Core Extensions for Non-balanced TU-Games*

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Abstract

A family of core extensions for cooperative TU-games is introduced. These solution concepts are non-empty when applied to non-balanced games yet coincide with the core whenever the core is non-empty. The extensions suggest how an exogenous regulator can sustain a stable and efficient outcome, financing a subsidy via individual taxes. Economic and geometric properties of the solution concepts are studied. When taxes are proportional, the proportional prenucleolus is proposed as a single-valued selection device. An application of these concepts to the decentralization of a public goods economy is discussed.

JEL Classification C71, H21, H41

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1 Introduction

The core is by far the most frequently applied cooperative solution concept. Various fields including economics, political science, and operations research have successfully employed the concept. Nevertheless, numerous real-world situations translate into models where the core is empty and thus, not applicable. This paper, based on the intuitive principles that motivate the original concept, provides an alternative to deal with empty-core games.

Although various standard solution concepts are never empty, they fail to coincide with the core when it is non-empty. Inclusion of the core (e.g. in the bargaining set and its numerous variations (Aumann and Maschler, 1964)) and non-empty intersection with the core (e.g. of the kernel (Davis and Maschler, 1965)) are well known results. In this paper we extend the core in such a way that, we argue, has considerable potential for applications. In particular, our concept is always non-empty, yet coincides with the core whenever the core is non-empty.

The main concern when the core is empty is that, even when overall cooperation is the only option to be efficient, agents find it in their best interest to form smaller coalitions. As emphasized by many authors, the lack of stability of the grand coalition provides a powerful argument for some form of regulation (Moulin, 1995). We suggest here a method to restore efficiency with a minimum amount of intervention. The concept is based on a corrective measure frequently used by governments: a subsidy financed by taxes.

A subsidy to the grand coalition is used to induce full cooperation and the necessary funds are raised by taxing individual agents. The subsidy and taxes are chosen so that every coalition's *after-tax* excess with respect to a proposed efficient allocation is non-positive, and thus no blocking coalition can form. Alternatively, our tax-financed subsidy arrangement can also be seen as a system of transfers that members of a community may agree on, in order to induce cooperation. Every efficient allocation can be "stabilized" through a large enough tax-financed subsidy. The *extended core* is the set of those efficient allocations that require a *minimal* subsidy.

To better illustrate the concept consider the following example taken from the literature on indivisible goods, an environment that often leads to empty-core examples (Zhao, 2000). Three travelers, named 1, 2, and 3, are willing to pay \$700, \$1000 and \$1200 respectively for a trip from New York to Los Angeles. A three-person jet charges \$1000 per trip while a smaller two-person jet charges only \$600. For each two-player coalition the best option is to take the smaller plane. The surpluses generated by $\{1,2\},\{1,3\}$ and $\{2,3\}$ in this case are \$1100, \$1300 and \$1600, respectively. The *maximum* total surplus, in the amount of \$1900, is obtained if the three travelers use the bigger plane. However, no distribution of \$1900 prevents deviation by some two-player coalition.

Suppose the government decides to give a subsidy of \$100 to the efficient provider, allowing the bigger jet to charge only \$900. Total surplus increases to \$2000 and the only core payoff vector after the subsidy is $(400, 700, 900)$. It arises when each traveler is charged \$300 for the ticket. To finance its \$100 expenditure, the government charges consumers, for example, 5% of their payoffs, leaving $(380, 665, 855)$ as the proposed solution. After taxes, no coalition has incentives to deviate from the proposed solution. Moreover, \$100 is the smallest subsidy needed to achieve stability.

Various generalizations of the core proposed in the literature are based on a similar idea of restoring stability through some form of "taxation". The strong and weak ε-cores of Shapley and Shubik (1966) are two such well-known examples, obtained by imposing a fixed and, respectively, per-capita cost to coalition formation. While Maschler et al (1979) do examine cases in which ε takes on large and even negative values, most of the related literature focuses on finding necessary conditions, usually regarding the number of agents, so that ε-cores are non-empty when ε is arbitrarily close to zero.[1] This kind of asymptotic result does not address the issue of what to do when faced with an empty core.

The main difference between this stream of literature and our approach lies in the way coalitions are "taxed". Taxation in the ε-core approach targets coalitions as a whole, rather than their individual members. For each coalition, the cost it incurs depends solely on its characteristics (such as size or total worth) but it is independent of the actual allocations of its members. This is an acceptable assumption as long as the coalitions' loss of value is interpreted as a cost to coalition formation or a lump sum tax to its members. Various other taxing instruments are however excluded by this approach. By contrast, our extended core solution relies on taxing individual players based on their particular payoff. Therefore, a (blocking) coalition may be taxed differently depending on the payoff its members receive. A system that taxes individuals based on their particular allocations rather than their association to a particular coalition comes closer, in our opinion, to the actual tax instruments frequently used by governments.

The paper is structured as follows. After establishing notation and definitions in Section 2, Section 3 shows that our concept and its refinements preserve several of the core's geometric and economic properties. Section 4 compares the extended core with other core-related solution concepts. Section 5 illustrates an application of the extended core to the decentralization of public good economies with minimal taxes and subsidies, and Section 6 concludes.

[1] A classical example is the seminal paper by Wooders and Zame (1984). For a review of the literature on the subject the reader is referred to Kannai (1992).

2 Definitions

We start by introducing the notation and basic definitions related to cooperative games. Let N be any finite and non-empty set of *agents*. Without loss of generality, let $N = \{1, 2, \ldots, n\}$. Define the *power set* of N as $2^N = \{S \mid S \subseteq N\}$. A *transferable utility (TU) game* on N is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Let Γ_+^N denote the family of TU-games on N with $v(S) \geq 0$ for all $S \subseteq N$. Similarly, let Γ_{++}^N denote the family of TU-games on N with $v(S) > 0$ for all $S \subseteq N$.

Any $S \in 2^N \setminus \{\emptyset\}$ is called a *coalition* and its *worth* is represented by $v(S)$. $|S|$ denotes the cardinality of S. A possible outcome for $v \in \Gamma_+^N$ is represented by a *payoff vector* $x \in \mathbb{R}^n$ that assigns agent i a payoff x_i. Given $x \in \mathbb{R}^n$ and $S \subseteq N$, let $x^S \in \mathbb{R}^{|S|}$ denote the restriction of x to S and $x(S) = \sum_{i \in S} x_i$, with the agreement that $x(\emptyset) = 0$. A payoff vector is *efficient* if $x(N) = v(N)$. Let $\mathbb{X}(v)$ be the set of efficient payoff vectors for game v. Given $v \in \Gamma_+^N$, $x \in \mathbb{X}(v)$ and $S \subseteq N$, let $e_v(S, x) = v(S) - x(S)$ denote the excess function. Coalition S is able to *improve upon* outcome $x \in \mathbb{R}^n$ if $e_v(S, x) > 0$. The *core* of a game is the set of efficient outcomes that cannot be improved upon by any coalition, i.e., $C(v) = \{x \in \mathbb{X}(v) \mid e_v(S, x) \leq 0 \ \ \forall S \subseteq N\}$. A TU-game with a non-empty core is said to be *balanced*.

Many variants of the core are studied in the literature. Given $\varepsilon \in \mathbb{R}$ and $v \in \Gamma_+^N$, the *strong ε-core* of v is $\varepsilon\text{-}C_s(v) = \{x \in \mathbb{X}(v) \mid e_v(S, x) \leq \varepsilon, \ \forall S \subsetneq N, S \neq \emptyset\}$; the *weak ε-core* of v is $\varepsilon\text{-}C_w(v) = \{x \in \mathbb{X}(v) \mid e_v(S, x) \leq |S|\varepsilon, \ \forall S \subsetneq N\}$. The strong and weak ε-cores have been studied by Shapley and Shubik (1966). Given $v \in \Gamma_+^N$, the *least core* (Maschler et al, 1979) of v is the smallest non-empty strong ε-core, i.e., $LC(v) = \bigcap_{\varepsilon \in \mathbb{R}} \{\varepsilon\text{-}C_s(v) \mid \varepsilon\text{-}C_s(v) \neq \emptyset\}$. Similarly, define the *weak least core* of v as the smallest non-empty weak ε-core, i.e., $LC_w(v) = \bigcap_{\varepsilon \in \mathbb{R}} \{\varepsilon\text{-}C_w(v) \mid \varepsilon\text{-}C_w(v) \neq \emptyset\}$.

Several single-valued selections from the core and its variants have been proposed. Given $v \in \Gamma_+^N$ and $x \in \mathbb{X}(v)$, let $\theta(v, x)$ be the vector that rearranges the coordinates of $(e_v(S, x))_{S \subsetneq N}$ in non-increasing order. Let $\preceq_L$ denote the lexicographic order. The *prenucleolus* (Schmeidler, 1969) of v is the set $\mathcal{N}(v) = \{x \in \mathbb{X}(v) \mid \forall y \in \mathbb{X}(v), \ \theta(v, x) \preceq_L \theta(v, y)\}$; the *per-capita prenucleolus* (Grotte, 2000) of v is denoted as $\overline{\mathcal{N}}(v)$ and defined by replacing $e_v(S, x)$ with $\overline{e}_v(S, x) = e_v(S, x)/|S|$; the *proportional prenucleolus* (Young et al, 1982) of $v \in \Gamma_{++}^N$, denoted by $\hat{\mathcal{N}}(v)$, is defined by replacing $e_v(S, x)$ with $\hat{e}_v(S, x) = e_v(S, x)/v(S)$.

Given $v \in \Gamma_+^N$ and $x \in \mathbb{X}(v)$, let $\theta_+(v, x)$ be the vector that rearranges the coordinates of $(\max\{e_v(S, x), 0\})_{S \subsetneq N}$ in non-increasing order. The *positive core* (Orshan and Sudhölter, 2001) of v is the set $\mathcal{C}_+(v) = \{x \in \mathbb{X}(v) \mid \forall y \in \mathbb{X}(v), \theta_+(v, x) \preceq_L \theta_+(v, y)\}$.

Given $v \in \Gamma_+^N$ and $T \geq 0$, define the *T-expansion* of v as $v^T \in \Gamma_+^N$ satisfying:

(a) $v^T(S) = v(S)$ for every $S \subsetneq N$, and (b) $v^T(N) = v(N) + T$. Given $v \in \Gamma_+^N$ and $x \in \mathbb{X}(v)$, define $T(v,x) = \min\{t(N) \mid t \in \mathbb{R}_+^n \text{ and } e_v(S,x) \leq t(S)\ \forall S \subseteq N\}$. $T(v,x)$ represents the minimum amount of resources needed to prevent coalitional deviations from x. It measures thus how far the payoff vector x is from being in the core.[2] Let $\overline{T}(v) = \min_{x \in \mathbb{X}(v)}\{T(v,x)\} = \min\{T \geq 0 \mid C(v^T) \neq \emptyset\}$ be the minimum amount of resources needed to stabilize *some* efficient payoff vector and denote by $\bar{v} = v^{\overline{T}(v)}$ the *minimal balanced expansion* of v.[3] Clearly, if the game has a non-empty core, $\overline{T}(v) = 0$ and for every $x \in C(v)$, $T(v,x) = \overline{T}(v) = 0$. If the core of the game is empty then $\overline{T}(v) > 0$ and the payoff vectors x for which $T(v,x) = \overline{T}(v)$ are those "closest" to prevent coalitional deviations. Given $v \in \Gamma_+^N$ we define the *extended core* of v as the set

$$EC(v) = \{x \in \mathbb{X}(v) \mid T(v,x) = \overline{T}(v)\}.$$

By definition, the extended core is always non-empty and it coincides with the core whenever the latter is non-empty. The extended core also inherits the fundamental property of the core, stability to coalitional deviations, once individual taxes are imposed on the agents. Indeed, for every $x \in EC(v)$, there exists a tax vector $t \in \mathbb{R}_+^n$ such that $x(S) \geq v(S) - t(S)$ for all $S \subseteq N$ and $\sum_{i \in N} t_i = \overline{T}(v)$.

The following examples illustrate the result of applying the extended core to TU-games with two and three players.

Example 1. *For the two-player TU-game v given by $v(\{1,2\}) = 1$, $v(\{1\}) = 0.8$ and $v(\{2\}) = 0.4$, $EC(v)$ is the segment with endpoints at $(0.6, 0.4)$ and $(0.8, 0.2)$.*

Example 2. *Consider a three-player TU-game v such that $v(\{1,2,3\}) = 8$, $v(\{1,2\}) = 7$, $v(\{1,3\}) = v(\{2,3\}) = 3, v(\{1\}) = v(\{2\}) = 0$, and $v(\{3\}) = 3$. Then $\overline{T}(v) = 2$ and $EC(v)$ is the trapezoid with vertices at $(7,-2,3)$, $(-2,7,3)$, $(7,0,1)$ and $(0,7,1)$.*

As illustrated by the previous examples, the extended core of a game can be a relatively large set. We now define various refinements of the extended core using the concept of a tax problem, also known as a claims problem. A survey of the literature on tax problems can be found in Thomson (2003).

Let $T \in \mathbb{R}_+$ and let $c = (c_i)_{i \in N} \in \mathbb{R}_+^n$ be a vector of claims. A pair (c,T) is called a *tax problem* whenever $\sum_{i \in N} c_i \geq T$. A *tax rule* is a function $t(\cdot,\cdot)$ that associates with each tax problem (c,T) a non-negative *tax vector*

[2] Other criteria for measuring this "distance" have been proposed by Kwon and Yu (1977).

[3] Whenever the core is empty, $\bar{v}(N)$ coincides with the minimum no-blocking payoff defined in Zhao (2001).

$(t_i)_{i\in N} \in \mathbb{R}^n_+$ such that $\sum_{i\in N} t_i = T$.[4] Given $v \in \Gamma^N_+$ and a tax rule $t(\cdot,\cdot)$ define the *t-extension of the core* of v as

$$EC^t(v) = \{\bar{x} - t(\bar{x}, \overline{T}(v)) \mid \bar{x} \in C(\bar{v})\}.$$

These solution concepts depend on what kind of rule is applied to distribute the tax burden. Every tax rule generates a refinement of the extended core which is a non-empty core extension. Although considerably smaller than the extended core when applied to empty-core games, the refinements are not, in general, single-valued. We analyze in the following several well-known tax rules and the core extensions they generate.

Let (c, T) be a tax problem.

1. The *proportional rule* $P(c,T)$ assigns to every $i \in N$ a tax $t_i = \lambda c_i$ where $\lambda = \frac{T}{\sum_{i\in N} c_i}$. The corresponding core extension is denoted by $EC^P(v)$.

2. The *equal awards rule* $EA(c,T)$ assigns to every $i \in N$ a tax $t_i = \frac{T}{n}$. The corresponding core extension is denoted by $EC^{EA}(v)$.

3. The *constrained equal awards rule* $CEA(c,T)$ assigns to every $i \in N$ a tax $t_i = \min\{c_i, \lambda\}$ where λ is a positive real number such that $\sum_{i\in N} t_i = T$. The corresponding core extension is denoted by $EC^{CEA}(v)$.

4. The *constrained equal losses rule* $CEL(c,T)$ assigns to every $i \in N$ a tax $t_i = \max\{0, c_i - \lambda\}$ where λ is a positive real number such that $\sum_{i\in N} t_i = T$. The corresponding core extension is denoted by $EC^{CEL}(v)$.

5. The *Talmud rule* $TAL(c,T)$ assigns to every $i \in N$: (a) If $\sum_{i\in N} \frac{c_i}{2} \geq T$, $t_i = \min\{\frac{c_i}{2}, \lambda\}$ where λ is a positive real number such that $\sum_{i\in N} t_i = T$. (b) If $\sum_{i\in N} \frac{c_i}{2} \leq T$, $t_i = c_i - \min\{\frac{c_i}{2}, \lambda\}$ where λ is a positive real number such that $\sum_{i\in N} t_i = T$. The corresponding core extension is denoted by $EC^{TAL}(v)$.

Figures 1 and 2 illustrate these core extensions for the previous examples.

3 Properties of the extended cores

Aside from being non-empty, efficient, and generalizing group rationality over non-balanced TU-games, the extended core and its refinements preserve a number of appealing features from their predecessor. We start by analyzing the geometric properties of our concepts.

[4]The standard definition of a tax rule (or a division rule for claims problems) includes the additional requirement that $t_i \leq c_i$ for every $i \in N$.

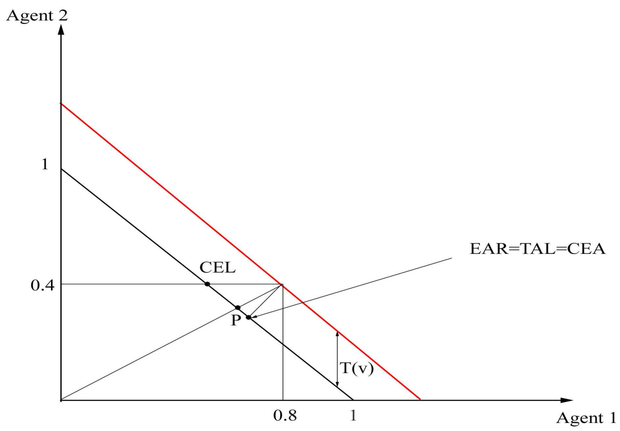

Figure 1: Tax-extended cores (Example 1)

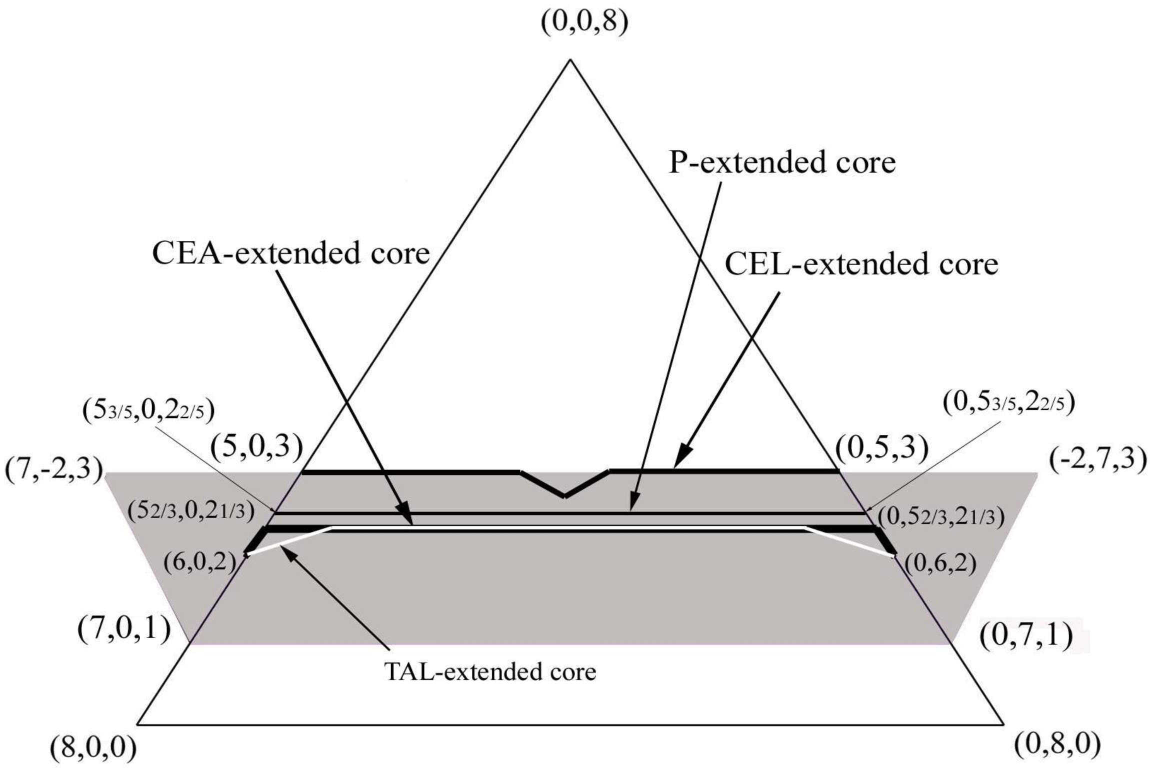

Figure 2: Tax-extended cores (Example 2)

Proposition 1. *The extended core is a continuous and convex-valued correspondence. Tax-extensions generated by continuous tax rules are also continuous, but not necessarily convex-valued. The proportional and equal-awards* *extensions of the core are convex-valued.*

Proof. Composition and addition preserve the continuity of compact-valued correspondences (Border, 1985). Since the core correspondence is continuous (Lucchetti et al, 1987), $EC(v)$ is continuous if the mapping $v \mapsto \bar{v}$ is continuous. Since $v(S) = \bar{v}(S)$ for every $S \subsetneq N$, this is equivalent to proving that $\tau : \Gamma^N_+ \to \mathbb{R}_+, \tau(v) = \bar{v}(N)$ is continuous. The function τ can be written as $\tau(v) = \min_{x \in \mathbb{R}^n}\{x(N) \mid x(S) \geq v(S), \forall S \subseteq N\}$. Its continuity is an immediate consequence of Berge's maximum principle (Border, 1985) once we observe that the domain of the payoff vector x can be restricted to a (sufficiently large) compact subset of $\mathbb{R}^n$ without affecting the solution of the minimization problem.

The arguments given in the previous paragraph imply that the correspondence that maps $v \in \Gamma^N_+$ to the pair $(C(\bar{v}), \overline{T}(v))$ is continuous. The function that maps the pair $(\bar{x}, \overline{T}(v))$ to the vector $\bar{x} - t(\bar{x}, \overline{T}(v))$ is also continuous for every continuous tax rule $t(\cdot,\cdot)$. The t-extension of the core, as the composition of the two, must also be continuous.

Since the core is a convex set and $EC(v) = C(\overline{v}) - \overline{T}(v) \cdot \Delta^{n-1}$, where Δ^{n-1} is the $(n-1)$-dimensional unit simplex, the extended core is convex-valued as well. Finally, the proportional and equal-awards extensions correspond to a contraction and, respectively, a translation of the convex set $C(\bar{v})$. Consequently they are also convex-valued.[5] $\square$

A correspondence $\sigma : \Gamma^N_+ \rightrightarrows \mathbb{R}^n$ is a *solution concept* if for every $v \in \Gamma^N_+$, $\sigma(v) \subseteq \mathbb{X}(v)$. It is *covariant under strategic equivalence* if for every $\alpha \in \mathbb{R}_+$, every $\beta \in \mathbb{R}^n_+$, and every $v \in \Gamma^N_+$ it satisfies $\sigma(\alpha v + \beta) = \alpha\sigma(v) + \beta$, where $\alpha v + \beta \in \Gamma^N_+$ is defined by $(\alpha v + \beta)(S) = \alpha v(S) + \beta(S)$ for any coalition $S \subseteq N$. It is well-known that the core is a covariant solution concept. It is clear from the definition that the extended core inherits this property. In general, with the exception of EC^{EA}, tax extensions of the core are *not* covariant.

We say that a tax rule $t(\cdot,\cdot)$ is *homogeneous* if for any $\alpha \in \mathbb{R}_+$ it satisfies $t(\alpha c, \alpha T) = \alpha t(c, T)$. The five tax rules we consider are homogeneous. It is clear that for this type of tax rules, the t-extensions of the core are well behaved with respect to multiplication by some $\alpha > 0$. However, tax-extensions are not, in general, covariant with respect to translations along a positive vector β. It turns out that taxing each individual on his/her *net* (instead of total) payoff generates a covariant core extension. In order to ensure the positivity of the sum of the claims in the tax problem, for the moment we restrict our domain to $\Gamma^N_* = \{v \in \Gamma^N_+ \mid \sum_i v(\{i\}) < v(N)\}$.[6] Given $v \in \Gamma^N_*$ and a homogeneous tax rule $t(\cdot,\cdot)$, define $EC^t_*(v) = \{\bar{x} - t((\bar{x}_i - v(\{i\}))_{i\in N}, \overline{T}(v)) \mid \bar{x} \in C(\bar{v})\}$.

[5] Figure 2 includes examples of tax-extensions that are not, in general, convex-valued.

[6] Notice that any two-player game with an empty core is not in this domain.

It is clear that EC^t_* is covariant. The "least-tax-core" proposed by Tijs and Driessen (1986), coincides with EC^P_*.

Let $\Pi(N)$ denote the set of permutations of N. Given $\pi \in \Pi(N)$, $x \in \mathbb{R}^n$ and $v \in \Gamma^N_+$, define $\pi x \in \mathbb{R}^n$ by $(\pi x)_i = x_{\pi(i)}$ for every $i \in N$, and $\pi v \in \Gamma^N_+$ by $(\pi v)(S) = v(\pi(S))$ for every $S \subseteq N$. A solution concept σ for Γ^N_+ is *anonymous* if $\sigma(\pi v) = \pi(\sigma(v))$ for every $\pi \in \Pi(N)$ and every $v \in \Gamma^N_+$. Clearly, the extended core and its five tax refinements are anonymous.

The characterization of the core solution concept using different families of axioms has been analyzed, among others, by Peleg (1986), Hwang and Sudhölter (2000) and Tadenuma (1992). It is of particular interest to verify if our core extensions preserve the properties analyzed by this branch of the literature. We focus here on consistency, super-additivity, individual rationality and monotonicity.

Let $v \in \Gamma^N_+$, $x \in \mathbb{X}(v)$ and $S \subsetneq N$. If coalition $N \setminus S$ decides to leave the original game, agents in S are left playing a *reduced game* $v^S_x : 2^S \to \mathbb{R}$ (which can be defined in various ways). We say that the (tax) extended core is *consistent* if for every $v \in \Gamma^N_+$, every $x \in EC(v)$ (resp. $x \in EC^t(v)$), and every $\emptyset \neq S \subsetneq N$, $v^S_x \in \Gamma^S_+$ and $x^S \in EC(v^S_x)$ (resp. $x^S \in EC^t(v^S_x)$). The most commonly used versions of reduced games are the max-reduced game (Davis and Maschler, 1965) and the complement-reduced game (Moulin, 1985).

Given $v \in \Gamma^N_+$, $x \in \mathbb{X}(v)$, and $S \subsetneq N$ define the *max-reduced game* relative to S at x as $v^S_{x,DM} : 2^S \to \mathbb{R}$, such that

$$v^S_{x,DM}(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ v(N) - x(N \setminus S) & \text{if } R = S, \\ \max_{Q \subseteq N \setminus S}(v(R \cup Q) - x(Q)) & \text{if } R \subsetneq S, R \neq \emptyset. \end{cases}$$

Define also the *complement-reduced game* relative to S at x as $v^S_{x,M} : 2^S \to \mathbb{R}$, such that

$$v^S_{x,M}(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ v(R \cup (N \setminus S)) - x(N \setminus S) & \text{if } R \subseteq S, R \neq \emptyset. \end{cases}$$

The following two examples show that none of the tax extensions of the core satisfies either type of consistency.

Example 3. *Consider a TU-game in which $N = \{1, 2, 3, 4\}$, $v(N) = 10$, $v(S) = 8$ if $|S| = 3$, $v(\{1, 2\}) = v(\{3, 4\}) = 5.4$, $v(S) = 2.5$ if $|S| = 2$ and $S \neq \{1, 2\}$ and $S \neq \{3, 4\}$, and $v(S) = 0$ for any other S. Then $\overline{T}(v) = 0.8$ and $(2.7, 2.7, 2.7, 2.7) \in C(\bar{v}) = \{(a, 5.4 - a, b, 5.4 - b) \mid a, b \in [2.6, 2.8]\}$. This implies that $y = (2.5, 2.5, 2.5, 2.5)$ is an element of all the tax-extensions we study. The max- and complement-reduced games with respect to $\{1, 2, 3\}$ at y coincide with the following game w: $w(S) = 7.5$ if $|S| = 3$, $w(S) = 5.5$ if*

$|S| = 2$, $w(\{3\}) = 2.9$ *and* $w(S) = 0$ *otherwise. Routine verification shows that* $(2.5, 2.5, 2.5) \in EC^{CEL}(w)$*, but is not included in* $EC^{P}(w)$*,* $EC^{EA}(w)$*,* $EC^{CEA}(w)$*, or* $EC^{TAL}(w)$*. Notice that* $\overline{T}(w) = 0.9$ *and* $C(\bar{w}) = \{(a, 5.5 - a, 2.9) \mid a \in [2.6, 2.9]\}$*.*

Example 4. *Consider a TU-game in which* $N = \{1, 2, 3\}$*,* $v(N) = 10$*,* $v(\{1, 2\}) = 7$*,* $v(\{1, 3\}) = 8$*,* $v(\{2, 3\}) = 9$*, and* $v(S) = 0$ *otherwise. Then* $\overline{T}(v) = 2$ *and* $C(\bar{v}) = \{(3, 4, 5)\}$*. This implies that* $EC^{CEL}(v) = \{(3, 3.5, 3.5)\}$*. Both versions of reduced game with respect to* $\{1, 2\}$ *at* $(3, 3.5, 3.5)$ *coincide with the following game* w*:* $w(\{1, 2\}) = 6.5$*,* $w(\{1\}) = 4.5$*, and* $w(\{2\}) = 5.5$*. Still,* $EC^{CEL}(w) = \{(3.25, 3.25)\}$*, contradicting consistency.*

A solution concept σ is *super-additive* if for every $v^1, v^2 \in \Gamma^N_+$, every $x^1 \in \sigma(v^1)$ and every $x^2 \in \sigma(v^2)$, $x^1 + x^2 \in \sigma(v^1 + v^2)$. The core itself is super-additive but, as shown in the following example, the extended core and its tax-refinements are not.

Example 5. *Let* $N = \{1, 2\}$ *and two TU-games* v^1 *and* v^2 *defined as follows: If* $|S| = 1$ *then* $v^1(S) = 5$ *and* $v^2(S) = 1$*. Also,* $v^1(N) = 6$ *and* $v^2(N) = 3$*. Then,* $(1, 5) \in EC(v^1), (1, 2) \in EC(v^2)$ *but* $(1, 5) + (1, 2) = (2, 7) \notin EC(v^1 + v^2)$*. Moreover, for every anonymous tax rule* $t(\cdot, \cdot)$*,* $(3, 3) \in EC^t(v^1)$*,* $(1, 2) \in EC^t(v^2)$*, but* $(3, 3) + (1, 2) = (4, 5) \notin EC^t(v^1 + v^2)$*.*

Given $v \in \Gamma^N_+$, $x \in \mathbb{R}^n$ is *individually rational* in v if for every $i \in N$, $x_i \geq v(\{i\})$. A solution concept σ is *individually rational* if $x \in \sigma(v)$ implies that x is individually rational. Clearly the core extensions proposed here do not satisfy individual rationality because they are even defined for games without any individually rational efficient payoffs. Still, it is worth asking if they satisfy this property for games that have a non-empty set of individually rational and efficient payoffs. The answer is again negative as can be checked by reviewing Example 2, in which the extended core and its refinements violate individual rationality for player 3.

Monotonicity is a property stating that if the worth of a coalition increases then, other things equal, the payoff to its agents should not decrease. The question can be raised of whether a tax-extension of the core, if single-valued, satisfies monotonicity. The answer to this question is negative due to a well-known result by Young (1985) which implies that no core selection device is monotonic if $n \geq 5$.

4 Relation to Other Solution Concepts

This section compares our extended cores to the ε-core (Shapley and Shubik, 1966) and other related solution concepts. While all these concepts are based on some form of coalition taxation, they differ in their design of taxes. The ε-cores (and related concepts) assume a constant tax per coalition or member. By contrast, the extended core relates taxes to the particular payoff of the coalition.

Figure 3 illustrates various solution concepts applied to Example 2.

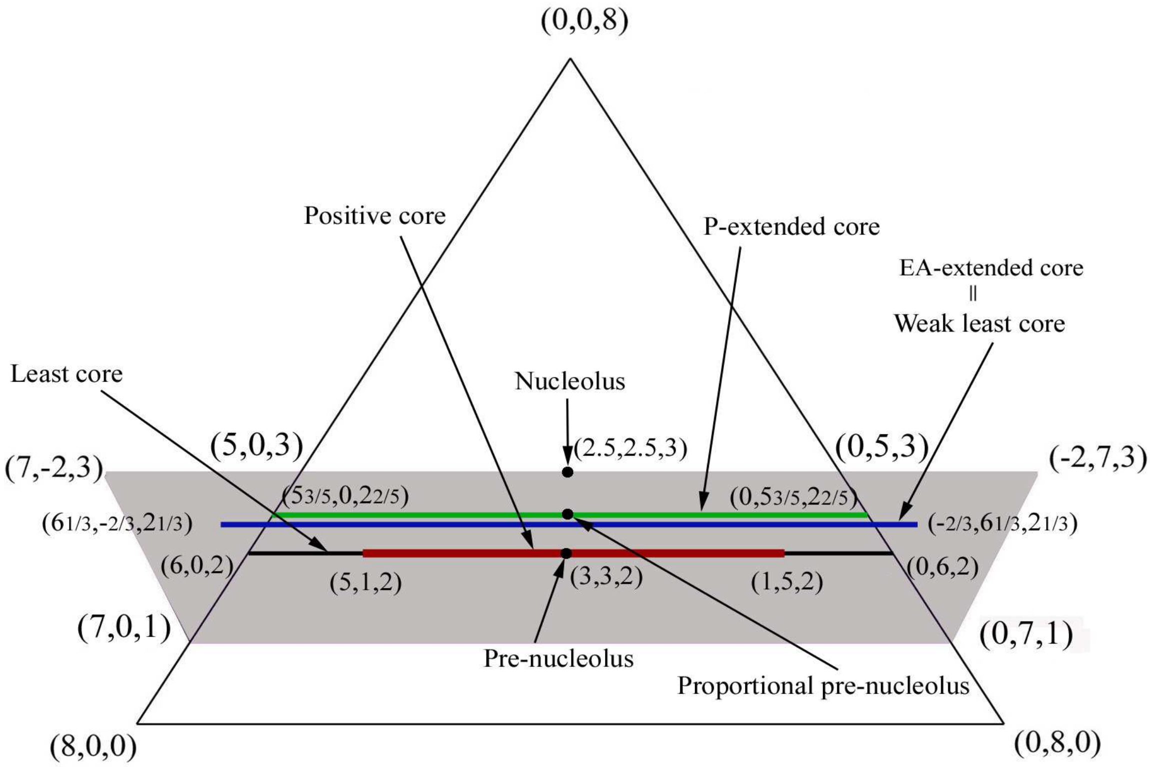

Figure 3: Tax-extended cores and other solution concepts (Example 2)

An analysis of Figure 3 shows that none of our tax-extended cores coincides with the least core. Moreover, the least core is neither a subset nor a super-set of any of our tax-extensions. The prenucleolus and the positive core are not contained in any of the tax-extended cores either.

Due to the individual nature of their taxation, our core extensions have stronger links with the weak ε-core and related solution concepts. Given $v \in \Gamma_+^N$, let $\bar{\varepsilon}(v)$ be the only ε satisfying $LC_w(v) = \varepsilon\text{-}C_w(v)$. Then $\overline{T}(v) = n\bar{\varepsilon}(v)$, and it is clear from the definitions that the EA-extended core coincides with the union of the weak least core and the core, thus containing the per-capita prenucleolus. Since the positive core (Orshan and Sudhölter, 2001) contains the per-capita prenucleolus and is a subset of the weak least core, we have the following list of inclusions: $\overline{\mathcal{N}}(v) \in \mathcal{C}_+(v) \subseteq EC^{EA}(v)$, for every $v \in \Gamma_+^N$.

A similar relation holds between the proportional core extension and the proportional prenucleolus.

Proposition 2. *For every $v \in \Gamma^N_{++}$, $\hat{\mathcal{N}}(v) \in EC^P(v)$.*

Proof. Define $h : \mathbb{X}(\bar{v}) \to \mathbb{X}(v)$ such that for every $y \in \mathbb{X}(\bar{v})$, $h(y) = \frac{v(N)}{\bar{v}(N)}y$. Clearly h is a bijection and $h(C(\bar{v})) = EC^P(v)$. The definition of $\hat{\theta}$ then implies that every $y, z \in \mathbb{X}(\bar{v})$ and every $S \subseteq N$ satisfy $\hat{\theta}_{\bar{v}}(S, y) \succeq_L \hat{\theta}_{\bar{v}}(S, z)$ if and only if $\hat{\theta}_v(S, h(y)) \succeq_L \hat{\theta}_v(S, h(z))$. Therefore $\hat{\mathcal{N}}(v) = h(\hat{\mathcal{N}}(\bar{v})) \in h(C(\bar{v})) = EC^P(v)$, as wanted. $\square$

5 Application: Public Good Economies

The following is an application of the extended core concept to the decentralization of efficient allocations in public good economies. We generalize the characterization of core allocations given in Mas-Colell (1980) for the case of a two-agent economy with an empty core.

Consider an economy $\mathcal{E}$ with 2 agents, one private good, M, and K public goods, $X_1, ..., X_K$. Good M can either be consumed or used for the production of the public goods. The input requirement for producing a vector of quantities $x \in \mathbb{R}^K_+$ of the public goods is $c(x)$, where $c : \mathbb{R}^K_+ \to \mathbb{R}_+$ is continuous, increasing in each argument, strictly convex and such that $c(0) = 0$. Consumer i has an initial endowment $\omega_i \in \mathbb{R}_+$ of the private good and preferences represented by $u_i(m, x) = m + \Phi_i(x)$ with $\Phi_i : \mathbb{R}^K_+ \to \mathbb{R}_+$ continuous, increasing in each argument, strictly quasi-concave and satisfying $\Phi_i(0) = 0$. An *allocation* is a vector $(m_1, m_2, x) \in \mathbb{R}^2_+ \times \mathbb{R}^K_+$. Its associated utility payoff vector is $(v_1, v_2) = (m_1 + \Phi_1(x), m_2 + \Phi_2(x))$. The allocation (m_1, m_2, x) is *feasible* if $m_1 + m_2 + c(x) \leq \omega_1 + \omega_2$. It is efficient if it is feasible and $x = \overline{x} = \text{argmax}_{z \in \mathbb{R}^K_+} \{\Phi_1(z) + \Phi_2(z) - c(z) \mid c(z) \leq \omega_1 + \omega_2\}$.

For every $i = 1, 2$ let $x^*_i = \text{argmax}_{x \in \mathbb{R}^K_+} \{\Phi_i(x) - c(x) \mid c(x) \leq \omega_i\}$ and $v^*_i = \omega_i + \Phi_i(x^*_i) - c(x^*_i)$. The TU-game associated with this economy is described by the characteristic function $v : 2^{\{1,2\}} \to \mathbb{R}_+$ where: $v(\{i\}) = v^*_i$ for $i = 1, 2$ and $v(\{1, 2\}) = \omega_1 + \omega_2 + \Phi_1(\overline{x}) + \Phi_2(\overline{x}) - c(\overline{x})$. The game has an empty core if and only if $\overline{T}(v) = v(\{1\}) + v(\{2\}) - v(\{1, 2\}) > 0$. To simplify exposition we are going to assume that $\overline{T}(v) \leq \omega_i - c(x^*_i)$ for $i = 1, 2$.

We define a *valuation system* as a vector $p = (p_1, p_2)$ of upper semicontinuous, non-negative functions $p_i : \mathbb{R}^K_+ \to \mathbb{R}_+$. Valuation systems generalize Lindahl prices and can be interpreted as personalized, non-linear prices. An allocation $(\widetilde{m}_1, \widetilde{m}_2, \widetilde{x})$, a valuation system $\widetilde{p} = (\tilde{p}_1, \widetilde{p}_2)$, and profit shares $\widetilde{\Pi}_i \in \mathbb{R}_+$ for $i = 1, 2$ constitute a *valuation equilibrium* for economy $\mathcal{E}$ if: **(a)** $(\widetilde{m}_i, \tilde{x}) \in \text{argmax}\{m_i + \Phi_i(x) | m_i + \widetilde{p}_i(x) \leq \omega_i + \widetilde{\Pi}_i\}$ for $i = 1, 2$. **(b)**

$\tilde{x} \in \text{argmax}\{\widetilde{p}_1(x)+\widetilde{p}_2(x)-c(x)|x \in \mathbb{R}^K_+\}$. **(c)** $\widetilde{\Pi}_1+\widetilde{\Pi}_2 = \widetilde{p}_1(\tilde{x})+\widetilde{p}_2(\tilde{x})-c(\tilde{x})$. **(d)** $\widetilde{m}_1 + \widetilde{m}_2 + c(\widetilde{x}) = \omega_1 + \omega_2$. We let $\widetilde{\Pi} = \widetilde{\Pi}_1 + \widetilde{\Pi}_2$ denote the *public firm's profit* associated with a given valuation equilibrium. Clearly, $\widetilde{\Pi} \geq 0$.

Any efficient allocation can be decentralized as a valuation equilibrium with non-negative profit for the public goods firm (Mas-Colell, 1980). It easily follows from the previous definition that such allocation can also be decentralized as a valuation equilibrium with arbitrarily larger profits. Since equilibrium profits cannot be negative, for every efficient allocation there is a minimum level of profit needed to support it as a valuation equilibrium. Mas-Colell shows that an efficient allocation generates core payoffs if and only if it can be supported as a valuation equilibrium with zero profit for the public firm. We now generalize this result by proving that an efficient allocation generates extended core payoffs if and only if it can be supported as a valuation equilibrium in which firm's profit is equal to $\overline{T}(v)$. Moreover, $\overline{T}(v)$ is the minimum level of profits that can be generated at a valuation equilibrium. Since firm's profits are distributed to consumers, they serve the role of wealth transfers. Our result therefore shows that allocations in the extended core can be decentralized with a minimal system of transfers.

Proposition 3. *At any valuation equilibrium the firm's profit satisfies* $\widetilde{\Pi} \geq \overline{T}(v)$. *Additionally, if* $(\widetilde{m}_1, \widetilde{m}_2, \widetilde{x})$ *is a feasible allocation and* $(\widetilde{v}_1, \widetilde{v}_2)$ *its associated payoff vector then* $(\widetilde{v}_1, \widetilde{v}_2) \in EC(v)$ *if and only if* $(\widetilde{m}_1, \widetilde{m}_2, \widetilde{x})$ *can be supported as a valuation equilibrium with profits* $\widetilde{\Pi} = \overline{T}(v)$.

Proof. Suppose that $(\widetilde{m}_1, \widetilde{m}_2, \widetilde{x})$ can be supported as a valuation equilibrium with valuation system $\widetilde{p}$ and profit shares $(\widetilde{\Pi}_1, \widetilde{\Pi}_2)$, $\widetilde{\Pi}_1 + \widetilde{\Pi}_2 = \widetilde{\Pi}$. As shown in Mas-Colell (1980), the allocation must be efficient, which implies that $\widetilde{x} = \overline{x}$ and $\widetilde{v}_1 + \widetilde{v}_2 = v_1^* + v_2^* - \overline{T}(v)$. If $\widetilde{\Pi} < \overline{T}(v)$ then, for at least some $i \in \{1, 2\}$, $\widetilde{v}_i < v_i^* - \widetilde{\Pi}_{3-i}$. Therefore $(\omega_i - c(x_i^*) - \widetilde{\Pi}_{3-i}, x_i^*)$ cannot be affordable[7] for i and thus $\omega_i - c(x_i^*) - \widetilde{\Pi}_{3-i} + \widetilde{p}_i(x_i^*) > \omega_i + \widetilde{\Pi}_i$ or, equivalently, $\widetilde{p}_i(x_i^*) - c(x_i^*) > \widetilde{\Pi}$. Since $\widetilde{p}_{3-i}(x_i^*) \geq 0$, we obtain $\widetilde{p}_1(x_i^*) + \widetilde{p}_2(x_i^*) - c(x_i^*) > \widetilde{\Pi}$, a contradiction.

Thus $\widetilde{v}_i \geq v_i^* - \widetilde{\Pi}_{3-i}$ for $i = 1, 2$ and $\widetilde{\Pi} \geq \overline{T}(v)$. If equality holds then $\widetilde{v}_i = v_i^* - \widetilde{\Pi}_{3-i}$ for $i = 1, 2$, which proves that $(\widetilde{v}_1, \widetilde{v}_2) \in EC(v)$ with the associated taxes $t_i = \widetilde{\Pi}_{3-i}$, $i = 1, 2$.

Conversely, let $(\widetilde{v}_1, \widetilde{v}_2) \in EC(v)$ and define $t_i = v_i^* - \widetilde{v}_i$, for $i = 1, 2$, as the associated taxes. Clearly, $\widetilde{x} = \overline{x}$ and $t_1 + t_2 = \overline{T}(v)$. Define $\widetilde{p}_i(x) = \max\{0, \overline{T}(v) + \Phi_i(x) - [\Phi_i(x_i^*) - c(x_i^*)]\}$ and $(\widetilde{\Pi}_1, \widetilde{\Pi}_2) = (t_2, t_1)$. It is straightforward to verify that $(\widetilde{m}_i, \overline{x})$ maximizes utility for agent i subject to her budget constraint. Let now x be an arbitrary vector in $\mathbb{R}^K_+$. If $\widetilde{p}_i(x) > 0$ for $i = 1, 2$,

[7] Since $\overline{T}(v) \leq \omega_i - c(x_i^*)$ and $\widetilde{\Pi}_{3-i} \leq \widetilde{\Pi} < \overline{T}(v)$, the allocation $(\omega_i - c(x_i^*) - \widetilde{\Pi}_{3-i}, x_i^*)$ is in i's consumption set.

then $\widetilde{p}_1(x)+\widetilde{p}_2(x)-c(x) = \sum_{i=1}^{2}[\overline{T}(v)+\Phi_i(x)] - \sum_{i=1}^{2}[\Phi_i(x_i^*)-c(x_i^*)]-c(x) \leq \overline{T}(v)$. If, for at least one i, $\widetilde{p}_i(x) = 0$ then, since $\widetilde{p}_i(x) \leq c(x)+\overline{T}(v)$, it follows that $\widetilde{p}_1(x)+\widetilde{p}_2(x)-c(x) \leq \overline{T}(v)$. Moreover, $\widetilde{p}_1(\overline{x})+\widetilde{p}_2(\overline{x})-c(\overline{x}) = \overline{T}(v)$. This shows that $\overline{x}$ maximizes firm's profits under the given valuation system. □

Example 6. *Consider a public good economy with two agents, one private good, M, and two public goods, X_1, X_2. Let $c(x) = (x_1+x_2)^2$, $\omega_i = 1$ and $\Phi_i(x) = x_i^{0.1}x_{3-i}^{0.9}$ for $i = 1,2$. The associated TU-game v has an empty core and $\overline{T}(v) = 0.0110$.*[8] *Consider the efficient allocation $(\widetilde{m}_1, \widetilde{m}_2, \widetilde{x}) = (\frac{7}{8}, \frac{7}{8}, (\frac{1}{4}, \frac{1}{4}))$ with associated payoffs $(\widetilde{v}_1, \widetilde{v}_2) = (1\frac{1}{8}, 1\frac{1}{8})$. Notice that for any of the tax rules $t(\cdot,\cdot)$ we study, $EC^t(v) = \{(1\frac{1}{8}, 1\frac{1}{8})\}$ and $t_1 = t_2 = 0.0055$. Proposition* 3 *guarantees that $\widetilde{p}_i(x) = \max\{0, \Phi_i(x) - 0.1195\}$ for $i = 1,2$ and $(\widetilde{\Pi}_1, \widetilde{\Pi}_2) = (0.0055, 0.0055)$ support the allocation as a valuation equilibrium.*

6 Conclusion

This work helps establish a new perspective towards the study of games with an empty core. The question of how to handle empty-core situations is of great importance because the number of games in which the core cannot be applied is considerable. Still, the issue is frequently avoided. The usual approach is to find enough (and sometimes arbitrary) assumptions so that the core of a game is non-empty. In extreme cases, balancedness is supposed without further justification. The full potential of cooperative-game applications has been limited by focusing on balanced games.

Defining a non-empty core extension is fairly easy. For instance, whenever a game has an empty core one could endow all agents with equal payoffs. However, such a solution concept would not be interesting due to the *ad hoc* rule used to extend the core. Our objective was to show that the extended core (and each of its tax-refinements) is a "natural" extension of the core. We showed that, apart from preserving the stability property of the core (after appropriate taxes are levied), the extended core also inherits the following important properties of the core: continuity, convexity, anonymity and symmetry. Moreover, the proportional (resp. per-capita) prenucleolus is an element of the proportional (resp. equal awards) extended core.

The extended core is a simple, intuitive, and easy-to-compute solution concept, encouraging its application in empty-core environments. While the standard, convex case of a competitive economy generates a balanced cooperative game, many frictions lead to empty-core games that preclude an efficient outcome. We have given here such an example of a multiple public goods economy and showed how our notion of extended core can be used to decentralize

[8] When necessary, we round figures to four decimal places.

particular efficient allocations. Another example, in the context of spatial location of local public goods facilities, is given by Dréze et al (2007). The authors use a notion equivalent to our proportional extended core refinement to estimate the minimum subsidy needed to enforce stability of their equilibrium solution. Economies with externalities, increasing returns to production or oligopolies are other examples of situations in which intervention is needed to restore efficiency. Such environments seem to be a natural place for applying the tools described in this paper.

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