his time, or other resource, among several coalitions. For example, people might work for more than one firm, invest in multiple ventures, or belong to a number of clubs. Such situations are not covered by solution concepts specifically designed to deal with partitions. We depart from the traditional approach to coalition formation by allowing a single player to participate in multiple coalitions.

Our first mechanism starts by players simultaneously announcing a payoff for themselves and a distribution of their time across coalitions. For example, a player might divide her available time among two or more non-disjoint coalitions. A coalition forms when all of its members choose to spend a positive fraction of their time on it, and their demands are less than or equal to its worth. The coalition is active for a period of time equal to the minimum participation by one of its members. A player who participates in a coalition that operates a fraction $\lambda$ of time gets a fraction $\lambda$ of his demand. It is assumed that time not spent in an active coalition is spent alone. Along the lines of (von Neumann and Morgenstern 1944, Kalai, Postlewaite, and Roberts 1979, Hart and Kurz 1983, Borm and Tijs 1992, Young 1998), we focus on strong Nash equilibria. We show that strong Nash equilibrium strategies in our game correspond to aspiration core payoffs and, conversely, any aspiration core vector can be generated by a strong Nash equilibrium. In equilibrium, players organize into efficient, overlapping coalition structures.

Our second game illustrates a probabilistic approach to the overlapping coalition issue. Instead of claiming that non-disjoint coalitions form simultaneously, they are assigned a positive probability of actually forming. Ex-post, a single coalition forms. For example, if we consider the three-player simple majority game[2] the prediction is that coalitions $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$, will form, each with a probability of $\frac{1}{3}$. The set of coalitions that may form is a balanced family, and the vectors of probabilities and balancing weights are proportional to each other.

As opposed to the first game, the second is not played by the original players, but by (at least) three principals. This type of construction is similar to (Aumann 1989, Pérez-Castrillo 1994, Gómez 2003, Morelli and Montero 2003). Our story is as follows: one principal is the owner of a set of machines (the players of the original cooperative game), but lacks the knowledge to operate them. Instead, she rents them to potential entrepreneurs in the market, who will set up firms (i.e., create "coalitions" of machines) and operate them to produce the output. The game has two stages. In the first stage, the interested entrepreneurs submit bids to the owner. A bid represents the *total* amount an entrepreneur is willing to pay to get the machines. The entrepreneur with the highest bid wins and enters stage 2, while the others leave the game. In the second stage the owner and the winning entrepreneur play the following simultaneous-move zero-sum game. The owner puts a price on each machine so that the sum equals the winning bid, while the entrepreneur selects a set of machines that she wants to buy. The transaction takes place, with the entrepreneur paying for the machines she wants at the prices set by the owner. The game implements the aspiration core allocations in subgame perfect equilibria.

Our mechanisms are related to the proposal-making bargaining game introduced by Selten (1981) and extended by Bennett (1997) which, for a very particular class of games,[3] supports aspirations (semi-stable payoff vectors in Selten's (1981) terminology) as outcomes of stationary subgame perfect equilibria. With additional restrictions, the outcome is refined to the set of

[2]In this game coalitions of two and three players have a worth equal to one and the remaining coalitions are worth zero.

[3]Selten (1981) requires that the complement of any productive coalition must have zero worth.

partnered aspirations (or stable payoff vectors in Selten's (1981) terminology). However, our approach applies to a much more general class of games, generates efficient outcomes, and coincides with the core in balanced games.

The paper is organized as follows. Section 2 introduces notation and basic definitions. Section 3 describes the mechanism that implements the aspiration core in strong Nash equilibria, while section 4 focuses on the subgame perfect implementation. Section 5 concludes.

# 2 Definitions and Notation

Let $N = \{1, ..., n\}$ be a finite set of players, where $n \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{N}$ be the collection of all non-empty subsets of $N$ and, for every $i \in N$, let $\mathcal{N}_i = \{S \in \mathcal{N} \mid S \ni i\}$. Let $\Delta_N$ be the unit simplex in $\mathbb{R}^n$, $\Delta_{\mathcal{N}}$ the unit simplex in $\mathbb{R}^{2^n-1}$ and $\Delta_{\mathcal{N}_i} = \{\lambda \in \Delta_{\mathcal{N}} \mid \lambda_S = 0 \text{ if } i \notin S\}$. For any $S \in \mathcal{N}$, let $\mathbf{e}_S \in \Delta_{\mathcal{N}}$ be the vertex of $\Delta_{\mathcal{N}}$ corresponding to coalition $S$. A *TU-game* (or simply a *game*) with a finite set of players $N$ is a mapping $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. For any $S \subseteq N$, $v(S)$ is called the *worth of coalition* $S$. The *zero-normalization* of $v$ is a TU-game, $v_0$, with the same set of players $N$, such that for every $S \subseteq N$, $v_0(S) = v(S) - \sum_{i \in S} v(\{i\})$.

A possible outcome of the game $v$ is represented by a *payoff vector* $x \in \mathbb{R}^n$ that assigns to every $i \in N$ a payoff $x_i$. Given $x \in \mathbb{R}^n$ and $S \subseteq N$, let $x(S) := \sum_{i \in S} x_i$, with the agreement that $x(\emptyset) = 0$. A payoff vector $x \in \mathbb{R}^n$ is *feasible* for coalition $S$ if $x(S) \leq v(S)$. It is *aspiration feasible* if for every $i \in N$, there exists $S \subseteq N$ with $i \in S$ such that $x$ is feasible for $S$. We say that a coalition $S$ is able to *improve upon* the outcome $x \in \mathbb{R}^n$ if $x(S) < v(S)$. A vector $x \in \mathbb{R}^n$ is called *stable* if it cannot be improved upon by any coalition. The *core* of a game $v$, denoted $C(v)$, is the set of stable outcomes that are feasible for $N$ i.e.,

$$C(v) := \{x \in \mathbb{R}^n \mid x(S) \geq v(S) \; \forall S \subseteq N, \; x(N) = v(N)\}.$$

A stable payoff vector $x \in \mathbb{R}^n$ that is aspiration feasible is called an *aspiration*. We denote by $Asp(v)$ the set of aspirations of game $v$. It is known that for any TU-game $v$, $Asp(v)$ is a non-empty, compact and connected set (Bennett and Zame 1988). The *generating collection* of an aspiration $x$ is the family of coalitions $S$ that can attain $x$, i.e.,

$$\mathcal{GC}(x) := \{S \in \mathcal{N} \mid x(S) = v(S)\}.$$

A collection of coalitions $\mathcal{B} \subseteq 2^N$ is called *balanced* (respectively *weakly balanced*) if every $S \in \mathcal{B}$ is associated with a positive (resp. non-negative) number $\lambda_S$ such that for every $i \in N$, $\sum_{S\in\mathcal{B},S\ni i} \lambda_S = 1$. The numbers $\lambda_S$ are called *balancing weights*. It is customary to interpret the balancing weight $\lambda_S$ as the fraction of resources each player (in $S$) devotes to coalition $S$, or as the fraction of time coalition $S$ is active (see, for example, (Kannai 1992) ). Thus, a pair $(\mathcal{B}, \lambda)$ consisting of a balanced family $\mathcal{B}$ and balancing weights $\lambda := (\lambda_S)_{S\in\mathcal{B}}$ can be interpreted as a feasible *overlapping coalition structure*, that is, a family of coalitions that can co-exist if players can divide their resources/time. The *total surplus* generated by an overlapping coalition structure $(\mathcal{B}, \lambda)$ is $b(\mathcal{B}, \lambda) := \sum_{S\in\mathcal{B}} \lambda_S v(S)$. An overlapping coalition structure $(\mathcal{B}, \lambda)$ is called *stable* if there exists a stable $x \in \mathbb{R}^n$ such that $x(N) \leq b(\mathcal{B}, \lambda)$.

For every TU-game $v$ we define $\bar{b}(v)$ as the maximum total surplus generated by an overlapping coalition structure:

$$\bar{b}(v) := \max\{b(\mathcal{B}, \lambda) \mid \mathcal{B} \text{ is balanced w.r.t. weights } \{\lambda_S\}\}. \tag{1}$$

A coalition structure $(\mathcal{B}, \lambda)$ is *efficient* if $b(\mathcal{B}, \lambda) = \bar{b}(v)$. It is known that $\bar{b}(v)$ is finite and $\bar{b}(v) = \min\{x(N) \mid x(S) \geq v(S)\ \forall S \subseteq N\}$, (Bennett 1983). This implies that an overlapping coalition structure is efficient if and only if it is stable. Moreover, the core of a game $v$ is non-empty if and only if $\bar{b}(v) \leq v(N)$ (Bondareva 1963, Shapley 1967). Such games are known as *balanced* games.

The *aspiration core* of a game $v$ is given by

$$AC(v) := \{x \in Asp(v) \mid \mathcal{GC}(x) \text{ is weakly balanced }\}.$$

The aspiration core is non-empty for every TU-game $v$ and it coincides with the core whenever the latter is non-empty. Two alternative but equivalent definitions of the aspiration core are $AC(v) = \{x \in Asp(v) \mid x(N) = \bar{b}(v)\}$ and $AC(v) = \arg\min\{x(N) \mid x \in \mathbb{R}^n,\ x(S) \geq v(S)\ \forall S \subseteq N\}$ (See (Bennett 1983)).

If $x \in AC(v)$ and $\mathcal{B} \subseteq \mathcal{GC}(x)$ is a balanced family with associated weights $\lambda$, then $b(\mathcal{B}, \lambda) = x(N) = \bar{b}(v)$ and thus $(\mathcal{B}, \lambda)$ is efficient. Reciprocally, every efficient overlapping coalition structure $(\mathcal{B}, \lambda)$ is stable, which implies that there exists $x \in AC(v)$ such that $x(N) = \bar{b}(v) = b(\mathcal{B}, \lambda)$ and thus $\mathcal{B} \subseteq \mathcal{GC}(x)$. Hence, the only efficient (and stable) overlapping coalition structures are balanced subsets of the generating collections of aspiration core allocations.

# 3 Strong Nash Equilibrium Implementation

Fix a TU-game $v$ with a finite set of players $N$. Given the covariance properties of the aspiration core, there is no loss of generality in assuming that $v$ is zero-normalized and thus, $v(\{i\}) = 0$ for all $i \in N$.[4] We associate with $v$ a non-cooperative game $\Gamma(v)$, defined as follows. The player set of $\Gamma(v)$ is $N$. Each player $i \in N$ is endowed with one unit of a resource (for example, time) and has to decide how to allocate it among different coalitions. In other words, player $i$ chooses a resource allocation vector $\lambda_i \in \Delta_{\mathcal{N}_i}$. Additionally, player $i$ chooses a payoff for himself, $x_i \in \mathbb{R}$. Thus, player $i$'s strategy space is $\mathcal{S}_i = \Delta_{\mathcal{N}_i} \times \mathbb{R}$. Let $\mathcal{S} = \prod_{i \in N} \mathcal{S}_i$ be the *strategy space* of the game.

A coalition is productive only as long as all its members want to invest resources in it and their payoffs are feasible for $S$. More formally, given a strategy profile $s = (\lambda, x) \in \mathcal{S}$, a coalition $S \in \mathcal{N}$ generates a total surplus for its members of $\lambda_0(S)v(S)$, where $\lambda_0(S) = \min_{i \in S} \lambda_i(S)$ if $x(S) \leq v(S)$ and $\lambda_0(S) = 0$ otherwise. The payoff to $i$ at $s \in \mathcal{S}$ is $p_i(s) := \alpha_i(s)x_i$, where $\alpha_i(s) := \sum_{S \ni i} \lambda_0(S)$ is the fraction of time spent productively. Notice that, by definition, $\alpha_i(s) \in [0, 1]$ and thus $p_i(s) \leq x_i$, for all $i \in N$ and $s \in \mathcal{S}$. We think of $1 - \alpha_i(s)$ as the amount of time agent $i$ is idle or equivalently, given that $v$ is zero normalized, time dedicated to coalition $\{i\}$. Accordingly, we define the *set of formed coalitions* as

$$\mathcal{F}(s) := \{S \in \mathcal{N} \mid \lambda_0(S) > 0\} \cup \big\{ \{i\} \mid \alpha_i(s) < 1\big\}.$$

For any strategic-form game with a finite player set $N$ and any strategy $s$ we say that a coalition $S$ *can improve upon* $s$ if there exist strategies $(t_i)_{i \in S} = t_S$ such that $p_i(s) < p_i(s_{-S}, t_S)$ for every $i \in S$. We say that the strategy profile $s$ is a *strong Nash equilibrium* if $s$ cannot be improved upon by any coalition, (Aumann 1967).

Assume $s^* = (\lambda^*, x^*) \in \mathcal{S}$ is a strong Nash equilibrium of $\Gamma(v)$. To simplify notation, for any $i \in N$, let $\alpha_i(s^*) = \alpha_i^*$, $p_i(s^*) = p_i^*$, and $\mathcal{F}(s^*) = \mathcal{F}^*$.

**Proposition 3.1** *If $s^*$ is a strong Nash equilibrium of $\Gamma(v)$, then $p^*$ is an aspiration of $v$.*

[4] The aspiration core is covariant with respect to affine transformations of the game, that is $AC(\alpha v + \beta) = \alpha AC(v) + \beta$ for every game $v$ and every $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}^n$.

**Proof.** First, we show that $p^* = (p_i^*)_{i \in N}$ is aspiration feasible. For any $j \in N$, if $\alpha_j^* = 0$ then $p_j^* = v(\{j\}) = 0$ and thus $p^*$ is aspiration feasible for $j$. If $\alpha_j^* > 0$, there must exist $S \in \mathcal{N}$ with $j \in S$ and $\lambda_0(S) > 0$. This implies that $p^*(S) \leq x^*(S) \leq v(S)$, and thus $p^*$ is aspiration feasible.

We show next that $p^*$ is stable. Suppose not, so $\exists S \in \mathcal{N}$ such that $p^*(S) < v(S)$. Define for every $i \in S$ $\hat{x}_i = p_i^* + \frac{\varepsilon}{|S|}$, where $\varepsilon = v(S) - p^*(S) > 0$. In this case $S$ improves upon $s^*$ by using strategies $(\mathbf{e}_S, \hat{x}_i)_{i \in S}$, which is a contradiction. We conclude that $p^*$ is an aspiration of $v$. ■

**Theorem 3.2** *If $s^* = (\lambda^*, x^*)$ is a strong Nash equilibrium of $\Gamma(v)$, then $p^* \in AC(v)$ and $\mathcal{F}^*$ is a balanced subset of $\mathcal{GC}(p^*)$. Conversely, if $y^* \in AC(v)$ then there exists a strong Nash equilibrium $s^*$ of $\Gamma(v)$ with $p^* = y^*$ and $\mathcal{F}^* = \mathcal{GC}(y^*)$.*

**Proof.** Let $s^*$ be a strong Nash equilibrium. According to Proposition 3.1, $p^*$ is an aspiration of $v$. To show $p^* \in AC(v)$ it is therefore sufficient to prove that $\mathcal{GC}(p^*)$ is balanced. To do so, we show that $\mathcal{F}^* \subseteq \mathcal{GC}(p^*)$ and that $\mathcal{F}^*$ is itself balanced.

Recall that $\mathcal{F}^* = \{S \in \mathcal{N} | \lambda_0^*(S) > 0\} \cup \{\{i\} | \ \alpha_i^* < 1\}$. If $\lambda_0^*(S) > 0$, then $x^*$ is feasible for $S$. As $p^*(S) \leq x^*(S)$ and $p^*$ is an aspiration, $S \in \mathcal{GC}(p^*)$. If $S = \{i\}$ and $\alpha_i^* < 1$, we show that $p_i^* = v(\{i\}) = 0$. Assume $\alpha_i^*, x_i^* > 0$, otherwise the result holds trivially. If $p_i^* > 0$, there exists $T \ni i$ such that $\lambda_0^*(T) > 0$. As $\alpha_i^* < 1$ we have $p^*(T) = \sum_{j \in T} \alpha_j^* x_j^* < \sum_{j \in T} x_j^* \leq v(T)$, which is impossible as $p^*$ is an aspiration. Therefore, $\mathcal{F}^* \subseteq \mathcal{GC}(p^*)$.

Finally, associate weights $\{\delta_S\}_{S \in \mathcal{F}^*}$ as follows. If $\lambda_0^*(S) > 0$ and $S \notin \{\{i\} | \ \alpha_i^* < 1\}$, let $\delta_S := \lambda_0^*(S)$. If $\alpha_i^* < 1$, let $\delta_{\{i\}} = 1 - \alpha_i^*$. Clearly, $\mathcal{F}^*$ is a balanced family of coalitions with the above balancing weights.

Conversely, let $y^* \in AC(v)$, $\mathcal{GC}(y^*)$ its corresponding balanced family of coalitions and $\{\delta_S\}_{S \in \mathcal{GC}(y^*)}$ some balancing weights. Consider the strategy profile $s^* = (\lambda^*, y^*)$ such that $\lambda_i^*(S) = \delta_S$ if $S \in \mathcal{GC}(y^*)$ and $i \in S$, and $\lambda_i^*(S) = 0$ otherwise. Clearly $\alpha_i^* = 1$ for every $i \in N$, so $p^* = y^*$. The strategy profile $s^*$ is a strong Nash equilibrium because, if $\hat{s} = (\hat{\lambda}, \hat{y})$ improved upon $s^*$ on $\hat{S}$, then $v(\hat{S}) \geq \hat{y}(\hat{S}) > y^*(\hat{S})$, so $y^*$ would not be an aspiration. ■

The theorem implies that the coalition structure $(\mathcal{F}^*, \delta)$ is stable and efficient. Therefore, the game delivers not only aspiration core allocations, but also the efficient coalition structures that support those allocations.

A related strategic game was proposed by Kalai, Postlewaite, and Roberts

(1979) in the context of a public good economy. They implement core allocations in strong Nash equilibrium. An analogous result can be obtained by applying Theorem 3.2 in the particular case the game $v$ is balanced.

**Corollary 3.3** *If $v$ is balanced and $s^*$ is a strong Nash equilibrium, then $p(s^*) \in C(v)$. Conversely, if $x^* \in C(v)$ then there exists a strong Nash equilibrium $s^*$ with $p(s^*) = x^*$.*

The non-cooperative game described above relied on the assumption that players can divide their resources/time among various coalitions. A natural question to ask is whether the aspiration core may still be used to predict coalition formation when agents' resources are assumed to be indivisible. Second, it may be argued that strong Nash equilibrium is a very restrictive solution concept as it assumes that agents are able to coordinate strategically. Unfortunately, Nash equilibrium outcomes of the non-cooperative game $\Gamma(v)$ coincide with the set of individually rational payoff vectors and it is therefore too weak as a solution concept for this case. To address these concerns we define next a second mechanism that does not require the co-existence of non-disjoint coalitions and implements the aspiration core in subgame perfect equilibrium.

# 4 The Bidding Game

The game of this section illustrates the role of the aspiration core solution concept in the context of firm formation. It relates stability of aspirations to firm profitability and exploits the fact that coalition structures that support aspiration core allocations are precisely those that maximize the total surplus.

Given a cooperative TU-game $v$, the following two-stage mechanism, denoted by $\Psi(v)$, describes a situation in which an auxiliary set of individuals, called $B_1, B_2$ and $X$, compete over the $n$ players in the cooperative game. $X$ is the owner of a set of machines (the players of the original cooperative game, $v$), but she lacks the knowledge to operate them. Instead, she rents them to the potential entrepreneurs in the market, who will set up firms (i.e., "coalitions" of machines) to produce the output. As in the previous section, we assume that $v$ is zero-normalized without affecting the generality of our results. The description of the game is the following:

**Stage 1:** Principals $B_1$ and $B_2$ simultaneously bid amounts $b_1, b_2 \geq 0$. Let $b = \max_{i=1,2} b_i$ and label the winning bidder principal $B$. If bids are equal, a winner is selected at random. The losing bidder gets zero payoff and leaves the game.

**Stage 2:** Principals $B$ and $X$ engage in the following zero-sum game: $B$ chooses a coalition $S \in \mathcal{N}$ and $X$ chooses $x \in b\Delta_N$. Payoffs for $B$ are determined according to the excess function $u_B(x,S) = v(S) - x(S)$. Payoffs for $X$ are given by $u_X = -u_B$. Denote the zero-sum game by $\Omega(b)$.

This mechanism is reminiscent of the game Aumann (1989) used to derive an alternative proof to Bondareva-Shapley theorem (see also (Gómez 2003)). The role of stage 1 in our mechanism is to pin down the total surplus achieved by the most efficient coalition structure. This step is not needed if the objective is to implement the core, because the grand coalition is always the most efficient coalition structure in balanced games.

A strategy profile for $\Psi(v)$ is $((b_1, \tau_1(b)), (b_2, \tau_2(b)), x(b))$ in which principal $B_i$ bids $b_i \geq 0$ and, for any $b \geq 0$, chooses a mixed strategy $\tau_i(b) \in \Delta_{\mathcal{N}}$ in the second-stage game $\Omega(b)$. If the winning bid in stage 1 is $b$, Principal $X$ chooses the vector $x(b) \in b\Delta_N$ in $\Omega(b)$. We now show that this mechanism implements the aspiration core of game $v$ in subgame perfect equilibria (SPE).

We begin by analyzing the equilibria of the subgames that start after a winning bid $b$. The following proposition is a generalization of the min-max theorem to zero-sum semi-infinite games (i.e., games in which exactly one player has an infinite strategy set). We refer the reader to (Raghavan 1994) for a proof of this result.

**Proposition 4.1** *For any $b \geq 0$ the semi-infinite zero-sum game $\Omega(b)$ has a value $\omega(b) \in \mathbb{R}$. Moreover, there exists a Nash equilibrium of $\Omega(b)$ in which $X$ plays a pure strategy.*

For ease of notation, denote the amount $\bar{b}(v)$ defined in (1) by $\bar{b}$.

**Proposition 4.2** *The value $\omega(b)$ is a continuous and strictly decreasing function of $b$, with $\omega(\bar{b}) = 0$.*

**Proof.** That $\omega(b)$ is strictly decreasing follows immediately from the fact that $\max_{x \in b\Delta_N} \sum_{S \in \mathcal{N}} \lambda_S x(S)$ is strictly increasing in $b$, for every $\lambda \in \Delta_{\mathcal{N}}$. Continuity is an immediate consequence of Berge's maximum theorem (Border 1985, p. 64).

Since $\bar{b} = \max\{\sum_{S \in \mathcal{B}} \lambda_S v(S) \mid \mathcal{B}$ is balanced w.r.t. weights $\{\lambda_S\}\}$, there exists a balanced family $\mathcal{B}$ with weights $\{\bar{\lambda}_S\}$ such that $\sum_{S \in \mathcal{B}} \bar{\lambda}_S v(S) = \bar{b}$.

Let $\bar{\Lambda} := \sum_{S\in\mathcal{N}} \bar{\lambda}_S$ and consider the mixed strategy $\bar{\tau}(b) \in \Delta_{\mathcal{N}}$ that assigns probability $\bar{\tau}_S(\bar{b}) := \frac{\bar{\lambda}_S}{\bar{\Lambda}}$ to $S \in \mathcal{B}$ and zero to any other $S \in \mathcal{N}$. Then, for any strategy $x \in \bar{b}\Delta_N$, we have

$$u_B(x, \bar{\tau}(\bar{b})) = \sum_{S\in\mathcal{B}} \frac{\bar{\lambda}_S}{\bar{\Lambda}}[v(S) - x(S)] = \frac{\bar{b} - x(N)}{\bar{\Lambda}} = 0.$$

Strategy $\bar{\tau}(\bar{b})$ guarantees Player $B$ a non-negative profit, so $\omega(\bar{b}) \geq 0$. On the other hand, $\bar{b} = \min\{x(N) \mid v(S) \leq x(N)\ \forall S \subseteq N\}$ implies that $\exists \bar{x} \in \mathbb{R}^n$ such that $\bar{x}(N) = \bar{b}$ and $u_B(\bar{x}, S) \leq 0$ for every $S \subseteq N$. Consequently $\omega(\bar{b}) \leq 0$. We conclude $\omega(\bar{b}) = 0$, as we wanted. ■

**Proposition 4.3** *The mechanism $\Psi(v)$ has at least one subgame perfect equilibrium.*

**Proof.** Proposition 4.1 implies the existence of at least one equilibrium $(\hat{x}(b), \hat{\tau}(b))$ for every zero-sum game $\Omega(b)$. We claim that the strategy profile $((\bar{b}, \hat{\tau}(b)), (\bar{b}, \hat{\tau}(b)), \hat{x}(b))$ is an SPE. Indeed, by definition, second-stage strategies are equilibria of $\Omega(b)$. Additionally, according to Proposition 4.2, no player can generate positive profits by deviating from his first stage strategy when the opponent bids $\bar{b}$. ■

**Proposition 4.4** *In any SPE, principals $B_1$ and $B_2$ choose bids equal to $\bar{b}$, yielding zero profits for Principals $B_1$, $B_2$ and $X$.*

**Proof.** Let $((b_1^*, \tau_1^*(b)), (b_2^*, \tau_2^*(b)), x^*(b))$ be an SPE strategy profile for this mechanism. Then $b_1^* = b_2^*$, otherwise, by strict monotonicity of $\omega(b)$, the highest bidder has an incentive to bid less. Let $b^* = b_1^* = b_2^*$. A bid $b^* > \bar{b}$ cannot be part of an equilibrium because, according to Proposition 4.2, $\omega(b^*) < 0$ and thus any bidder would prefer to stay out by biding 0. If $b^* < \bar{b}$, since $\omega(b^*) > 0$, any bidder has a profitable deviation in announcing $b^* + \varepsilon$ for $\varepsilon > 0$ small enough to guarantee $\omega(b^* + \varepsilon) > \frac{1}{2}\omega(b^*)$. Such $\varepsilon > 0$ always exists because $\omega(b)$ is continuous and $\omega(b^*) > 0$. Thus $b^* = \bar{b}$ and $\omega(b^*) = 0$. ■

**Theorem 4.5** *If $((b^*, \tau_1^*(b)), (b^*, \tau_2^*(b)), x^*(b))$ is an SPE for $\Psi(v)$ then $x^*(b^*) \in AC(v)$. Conversely, if $\bar{x} \in AC(v)$, then there exists an SPE strategy profile $((b^*, \tau_1^*(b)), (b^*, \tau_2^*(b)), x^*(b))$ such that $x^*(b^*) = \bar{x}$.*

**Proof.** The previous proposition shows $b^* = \bar{b}$. By definition of $\bar{b}$ we know that any vector $x$ such that $x(N) < \bar{b}$ cannot be an aspiration as it will be blocked by some coalition. It is then enough to show that $x^*(\bar{b})$, $x^*$ for short, is an aspiration. If $x^*(S) < v(S)$ for some $S \in \mathcal{N}$ then the winning bidder could make positive profits choosing $S$, which would contradict Proposition 4.4. Thus, $x^*$ is stable. Second, if there ia an $i \in N$ such that $x^*(S) > v(S)$ for every $S \ni i$, then coordinate $x_i^*$ could be slightly reduced so that inequalities $x^*(S) \geq v(S)$ are still satisfied, and definition of $\bar{b}$ would be contradicted. Hence, $x^*$ is also aspiration feasible and therefore an aspiration core allocation.

Conversely, let $\bar{x} \in AC(v)$. Let $b^* = \bar{b}$ and $(\tau(b), x(b))$ be a Nash equilibrium of the game $\Omega(b)$. For every $b \neq \bar{b}$ and $i = 1, 2$ define $\tau_i^*(b) = \tau(b)$. As $\mathcal{GC}(\bar{x})$ is balanced, denote its balancing weights by $\{\bar{\lambda}_S\}_S$. Let $\bar{\Lambda} = \sum_{S \in \mathcal{GC}(\bar{x})} \bar{\lambda}_S$. Define $\tau_{i,S}^*(\bar{b}) = \frac{\bar{\lambda}_S}{\bar{\Lambda}}$ if $S \in \mathcal{GC}(\bar{x})$ and zero otherwise. Finally, define $x^*(b) = x(b)$ for $b \neq \bar{b}$ and $x^*(\bar{b}) = \bar{x}$.

We show next that both Principals play a best response in the subgame $\Omega(\bar{b})$. For Principal $B$, giving positive weight to any $T \notin \mathcal{GC}(\bar{x})$ offers a lower payoff (excess), as $\bar{x}$ is an aspiration. For principal $X$, similar calculations to those performed in Proposition 4.2 show that $u_X(\tau^*(\bar{b}), x)$ is independent of $x \in \bar{b}\Delta_N$. Furthermore, we can now use the argument in Proposition 4.3 to show that players cannot benefit from choosing a bid different from $\bar{b}$. We conclude that the strategy profile $((b^*, \tau_1^*(b)), (b^*, \tau_2^*(b)), \bar{x})$ is an SPE. ■

The game suggests a natural probabilistic interpretation of an overlapping coalition structure. Instead of claiming that non-disjoint coalitions form simultaneously, each coalition is assigned a positive probability of actually forming. We call this a *probabilistic coalition structure*. Ex-post, a single coalition forms. As shown bellow, the set of coalitions that have a positive probability of forming is a balanced family and the associated vector of probabilities is proportional to the vector of balancing weights.

Let $((\bar{b}, \tau_1^*(b)), (\bar{b}, \tau_2^*(b)), x^*(b))$ be an SPE for $\Psi(v)$ and define

$$\mathcal{P}_k := \{S \mid \tau_k^*(\bar{b})(S) > 0\}$$

to be the associated family of *productive coalitions* (or *firms*) that can form, depending on the identity, $k = 1, 2$, of the winning bidder. Each firm $S \in \mathcal{P}_k$ forms with probability $\tau_k^*(\bar{b})(S)$. For every $k = 1, 2$, define $T_k^* :=$

$\max_{i\in N}\sum_{S\ni i}\tau_k^*(\bar{b})(S)$ and let

$$\mathcal{I}_k := \{i \in N \mid \sum_{S\ni i}\tau_k^*(\bar{b})(S) = T_k^*\}$$

be the set of players (machines) most likely to be chosen by the winning bidder $B_k$. For every $S \in \mathcal{P}_k$, define the participation coefficient $\lambda_S := \frac{\tau_k^*(\bar{b})(S)}{T_k^*}$. Clearly, for every $i \in N$, $\sum_{\mathcal{P}_k\ni S\ni i}\lambda_S \leq 1$, with equality if and only if $i \in \mathcal{I}_k$. If $N \setminus \mathcal{I}_k \neq \emptyset$, define $\lambda_{\{i\}} := 1 - \sum_{\mathcal{P}_k\ni S\ni i}\lambda_S$ for every $i \notin \mathcal{I}_k$ and let $\widetilde{\mathcal{P}}_k := \mathcal{P}_k \cup \{\{i\}|i \notin \mathcal{I}_k\}$.

**Proposition 4.6** $(\widetilde{\mathcal{P}}_k, \lambda)$ *is an efficient and stable coalition structure. If* $x^*(\bar{b}) >> 0$ *then* $\mathcal{I}_k = N$.

**Proof.** According to Theorem 4.5, $x^* := x^*(\bar{b}) \in AC(v)$ and thus $v(S) \leq x^*(S)$ for all $S \subseteq N$. On the other hand, Proposition 4.2 implies that $\omega(\bar{b}) = \sum_{S\in\mathcal{N}}\tau_k^*(\bar{b})(S)(v(S)-x^*(S)) = 0$ and thus $\tau_k^*(\bar{b})(S) > 0$ only if $v(S) = x^*(S)$, which proves that $\mathcal{P}_k \subseteq \mathcal{GC}(x^*)$.

In addition, if $i \notin \mathcal{I}_k$ then, since $x^* \in \operatorname{argmax}_{x\in\bar{b}\Delta_N}\sum_i x_i(\sum_{S\ni i}\tau_k^*(\bar{b})(S))$, it must be that $x_i^* = 0$ and thus $\{i\} \in \mathcal{GC}(x^*)$. This implies that $\widetilde{\mathcal{P}}_k \in \mathcal{GC}(x^*)$ and, since $\widetilde{\mathcal{P}}_k$ is balanced by definition, $(\widetilde{\mathcal{P}}_k, \lambda)$ is an efficient and thus stable overlapping coalition structure. Moreover, if $x^* >> 0$ then agent $X$'s maximization problem implies that $\mathcal{I}_k = N$ and thus $\mathcal{P}_k = \widetilde{\mathcal{P}}_k$. ■

There is a simple one-to-one and onto relationship between the set of efficient overlapping coalition structures as defined in Section 2 and the probabilistic coalition structures that assign each player in $N$ equal probability of participating in a coalition. Indeed, if $(\mathcal{B}, \lambda)$ is an overlapping coalition structure, then this can be mapped into a probabilistic coalition structure $(\mathcal{B}, \pi)$ in which each coalition $S$ have a probability $\pi_S := \frac{\lambda_S}{\Lambda}$ of forming, where $\Lambda := \sum_{S\in\mathcal{B}}\lambda_S > 0$. Reciprocally, if $(\mathcal{B}, \pi)$ is a probabilistic coalition structure with $\pi \geq 0$, $\sum_{S\in\mathcal{B}}\pi_S = 1$ and $\sum_{S\ni i}\pi_S = \sum_{S\ni j}\pi_S$ for every $i, j \in N$, $i \neq j$, then $(\mathcal{B}, \lambda)$ is an overlapping coalitions structure, where $\lambda_S := \frac{\pi_S}{\Pi}$, with $\Pi := \sum_{S\ni i}\pi_S$. The results above show that the equilibrium family $\mathcal{P}_k$ is exactly the family of coalitions that would be selected with a positive probability by a social planner who is concerned with maximizing the expected total surplus and who is fair to the players, in the sense of giving each of them an equal probability of participating in a formed coalition.

## 5 Concluding Remarks

This paper has proposed a generalization of the bargaining procedures leading to outcomes in the core and investigated their predictions in case the formation of the grand coalition is not efficient. Even though coalition formation is endogenous, the core, when non-empty, still arises as the unique equilibrium outcome. In different contexts, Morelli and Montero (2003), Sun, Trockel, and Yang (2008), and Zhou (1994) discuss the need to determine the payoffs of a cooperative game without the assumption that a given coalition structure (e.g., the grand coalition) will form. The premise of those papers and ours is that the payoff and coalition formation processes should occur simultaneously and take feedback from each other. Our two mechanisms exploit this idea and deliver, as their equilibrium outcomes, the aspiration core allocations and their supporting family of coalitions as the only efficient and stable overlapping (or probabilistic) coalition structures.

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