OWNERSHIP STRUCTURE AND EFFICIENCY IN LARGE ECONOMIES

Camelia Bejan * Florin Bidian †‡

September 29, 2010

Abstract

We analyze the limit behavior of sequences of oligopolistic equilibria in which firms follow objectives consistent with their shareholders' interests. We show that convergence to a competitive outcome may fail for some distributions of firms' shares across consumers, and provide a characterization of the class of ownership structures that lead to Walrasian equilibrium allocations in the limit.

JEL classification numbers: D21, D43, D51.
Keywords: Firm's objective, Strategic foundations of competitive equilibrium, Ownership structure.

1 Introduction

Perfectly competitive (or price taking) behavior is believed to arise – and is generally justified in the literature – when the number of economic agents that interact in the market is large, and each agent is small relative to the whole economy. There are, however, examples that show how monopoly profits and inefficient allocations can persist in equilibrium, even with an arbitrarily large number of small, competing agents. In an environment without uncertainty (or with uncertainty but a complete set of contingent securities) this happens if, as the economy grows larger, the sequence of its (oligopolistic) equilibria approaches a *critical* equilibrium point of the limit economy (**?**). The results of this paper uncover yet another possible source of inefficiency in large economies: the firms' ownership structures. If firms pursue their shareholders' interests, the way shares are allocated across consumers plays an important role in achieving competitive behavior in the limit.

*Rice University, MS 22, PO Box 1892, Houston, TX 77251-1892. E-mail: camelia@rice.edu
†University of Minnesota. E-mail: bidi0004@umn.edu
‡We greatly benefited from the discussions with Jan Werner, Beth Allen, Andrew McLennan and Andreu Mas-Colell, as well as the participants of the MathEcon workshop at the University of Minnesota and the Economic Theory Workshop at HEC, Paris, France. We also thank an anonymous referee for very useful comments. All remaining errors are ours.

For a firm that has market power, the choice of a production plan affects shareholders' real wealth in two ways: through the profits it generates (we will call this the *income effect*), and through the change in market prices it induces (we call this the *price effect*). It is well-known that these can be opposite effects (see, for example, **?**, **?**, **?**) and thus the production plan that maximizes firm's profit, under some price normalization,[1] may not maximize the welfare of firm's shareholders.

One would expect a firm's production choice to be consistent with its shareholders' interests but, typically, no production plan will be unanimously supported by all shareholders. We say that a production plan chosen by a firm is compatible with its shareholders' interests (given the production plans chosen by the other firms) if no other production plan makes all shareholders better off (provided that the other firms do not change their plans). Such a production plan is therefore efficient (or Pareto undominated) from the point of view of the firm's shareholders and will be called S-efficient (with S standing for "shareholders"). We are interested in the strategic interaction of a large number of firms whose objective is compatible with their shareholders' interests, in the sense of selecting S-efficient production plans. The Cournot-Nash equilibria of such game played by the firms must then have the property that every firm's equilibrium production plan is S-efficient given the production plans of the others. We call such equilibrium a Cournot S-equilibrium. Although we assume, for simplicity, that the interests of *all* shareholders govern the decisions of a firm, our results also hold under the weaker assumption that a firm's objective is shaped by the interests of a smaller "control group" such as the Board of Directors.

We study the limit behavior of Cournot S-equilibrium production plans of a sequence of private ownership economies and show that, depending on the ownership structure, the equilibria may or may *not* approach a Walrasian equilibrium of the limit economy. Convergence to a Walrasian equilibrium obtains if the fraction of each firm owned by its (controlling) shareholders is bounded away from zero. The result is fairly intuitive. In economies with many competing firms, the price effect of each firm's action on its shareholders' welfare becomes almost negligible. However, if the ownership of a given firm is dispersed among a large number of shareholders, so that each of them holds only a tiny fraction of the firm, the income effect of that firm's choices on their wealth must be negligible as well. Thus the price effect, albeit becoming negligible itself, may still dominate the income effect. As a result, shareholders may disapprove the maximization of profits in arbitrarily large economies. Our results suggest that, while perfect portfolio diversification might be optimal from an investor's point of view (as suggested by CAPM-style models) it may not lead to efficiency economy-wide when firms pursue their shareholders' interests.

One of the major difficulties in studying the limit behavior of a sequence of Cournot S-equilibria is defining a notion of "closeness" on the space of private

[1] Profit maximization is not well-defined in this context unless it is specifically linked to a particular price normalization. For a discussion of this well-known issue the reader is referred to **?** or **?**.

ownership production economies. For the case of pure-exchange, representing an economy as a distribution on the space of agents' characteristics (**?**, **?**) enables the use of weak convergence of measures to define a topology on the space of economies. For a production economy, the space of characteristics must be enlarged to include firms' production sets *and* ownership structure. However, as opposed to preferences, endowments or production sets, an ownership structure is intrinsically related to a space of consumers and a space of firms, and it is not obvious how such ownership structure can be included in a space of characteristics that is agent-independent. Even when restricting attention to economies with a finite number of types of consumers and firms, the separation of ownership from the actual *names* of consumers and firms is difficult, unless one is willing to make very restrictive symmetry assumptions on the ownership structure. A familiar example of such symmetry requirement is that every consumer of type i owns equal shares in all the firms of type j (**?**, **?**). As some of our results show, focusing on such specific ownership structure, one is bound to miss important insights that are revealed only in economies with more general ownership structures.

We construct here a general framework that embeds any private ownership production economy and allows for a natural topological structure, which generalizes other topologies defined in the literature, over more restrictive spaces of economies. We also show that a continuum production economy is a good approximation for a large finite economy, since it can be written as the limit of finite economies.

? proves, in related work, that if a firm maximizes profits (under a specific price normalization), then each shareholder's gain from switching to his most preferred production plan diminishes as the economy grows larger. Thus, Hart's result implies that profit maximization by oligopolistic firms is *approximately* in the best interest of each firm's shareholder if the economy is large enough. His proof depends crucially on the "assumption that all consumers are typical", which is a condition imposed on the controlling agents' (endogenous) wealth. We show that this condition is extremely restrictive, holding only under special circumstances, and proceed by giving a proof of the main result in **?** which dispenses entirely with the assumption that consumers are typical, without adding any other assumptions (Lemma 5.1). Then we use this fact to show that Cournot S-efficient equilibria are approximately profit maximizing in large economies (Theorem 5.6), and to give conditions under which the limit of a sequence of converging Cournot S-efficient equilibria in converging economies is a Walrasian equilibrium of the limit economy (Theorems 5.6, 5.7).

The paper is organized as follows. We start, in section 2, by giving an example that illustrates the main points of the paper. In section 3 we set up a general framework for describing private ownership production economies and show how the standard Arrow-Debreu economies and their replicas can be embedded in this framework. The Cournot S-equilibrium concept and some of its properties in continuum economies are described in section 4. Section 5 defines a topology on the space of private ownership economies and provides conditions on the ownership structure such that approximate and exact convergence of Cournot

S-efficient equilibria to a Walrasian equilibrium obtains. Section 6 concludes.

2 An illustrative example

Let $\mathcal{E}_1$ be an Arrow-Debreu economy with two goods, two consumers and one firm. Consumers have identical preferences over consumption of the two goods, represented by the utility $u : \mathbb{R}^2_+ \to \mathbb{R}$, $u(c_1, c_2) = \ln c_1 + \ln c_2$. The endowment of goods of consumer 1, respectively 2, are $e^1 = (4, 4)$, respectively $e^2 = (4, 2)$. The first consumer is the sole owner of the firm, whose production set is $Y := \{(-\alpha, \alpha) \mid \alpha \in [0, 1]\}$. We will refer to the economy $\mathcal{E}_1$ as the *prototype economy.*

It is assumed throughout that consumers are price takers in all markets while every firm behaves strategically, internalizing the effect of its choices of production plans on the market prices. Unlike the standard Cournot-Walras model (**?**), firms do not maximize profits, but rather choose production plans that are non-dominated from the point of view of their shareholders, taking as given the choices of other firms, but internalizing the effect of its own choice on the equilibrium market prices. We call such equilibria Cournot S-efficient equilibria (or simply Cournot S-equilibria). The term hints to the fact that such a production plan is efficient from the point of view of the shareholders who, while price takers as consumers, are aware of the market power of the firms they own. Cournot S-efficient equilibria are typically different from the standard Cournot-Walras equilibria due to price effects on shareholders' wealth.

Without loss of generality, we normalize prices to lie in the unit simplex.[2] Given a choice of a production plan $(-\alpha, \alpha)$, in the resulting competitive exchange equilibrium, (normalized) prices are $\left(\frac{6+\alpha}{14}, \frac{8-\alpha}{14}\right)$, and first consumer's utility is $v(\alpha) = 2\ln(28 + \alpha(1 - \alpha)) - \ln(6 + \alpha) - \ln(8 - \alpha)$. As the sole owner of the firm, consumer 1 would want the firm to choose a production plan $(-\alpha^*, \alpha^*)$ with $\alpha^* \in [0, 1]$ that maximizes his utility (which is strictly concave in α). The first order conditions show that α^* is the unique solution of the equation $\alpha^3 - 3\alpha^2 - 67\alpha + 20 = 0$ that belongs to $[0, 1]$, that is $\alpha^* \approx 0.3$.

Hence, the unique equilibrium of the prototype economy in which the firm acts strategically in the market but follows an objective that is consistent with its owner's interests corresponds to a production vector $(-\alpha^*, \alpha^*) \approx (-0.3, 0.3)$ and the equilibrium prices $\left(\frac{6+\alpha^*}{14}, \frac{8-\alpha^*}{14}\right) \approx (0.45, 0.55)$.

To study the limit behavior of Cournot S-equilibria we construct sequences of replica economies, in the spirit of **?**. An n-fold replica of $\mathcal{E}_1$, denoted $\mathcal{E}_n$, is an economy with $2n$ consumers and n firms. All firms, indexed by $j \in \{1, ..., n\}$, have the same production set as the firm of the prototype economy. Consumers are indexed by (i, k) with $i \in \{1, 2\}$ and $k \in \{1, .., n\}$. We will refer to i as the *type* and to k as the *name* of the consumer (i, k). Every consumer of type i has the same preferences and endowment of goods as consumer i of the prototype

[2] Our results are independent of this normalization since the objective of each firm is formulated in terms of shareholders' indirect utilities, which only depend on *relative* equilibrium prices and thus are immune to the normalization chosen.

economy. Similarly, a *continuum replica* (also called the *limit* replica) $\mathcal{E}_\infty$, is an economy with a continuum of identical firms and consumers of each type.

Due to log-utilities, the exchange equilibrium prices following a choice of production plans by the firms in any of these replicas do not depend on the way shares are distributed across consumers. For the n-fold replica economy, given a production plan $y = ((-\alpha_j, \alpha_j))_{j=1}^n$, let

$$\kappa(y) := \frac{1}{n} \sum_{j=1}^{n} \alpha_j. \tag{2.1}$$

Simple computations reveal that the unique exchange equilibrium price vector following the choice of production plan y depends only on the average production and it is equal to $\left(\frac{6+\kappa(y)}{14}, \frac{8-\kappa(y)}{14}\right)$. The same formula is valid in the continuum replica economy $\mathcal{E}_\infty$, with a proper reinterpretation of κ, that is,

$$\kappa(y) := \int_{[0,1]} \alpha(j) d\lambda(j), \tag{2.2}$$

where λ is the Lebesgue measure on $[0, 1]$. For a continuum replica, a feasible production plan is a Lebesque measurable function $y : [0, 1] \to Y$.

Cournot S-equilibrium production plans of the n-fold replica economy depend on the ownership structure, which is the deciding factor in whether Cournot S-equilibria of large economies become close to Walrasian equilibria of the limit economy $\mathcal{E}_\infty$. There are various ways to replicate the ownership of firms' shares. We will outline here two different types of ownership structures, and show that they bear very different implications on the issue whether Cournot S-equilibrium allocations approach the competitive allocations of the limit economy.

1. **Concentrated ownership replication.** In this replication, every consumer of type 1 is the sole owner of the firm with the same name (i.e., consumer $(1, j)$ is the sole owner of firm j), and all consumers of type 2 have no firm ownership. We denote (finite and continuum) replicas bearing this ownership structure by $\mathcal{E}_n^c$ and $\mathcal{E}_\infty^c$.

   Following a choice $y = ((-\alpha_j, \alpha_j))_{j=1}^n \in \mathbb{R}^{2n}$ of production plans by the firms, the wealth and utility of consumer $(1, j)$ in such replica are given by:

$$w(\kappa(y), \alpha_j) = \frac{1}{7} \left(28 + \alpha_j(1 - \kappa(y))\right), \tag{2.3}$$

$$V(\kappa(y), \alpha_j) = 2 \ln \left(28 + \alpha_j(1 - \kappa(y))\right) - \ln[(6 + \kappa(y))(8 - \kappa(y)]). \tag{2.4}$$

   Thus, in accordance with its owner's preferences, each firm chooses $\alpha \in [0, 1]$ to maximize $V(\kappa, \alpha)$. Since the problem has a unique solution, Cournot S-equilibria of the economy $\mathcal{E}_n^c$ must be symmetric, i.e., $\alpha_j = \kappa$ for $j = 1, .., n$. The first order condition implies that κ satisfies

$$\frac{2\kappa - 2}{\kappa^2 - \kappa - 28} = \frac{1}{n} \left(\frac{1}{\kappa - 8} + \frac{1}{\kappa + 6} - \frac{2\kappa}{\kappa^2 - \kappa - 28} \right). \tag{2.5}$$

According to the implicit function theorem, the solution of (2.5), denoted $\kappa(1/n)$, is a continuous function of $1/n$, hence when $n \to \infty$, $\kappa(1/n)$ converges to the solution of $(2\kappa - 2)/(\kappa^2 - \kappa - 28) = 0$, which is $\kappa^* = 1$. This corresponds to every firm choosing the competitive production plan. Thus, the sequence of Cournot S-efficient equilibria of $\mathcal{E}_n^c$ converges to the Walrasian equilibrium of $\mathcal{E}_\infty^c$.

2. **Diffuse ownership replication.** In this replication, every firm is equally owned by *all* consumers of type 1, while consumers of type 2 still have no ownership. The finite n-fold replica will be denoted by $\mathcal{E}_n^d$, while the continuum replica will be denoted by $\mathcal{E}_\infty^d$.

   Since each firm in $\mathcal{E}_n^d$ is owned by n identical consumers, at a Cournot S-equilibrium, firms maximize the utility of their representative owner. The wealth and utility of consumer $(1, j)$ in $\mathcal{E}_n^d$ depend only on the average production, $\kappa(y)$, and have the same expressions as (2.3),(2.4) with α_j replaced by $\kappa(y)$. Thus owners of each firm are identical in terms of preferences and wealth, and therefore finding S-efficient allocations amounts to maximizing the utility of the representative consumer, which reduces to the case analyzed for the prototype economy. Hence a production plan y_n is a Cournot S-equilibrium plan if and only if it satisfies $\kappa(y_n) = \alpha^* \approx 0.3$. In particular, the production plan y_n^* in which all firms choose $(-\alpha^*, \alpha^*)$ is a Cournot S-equilibrium. Hence, the monopolistic choice persists in arbitrarily large economies.

Note that every consumer of type 1 has the same total ownership of shares in the two examples. In the concentrated ownership example, each consumer remains the sole owner of a firm, irrespective of the size of the economy. Thus, even as the economy grows larger, that firm's production choice significantly affects its owner's budget constraint and thus the income effect of a firm's production choice remains significant in arbitrarily large economies. On the other hand, the price effects vanish and are dominated by the income effect in sufficiently large economies. Thus, every type 1 consumer would want his firm to choose a production plan close to the profit maximizing plan at the limit competitive price. By contrast, in the diffuse ownership example, a type 1 consumer's ownership in any firm diminishes as the economy grows larger. Thus, the income effect of a firm's choice vanishes as well and never dominates the price effect. This is the mechanism through which a monopolistic equilibrium can persist in arbitrarily large economies.

The two examples illustrate that what drives the competitive behavior is not only a vanishing price effect, but also the *relationship* between the price and the income effect. In large economies the price effect of a change in production by a firm becomes negligible. If, at the same time, the income effect becomes negligible (as it happens with the diffuse ownership), then the price effect may still dominate the income effect and non-competitive outcomes can persist in arbitrarily large (and even the atomless limit) economies . If both the income and the price effects vanish, shareholders become indifferent among their firm's choices *in the limit*. However, as our example shows, shareholders are *not* indifferent among firm's choices along the sequence.

It should also be noted that the relevance of ownership structure in this framework is driven solely by firms' behavior, through their choice of production plans in accordance with their shareholders' interests. Whether a firm's shareholders' interests are aligned (as in this example) or not (as in the main results and the examples contained in section 5) is inconsequential.

3 Finite-type production economies

Let $\mathcal{I} := \{1, ..., I\}$ be the set of consumers' types and $\mathcal{J} := \{1, ..., J\}$ be the set of firms' types, where I, J are positive integers. For every $j \in \mathcal{J}$ let $Y_j \subseteq \mathbb{R}^L$ be the production set of a type-j firm. Y_j is assumed to satisfy the following standard conditions: (a) Y_j is closed, convex and contains the origin and (b) $Y_j \cap \mathbb{R}^L_+ = \{0\}$ (i.e., Y_j excludes "free lunches"). For every $i \in \mathcal{I}$, let $(\mathbb{R}^L_+, u^i, e^i)$ be the characteristics of a type i consumer, where $\mathbb{R}^L_+$ is the consumption set, $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ a utility representation of his preferences, and $e^i \in \mathbb{R}^L_{++}$ the endowment of goods. It is assumed that the utility functions u^i are continuous, monotonic and strictly quasi-concave.

The *space of firms* is $(\Omega_F, \mathcal{G})$, where $\Omega_F := \mathcal{J} \times [0, 1]$ and $\mathcal{G}$ is a finite or countably generated σ-algebra on Ω_F such that $2^{\mathcal{J}} \times [0, 1] \subset \mathcal{G}$; thus the projection $\jmath$ of Ω_F on $\mathcal{J}$, defined as $\jmath((j, a)) = j$, for all $(j, a) \in \Omega_F$, is measurable. A *firm* is an atom of the σ-algebra $\mathcal{G}$.[3] For every $t \in \Omega_F$, the unique atom that contains t is denoted by $\mathcal{G}(t)$, and is called *firm* $\mathcal{G}(t)$, or simply *firm* t, when no confusion can arise.[4] Since $2^{\mathcal{J}} \times [0, 1] \subseteq \mathcal{G}$, every atom's projection on $\mathcal{J}$ must be a singleton, and therefore any firm $\mathcal{G}(t)$ can be written as a pair (j, A) for some $j \in \mathcal{J}$ and $A \subset [0, 1]$. We will refer to $j = \jmath(t)$ as the *type* of firm t.

The *consumers'* side of the economy is represented by the probability space $(\Omega_C, \mathcal{F}, \mu_C)$, where $\Omega_C := \mathcal{I} \times [0, 1]$, $\mathcal{F}$ is a σ-algebra on Ω_C such that $(\Omega_C, \mathcal{F})$ is a Polish space,[5] and μ_C is a probability measure on $\mathcal{F}$. We assume that $2^{\mathcal{I}} \times [0, 1] \subset \mathcal{F}$, hence the projection function $\imath$ of Ω_C on $\mathcal{I}$ is measurable. A *consumer* is an atom of the σ-algebra $\mathcal{F}$. For every $s \in \Omega_C$, the unique atom that contains s will be denoted by $\mathcal{F}(s)$ and referred to as consumer $\mathcal{F}(s)$, or simply as consumer s.[6] The *type* of consumer $\mathcal{F}(s)$ is $\imath(s) \in \mathcal{I}$. The relative size of type-i consumers to the size of the economy is $\mu_C(\{i\} \times [0, 1])$.

The *ownership structure* of the economy is described by a measure kernel $\theta : \Omega_C \times \mathcal{G} \to \mathbb{R}_+$. Thus, for all $s \in \Omega_C$, $\theta(s, \cdot)$ is a finite measure on $\mathcal{G}$ (interpreted as consumer s's allocation of shares across firms) and, for every

[3] A non-empty set B is called an atom of the σ-algebra $\mathcal{G}$ if and only if $B \in \mathcal{G}$ and for all $C \in \mathcal{G}$, either $B \subseteq C$ or $B \cap C = \emptyset$ (**?**, p.87).

[4] Since $\mathcal{G}$ is countably generated, the atoms of $\mathcal{G}$ form a partition of Ω_F, and the atom $\mathcal{G}(t)$ equals the intersection of all sets in $\mathcal{G}$ containing t (see Appendix A for details).

[5] The space $(\Omega_C, \mathcal{F})$ is Polish if $\mathcal{F}$ is the Borel σ-algebra generated by a topology on Ω_C induced by a complete and separable metric.

[6] The Polish space assumption imposed on $(\Omega_C, \mathcal{F})$ implies that $\mathcal{F}$ is countably generated, and hence the results of Appendix A apply. See also footnote 4.

$B \in \mathcal{G}$, the map $\theta(\cdot, B)$ is $\mathcal{F}$-measurable. For every $s \in \Omega_C$, $\theta(s, \Omega_F)$ represents the total "number" of shares (in various firms) owned by consumer s. We assume that $\theta(\cdot, \Omega_F)$ is bounded. Note that this definition allows consumers of the same type to have different endowments of shares. Therefore, consumers of the same type are identical only in terms of their preferences and endowments of goods.

Let $\mu_C \otimes \theta$ be the measure on the product σ-algebra $\mathcal{F} \otimes \mathcal{G}$ defined by

$$(\mu_C \otimes \theta)(B) := \int_{\Omega_C} \int_{\Omega_F} \mathbf{1}_B(s,t)\theta(s,dt)\mu_C(ds), \quad B \in \mathcal{F} \otimes \mathcal{G}, \tag{3.1}$$

where $\mathbf{1}_B$ denotes the indicator function[7] of set B (**?**, p.20). Since $\theta(\cdot, \Omega_F)$ is bounded, $\mu_C \otimes \theta$ is a finite measure.

The *composition* $\mu_C\theta$ of μ_C and the kernel θ (**?**, p.21) defines a measure $\mu_F := \mu_C\theta$ on the the space of firms, given by

$$\mu_F(T) := \int_{\Omega_C} \theta(s,T)\mu_C(ds), \quad T \in \mathcal{G}. \tag{3.2}$$

Notice that $\mu_F(\cdot) = (\mu_C \otimes \theta)(\Omega_C \times \cdot)$ and thus μ_F is also a finite measure. We consider only economies with $\mu_F(\Omega_F) > 0$, otherwise only a μ_C-measure zero of consumers own shares and firms' choices become inconsequential and cannot affect the economy. Hence, there exists a probability kernel[8] $\gamma : \Omega_F \times \mathcal{F} \to [0,1]$, such that: (i) for every $t \in \Omega_F$, $\gamma(t, \cdot)$ is a probability measure on $\mathcal{F}$, (ii) for every $S \in \mathcal{F}$, $\gamma(\cdot, S)$ is $\mathcal{G}$-measurable, and (iii) for any $g : \Omega_C \times \Omega_F \to \mathbb{R}$ which is $\mathcal{F} \otimes \mathcal{G}$-measurable and $(\mu_C \otimes \theta)$-integrable,

$$\begin{aligned}\int_{\Omega_F} \left[\int_{\Omega_C} g(s,t)\gamma(t,ds)\right] \mu_F(dt) &= \int_{\Omega_C \times \Omega_F} g \, d(\mu_C \otimes \theta) \\ &= \int_{\Omega_C} \left[\int_{\Omega_F} g(s,t)\theta(s,dt)\right] \mu_C(ds). \end{aligned} \tag{3.3}$$

Moreover, γ is unique μ_F-a.s., in the sense that if γ' has the above properties, then for μ_F-a.e. $t \in \Omega_F$, $\gamma(t, \cdot) = \gamma'(t, \cdot)$.[9] For every $t \in \Omega_F$, the probability $\gamma(t, \cdot)$ represents firm t's *distribution of shares* across consumers.

The probability space of consumers $(\Omega_C, \mathcal{F}, \mu_C)$, together with the measurable space of firms $(\Omega_F, \mathcal{G})$ and an ownership structure described by the kernel θ from Ω_C to Ω_F defines a *private ownership production economy* $\mathcal{E}$,

$$\mathcal{E} := ((\Omega_C, \mathcal{F}, \mu_C); (\Omega_F, \mathcal{G}); \theta).$$

A *finite economy* is an economy for which the σ-algebras $\mathcal{F}$ and $\mathcal{G}$ are finite. An *atomless economy* is an economy for which the measure μ_F is atomless.[10]

[7] Given $B \subset \Omega_C \times \Omega_F$, $\mathbf{1}_B : \Omega_C \times \Omega_F \to \mathbb{R}$ is defined as $\mathbf{1}_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B \end{cases}$.

[8] See the Appendix B for a proof.

[9] Throughout the paper, "a.e." means "almost every(where)" and "a.s." means "almost surely".

[10] The measure μ_F on $(\Omega_F, \mathcal{G})$ is atomless, or nonatomic, if $\mathcal{G}$ has no μ_F-nonnull atoms (**?**, p.82).

Equation (3.2) implies that μ_F is atomless if and only if for every $t \in \Omega_F$, $\theta(s, \mathcal{G}(t)) = 0$ for μ_C-a.e. $s \in \Omega_C$. This definition focuses on the atomicity of firms, rather than consumers. The economy $\mathcal{E}$ can be atomless even if μ_C has atoms. This happens if for any atom $\mathcal{F}(s)$ of μ_C, $\theta(s, \cdot)$ is an atomless measure. Conversely, μ_C can be atomless without μ_F being so. Indeed, a firm t is an atom if $\theta(s, \mathcal{G}(t)) > 0$ for a positive μ_C-measure set of consumers s.

The prototypical Arrow-Debreu production economy with I consumers and J firms, in which the i-th consumer owns $s(i,j)$ shares of j-th firm can be represented as an economy $\mathcal{E}_1 = ((\Omega_C, \mathcal{F}_1, \mu_C), (\Omega_F, \mathcal{G}_1), \theta_1)$ with $\mathcal{F}_1 := 2^{\mathcal{I}} \times [0,1]$, $\mathcal{G}_1 := 2^{\mathcal{J}} \times [0,1]$, $\theta((i,[0,1]),(j,[0,1])) = s(i,j)$ and μ_C is equal to $\lambda_{\mathcal{I}} \otimes \lambda$, the product measure between $\lambda_{\mathcal{I}}$, the uniform probability on $\mathcal{I}$ (that is $\lambda_{\mathcal{I}}(i) = 1/I,\ \forall i \in \mathcal{I}$) and λ, the Lebesque measure on $[0,1]$. Thus we identify the i-th consumer, respectively the j-th firm of the Arrow-Debreu economy with the atom $(i,[0,1])$ of $\mathcal{F}_1$, respectively the atom $(j,[0,1])$ of $\mathcal{G}_1$.

Using sequences of replica economies to draw inferences about (strategic) equilibrium behavior in large economies is a technique introduced by **?**, for pure exchange economies, and also widely used in the literature for economies with production. An n-fold replica consists of n "clones" of each firm and each consumer of the prototype Arrow-Debreu economy. There are many ways to assign ownership of firms across consumers in replica economies. For example, a replica may be constructed such that each clone of a certain type holds the same number of shares in firms of the same industry (type); in the example of Section 2 we referred to this ownership structure as a "diffuse ownership" replication. This approach is advocated by **?**, **?**, **?**, and **?**, among others. However this is not the only way one can construct replicas of a particular economy, even when similarity of the clones is a concern. **?**, **?** and **?** assume that each clone of the prototype economy inherits the initial ownership structure. In this "concentrated ownership" replication (see Section 2), a clone of a consumer of type i owns $s(i,j)$ shares of the corresponding clone of firm j. The name captures the idea that ownership is segmented across the clones of the prototype economy, rather than being spread across multiple clones. We illustrate here how the concentrated and diffuse ownerships described above can be embedded in our framework. Section 5 contains an example of a different ownership structure and replication technique. However, our main results (of Sections 4 and 5) apply to general ownership structures and do not rely on the idea of replication.

For every $n \in \mathbb{N}$, let $H_n^1 := [0, 1/n]$ and for $k \in \{2, 3, \ldots, n\}$, let $H_n^k := \left(\frac{k-1}{n}, \frac{k}{n}\right]$. Denote by $\mathcal{H}_n$ the algebra generated by $\{H_n^1, \ldots, H_n^n\}$. For each $a \in [0,1]$, let $k(a) := \{k : a \in H_n^k\}$ and $\mathcal{H}_n(a) := H_n^{k(a)}$. Define $\mathcal{F}_n := 2^{\mathcal{I}} \otimes \mathcal{H}_n$, $\mathcal{G}_n := 2^{\mathcal{J}} \otimes \mathcal{H}_n$ and $\mu_C^n := \lambda_{\mathcal{I}} \otimes \lambda$. Thus consumers and firms in the *n-fold replica* are pairs of the form (i, H_n^k) and, respectively, (j, H_n^k), with $k = 1, ..., n$. By an abuse of notation we will often identify a point $a \in [0,1]$ with the interval $\mathcal{H}_n(a)$ and thus represent consumers and firms as pairs (i,a) and respectively (j,a). We will refer to the first component of such pair as the "type" and to the second as the "name" of the consumer/firm.

The *concentrated ownership* n-fold replica can be modeled as an economy

$$\mathcal{E}_n^c := \left((\Omega_C, 2^{\mathcal{I}} \otimes \mathcal{H}_n, \lambda_{\mathcal{I}} \otimes \lambda); (\Omega_F, 2^{\mathcal{J}} \otimes \mathcal{H}_n); \theta_n^c\right), \tag{3.4}$$

where

$$\theta_n^c((i,a),(j,A)) = s(i,j) \cdot \delta_a(A), \forall (i,a) \in \Omega_C, \forall (j,A) \in 2^{\mathcal{J}} \otimes \mathcal{H}_n, \tag{3.5}$$

and δ_a is the Dirac measure on $([0,1], \mathcal{B}[0,1])$ defined by $\delta_a(A) := \mathbf{1}_A(a)$, with $\mathcal{B}[0,1]$ being the Borel σ-algebra on $[0,1]$. The *diffuse ownership* n-fold replica can be described as the economy

$$\mathcal{E}_n^d := \left((\Omega_C, 2^{\mathcal{I}} \otimes \mathcal{H}_n, \lambda_{\mathcal{I}} \otimes \lambda); (\Omega_F, 2^{\mathcal{J}} \otimes \mathcal{H}_n); \theta_n^d\right), \tag{3.6}$$

with

$$\theta_n^d((i,a),(j,A)) = s(i,j) \cdot \lambda(A), \forall (i,a) \in \Omega_C, \forall (j,A) \in \mathcal{G}_n. \tag{3.7}$$

Note that for every consumer (i,a), the total number of shares owned by (i,a) is the same in the concentrated ownership and diffuse replica economies $\mathcal{E}_n^c, \mathcal{E}_n^d$. Thus, the total mass of the ownership distribution of a consumer stays the same. However, under the diffuse ownership specification, the support of the distribution becomes larger as the size of the economy increases.

Intuitively, the sequence of economies $(\mathcal{E}_n^c)$ "converges" to the atomless economy

$$\mathcal{E}^c := \left((\Omega_C, 2^{\mathcal{I}} \otimes \mathcal{B}[0,1], \lambda_{\mathcal{I}} \otimes \lambda); (\Omega_F, 2^{\mathcal{J}} \otimes \mathcal{B}[0,1]); \theta^c\right), \tag{3.8}$$

where $\theta^c((i,a),(j,A)) = s(i,j) \cdot \delta_a(A)$, for any $A \in \mathcal{B}[0,1]$. Similarly, the sequence of economies $(\mathcal{E}_n^d)$ "converges" to the atomless economy

$$\mathcal{E}^d := \left((\Omega_C, 2^{\mathcal{I}} \otimes \mathcal{B}[0,1], \lambda_{\mathcal{I}} \otimes \lambda); (\Omega_F, 2^{\mathcal{J}} \otimes \mathcal{B}[0,1]); \theta^d\right), \tag{3.9}$$

where $\theta^d((i,a),(j,A)) = s(i,j) \cdot \lambda(A)$, for any $A \in \mathcal{B}[0,1]$. We formalize the notion of convergence for *arbitrary* (i.e., not necessarily replica-type) sequences of finite economies in Section 5.

4 Cournot S-equilibrium

This section defines our notion of equilibrium for production economies in which consumers are price takers when making their consumption decisions, and firms interact strategically via a Cournot-type quantity competition but, rather than maximizing profits, they follow an objective that is consistent with their shareholders' interests. We call this new concept a *Cournot S-equilibrium*.

To simplify exposition, we make no distinction here between the consumers who own the firm and those who control it. However all our results remain true if we assume that a firm's decisions are controlled by a (predetermined) group of consumers (e.g., the Board of Directors).

Consider a production economy $\mathcal{E} = ((\Omega_C, \mathcal{F}, \mu_C); (\Omega_F, \mathcal{G}); \theta)$. An *allocation* for the economy $\mathcal{E}$ is a pair (c, y), such that $c : \Omega_C \to \mathbb{R}_+^L$ is $\mathcal{F}$-measurable,

$y : \Omega_F \to \mathbb{R}^L$ is $\mathcal{G}$-measurable and $\theta(s,\cdot)$-integrable for μ_C-almost all $s \in \Omega_C$, and $y(j,a) \in Y_j$ for all $(j,a) \in \Omega_F$. Hence for any $s \in \Omega_C$ and $t \in \Omega_F$, $c(s)$ represents the consumption bundle of agent s, and $y(t)$ is the production *per outstanding share* of firm t. We call c a consumption allocation and y a production plan for the economy $\mathcal{E}$. The allocation (c, y) is called *feasible* if

$$\int_{\Omega_C} c\, d\mu_C = \int_{\Omega_C} e\, d\mu_C + \int_{\Omega_F} y\, d\mu_F,$$

where $e(s) := e^{\imath(s)},\ s \in \Omega_C$ and $\mu_F = \mu_C\theta$. A given production plan y generates the *intermediate endowment* mapping $w_y : \Omega_C \to \mathbb{R}^L$ defined by

$$w_y(s) := e(s) + \int_{\Omega_F} y(t)\theta(s, dt), \quad s \in \Omega_C. \tag{4.1}$$

Note that (3.3) implies that w_y is μ_C-integrable and

$$\int_{\Omega_C} w_y\, d\mu_C = \int_{\Omega_C} e\, d\mu_C + \int_{\Omega_F} y\, d\mu_F.$$

When the market price vector is $p \in \Delta^{L-1}$ (with Δ^{L-1} denoting the unit simplex in $\mathbb{R}^L_+$), the budget constraint of a consumer $s \in \Omega_C$ is $\{x \in \mathbb{R}^L_+ \,|\, p\cdot x \le p\cdot w_y(s)\}$ and his consumption choice is $D^{\imath(s)}(p, w_y(s))$, where

$$D^i(p,z) := \arg\max\{u^i(x) \mid x \in \mathbb{R}^L_+,\ px \le pz\}, \quad \forall i \in \mathcal{I}, p \in \Delta^{L-1}, z \in \mathbb{R}^L_+.$$

The utility of consumer s at his optimal consumption choice, when faced with prices p and a production plan y, is $V^{\imath(s)}(p, w_y(s))$, where $V^i(p,z) := u^i\left(D^i(p,z)\right)$, for every $i \in \mathcal{I},\ p \in \Delta^{L-1},\ z \in \mathbb{R}^L_+$.

For any production plan y, denote by $\mathcal{E}(y)$ the associated pure-exchange economy in which consumers' endowments are given by w_y. The set of Walrasian equilibrium prices of the economy $\mathcal{E}(y)$ depends only on the *distribution of intermediate endowments across types* defined as $\mu_C \circ \widetilde{w}_y^{-1}$, where $\widetilde{w}_y : \Omega_C \to \mathcal{I} \times \mathbb{R}^L_+$, $\widetilde{w}_y(s) := (\imath(s), w_y(s))$, (**?**). Let $P(\mu_C \circ \widetilde{w}_y^{-1}) \subseteq \Delta^{L-1}$ be the set of Walrasian equilibrium price vectors of $\mathcal{E}(y)$. For brevity, we set $P(y) := P(\mu_C \circ \widetilde{w}_y^{-1})$.

For some production sets, in particular for those that exhibit free disposal, the economy $\mathcal{E}_y$ may have no Walrasian equilibrium. Certain lower bounds, or capacity constraints, need to be imposed on the firms' strategy sets to avoid this occurrence and make the problem meaningful. It is sufficient, for example to restrict firms' choices to production plans that generate positive intermediate endowments.[11] For such production plans, the main theorem in **?** implies that $P(\cdot)$ is not empty-valued. However, positivity of intermediate endowments is not necessary for the existence of a Walrasian equilibrium in the associated pure-exchange economy and therefore a much larger set than the one described above

[11] This happens, for example, if each production set is contained in the set $\{y \in \mathbb{R}^L | y_l \ge -\frac{\min_{i\in\mathcal{I}} e^i_l}{M}, \forall l = 1,\ldots,L\}$, where M is the upper bound on the kernels θ.

may still generate non-empty values for P. For the remaining of the paper we are going to abstract from the difficulties posed by the possible empty-values of P by assuming that the production sets $(Y_j)_{j\in\mathcal{J}}$ contain some capacity constraints that are tight enough to guarantee the existence of a competitive equilibrium for *every* production plan y.[12] Thus, the production sets $(Y_j)_{j\in\mathcal{J}}$ are assumed to be compact.[13]

Since the correspondence P is closed, has compact values,[14] and it is defined on a compact space,[15] P is weakly measurable and thus, according to Kuratowski-Ryll-Nardzewski theorem (**?**, Theorem 18.13), it has a measurable selection $\mathbf{p}$ (i.e., $\mathbf{p}$ is measurable and $\mathbf{p}(y) \in P(y)$).

A pair $(\bar{p}, \bar{y})$ of prices $\bar{p} \in \Delta^{L-1}$ and production plan $\bar{y}$ is a *Walrasian equilibrium* for the economy $\mathcal{E}$ if and only if $\bar{p} \in P(\bar{y})$ and, for μ_F-almost every $t \in \Omega_F$,

$$\bar{p} \cdot \bar{y}(t) = \max_{z \in Y_{\jmath(t)}} \bar{p} \cdot z.$$

We introduce next the concept of a *Cournot S-equilibrium*, which captures the idea that, although an individual consumer cannot affect market prices through his consumption decisions, he is aware of the effect that a firm that he owns has on market prices.

Definition 4.1. *A production plan y^* is called S-efficient for firm $t = (j, a)$, given the measurable price selection $\mathbf{p}$ from P if and only if there does not exist $z \in Y_j$ such that:*

$$\begin{aligned} &\gamma\left(t, \left\{s \mid V^{\imath(s)}(\mathbf{p}(\tilde{y}), w_{\tilde{y}}) \geq V^{\imath(s)}(\mathbf{p}(y^*), w_{y^*})\right\}\right) = 1, \\ &\gamma\left(t, \left\{s \mid V^{\imath(s)}(\mathbf{p}(\tilde{y}), w_{\tilde{y}}) > V^{\imath(s)}(\mathbf{p}(y^*), w_{y^*})\right\}\right) > 0, \end{aligned} \tag{4.2}$$

where $\tilde{y} : \Omega_F \to \mathbb{R}^L$ is defined as $\tilde{y} := y^* + (z - y^*)\mathbf{1}_{\mathcal{G}(t)}$ and $\mathbf{1}_{\mathcal{G}(t)}$ is the indicator function of the set $\mathcal{G}(t)$. A pair $(\mathbf{p}, y^*)$ consisting of a measurable selection $\mathbf{p}$ from P and a production plan y^* is called a Cournot S-equilibrium if, given the selection $\mathbf{p}$, y^* is S-efficient for μ_F-almost every $t \in \Omega_F$.

We will simply refer to a production plan y^* as being a Cournot S-equilibrium whenever there exists a measurable price selection $\mathbf{p}$ such that $(\mathbf{p}, y^*)$ is a

[12] One can dispense of this assumption by restricting the firms' strategy sets to a subset on which existence of a competitive equilibrium is guaranteed. With due care, all the results of this paper can be derived under such restriction, but the details of the construction are beyond the scope of this paper.

[13] It is well-known that boundedness of the production sets is a necessary condition for obtaining competitive behavior in large economies. Otherwise a firm could increase its scale as the economy grows larger and still be a dominant part of the aggregate production (see the discussion and additional references in **?**). Nevertheless, as we show in this paper, boundedness of the production plans is not *sufficient* to achieve competitive behavior in large economies: the ownership structure plays a crucial role towards that end.

[14] The standard reference is **?**. However, in our case an extension of the classical result is needed since the intermediate endowments may generate zero wealth (see, for example, **?**).

[15] Note that intermediate endowments must lie in a compact subset of $\mathbb{R}^L$, since the set of feasible production plans is bounded. The space of laws on $\mathbb{R}^L$ with support in a given compact is compact, when endowed with the weak convergence topology (**?**, Theorem 9.3.3).

Cournot S-equilibrium. Thus for a fixed price selection, a production plan is S-efficient for a firm if, given the choices of the other firms, there does not exist another production plan such that every shareholder of the firm is better off in the new market equilibrium. S-efficiency requires thus the firm to obey a unanimous vote of its shareholders and therefore it represents a *minimal* condition on preference aggregation. This makes S-efficiency a very weak condition, since different production choices made by a firm may generate equilibrium allocations for that firm's shareholders which are not Pareto comparable. It is therefore likely that the set of Cournot S-equilibria is large. Our example 5.2 in Section 5 supports this hypothesis. Note however that having such a permissive equilibrium concept only strengthens our main results. Convergence to Walrasian equilibria, as stated in Theorem 5.6 for example, still holds when firms follow a different objective that selects a strict subset of the S-efficient outcomes. Moreover, failure to converge to a Walrasian equilibrium outcome is not driven by the existence of a large set of S-efficient outcomes either. This is illustrated by the diffuse ownership example of Section 2, in which the Cournot S-efficient equilibrium is unique.

We examine first the relationship between Walrasian equilibria and Cournot S-equilibria in atomless economies. We start by showing that a measure zero of firms cannot affect the equilibrium price. Since every single firm (atom) is of measure zero in an atomless economy, an individual firm's change in production has a negligible price effect.

Lemma 4.2. *If the production plans y and y' associated to the economy $\mathcal{E} = ((\Omega_C, \mathcal{F}, \mu_C); (\Omega_F, \mathcal{G}); \theta)$ are equal μ_F-a.e ($\mu_F = \mu_C\theta$), then $P(y) = P(y')$.*

Proof. Since $\mu_F(\{y \neq y'\}) = 0$, for any set $S \in \mathcal{F}$ of consumers, (3.3) implies

$$\int_S \left[\int_{\Omega_F} y(t)\theta(s,dt)\right]\mu_C(ds) = \int_{\Omega_F} y(t)\gamma(t,S)\mu_F(dt) =$$
$$= \int_{\Omega_F} y'(t)\gamma(t,S)\mu_F(dt) = \int_S \left[\int_{\Omega_F} y'(t)\theta(s,dt)\right]\mu_C(ds).$$

Thus the intermediate endowments associated to the two production plans coincide μ_C-a.e., and therefore $P(y) = P(y')$. □

In atomless economies, a Walrasian equilibrium is also a Cournot S-equilibrium, but the choice of an S-efficient production plan may not necessarily lead to profit maximization (at the Walrasian prices) for *some* ownership structures.

Proposition 4.3. *Let $\mathcal{E} = ((\Omega_C, \mathcal{F}, \mu_C); (\Omega_F, \mathcal{G}); \theta)$ be an atomless economy.*

1. *If $(\bar{p}, \bar{y})$ is a Walrasian equilibrium for $\mathcal{E}$, then $(\bar{\mathbf{p}}, \bar{y})$ is a Cournot S-equilibrium for $\mathcal{E}$, where $\bar{\mathbf{p}}$ is any price selection such that $\bar{\mathbf{p}}(\bar{y}) = \bar{p}$.*

2. *If $(\bar{\mathbf{p}}, \bar{y})$ is a Cournot S-equilibrium for $\mathcal{E}$, then $\bar{y}$ is profit maximizing at prices $\bar{\mathbf{p}}(\bar{y})$ on the set of firms Ω_F^{max} defined by*

$$\Omega_F^{max}(\mathcal{E}) := \{t \mid \gamma(t, \{s \mid \theta(s, \mathcal{G}(t)) > 0\}) > 0\}. \tag{4.3}$$

Moreover, $\left(\bar{\mathbf{p}}, \bar{y} \cdot \mathbf{1}_{\Omega_F^{max}(\mathcal{E})} + y \cdot \mathbf{1}_{\Omega_F \backslash \Omega_F^{max}(\mathcal{E})}\right)$ *is a Cournot S-equilibrium, for any production plan y.*

Proof. 1. By Lemma 4.2, any deviation from $\bar{y}$ by a single firm will leave the price $\bar{p}$ unchanged. Hence, given the absence of price effects, the best choice of a production plan for each firm, from the perspective of its shareholders, is a profit maximizing plan.

2. Let $\bar{p} := \bar{\mathbf{p}}(\bar{y})$. Using Lemma 4.2 and Definition 4.1, it follows that $(\bar{\mathbf{p}}, \bar{y})$ is a Cournot S-equilibrium if and only if for every firm t, the set of its shareholders whose wealth can be increased by a deviation to a profit maximization choice is negligible. Formally, for any $\hat{y}$ which is a profit maximizing at prices $\bar{p}$,

$$\gamma\left(t, \{s \mid (\bar{p} \cdot \hat{y}(t) - \bar{p} \cdot \bar{y}(t))\theta(s, \mathcal{G}(t)) > 0\}\right) = 0, \text{ for } \mu_F\text{-a.e. } t \in \Omega_F. \tag{4.4}$$

Condition (4.4) requires that for μ_F-a.e. $t \in \Omega_F$, either $\bar{p} \cdot \hat{y}(t) = \bar{p} \cdot \bar{y}(t)$ or $\gamma\left(t, \{s \mid \theta(s, \mathcal{G}(t)) > 0\}\right) = 0$ (or both), and the conclusion follows. $\square$

The last part of the proposition points out that, in an atomless economy, any production plan is S-efficient for firms in $\Omega_F \backslash \Omega_F^{max}(\mathcal{E})$. Note that a firm belongs to the set $\Omega_F^{max}(\mathcal{E})$ if it satisfies two conditions. One is to have some shareholders whose portfolios are not fully diversified (in the sense of having a positive mass of their total share holdings invested in the firm and thus being affected significantly by that firm's profits). The second requirement is that the set of those non fully-diversified shareholders own (jointly) a positive fraction of the firm. These two conditions insure that the firm's choices have a non-negligible effect on the wealth of a group of its shareholders owning a positive fraction of the firm. Because the price effect is absent in an atomless economy, those shareholders unanimously approve profit maximization and thus any Cournot S-efficient production plan for a given firm has to be profit maximizing.

Note that $\Omega_F^{max}(\mathcal{E}) = \emptyset$ if the measures $\theta(s, \cdot)$ are atomless for every $s \in \Omega_C$. This happens, for instance, in the diffuse ownership economy $\mathcal{E}^d$ introduced in (3.9). At the other extreme lies the concentrated ownership economy $\mathcal{E}^c$ defined in (3.8), where $\Omega_F^{max}(\mathcal{E}^c) = \Omega_F$ and thus every Cournot S-equilibrium is profit maximizing.

5 Limit behavior of Cournot S-equilibrium

We investigate the behavior of Cournot S-equilibrium in large economies and establish under what conditions Cournot-S equilibrium production plans in a sequence of convergent economies approach profit maximizing production plans of the limit economy. To avoid repetition, throughout this section we will refer to a sequence of economies $(\mathcal{E}_n = ((\Omega_C, \mathcal{F}_n, \mu_C^n); (\Omega_F, \mathcal{G}_n); \theta_n))_{n \in \mathbb{N}}$ simply as $(\mathcal{E}_n)$, and to a limit economy $\mathcal{E} = ((\Omega_C, \mathcal{F}, \mu_C); (\Omega_F, \mathcal{G}); \theta)$ as $\mathcal{E}$. We use the notation $\mu_F^n := \mu_C^n \theta_n, \mu_F := \mu_C \theta$ (see (3.2)). For any sequence of economies $(\mathcal{E}_n)$, we assume that there exists $\bar{L} > 0$ such that $\theta_n(s, \Omega_F) \leq \bar{L}$, for all $s \in \Omega_C$ and $n \in \mathbb{N}$.

We prove first that a firm's choice of production plans bounded away from profit maximization in increasingly large economies is detrimental to a shareholder owning a fraction of a firm bounded away from zero.

Lemma 5.1. *Consider a sequence of economies $(\mathcal{E}_n)$ and for each $n \in \mathbb{N}$, let y_n be a production plan in $\mathcal{E}_n$ and $p_n \in P(y_n)$ be a pure exchange equilibrium of $\mathcal{E}_n(y_n)$. Assume that $P(\cdot)$ is a singleton at any adherent point of the sequence of intermediate endowment distributions induced by (y_n).*[16] *Let $\bar{t} \in \Omega_F$ be a firm such that $\mu_F^n(\mathcal{G}_n(\bar{t})) \to 0$. For each n, let $\hat{y}_n$ be another production plan differing from y_n only in the choice of firm $\bar{t}$, and let $\hat{p}_n \in P(\hat{y}_n)$ be an equilibrium of $\mathcal{E}_n(\hat{y}_n)$. Assume that there exists $\eta > 0$ such that $\hat{p}_n\hat{y}_n(\bar{t}) > p_n y_n(\bar{t}) + \eta$ for all n. Then for each $\varepsilon > 0$ there exists $\eta' > 0$ and $N \in \mathbb{N}$ such that for all consumers $\bar{s} \in \Omega_C$ whose holdings in firm $\bar{t}$ are bounded below by ε in all economies $(\mathcal{E}_n)$,*

$$V^{\imath(\bar{s})}(\hat{p}_n, w_{\hat{y}_n}(\bar{s})) > V^{\imath(\bar{s})}(p_n, w_{y_n}(\bar{s})) + \eta', \forall n \geq N. \tag{5.1}$$

Proof. Suppose the conclusion is not true. Then there exists $\varepsilon > 0$ and $i \in \mathcal{I}$, such that for every $k \in \mathbb{N}$ there exists $s_k \in \Omega_C$ and $n_k \in \mathbb{N}$ such that $\imath(s_k) = i$, $\theta_{n_k}(\bar{s}_k, \mathcal{G}_{n_k}(\bar{t})) \geq \varepsilon$ and

$$V^i(\hat{p}_{n_k}, w_{\hat{y}_{n_k}}(\bar{s}_k)) < V^i(p_{n_k}, w_{y_{n_k}}(\bar{s}_k)) + \frac{1}{k}. \tag{5.2}$$

Since production plans take value in a compact and the kernels (θ_n) are uniformly bounded from above, intermediate endowments also lie in a compact. Therefore, on a subsequence, $w_{y_{n_k}}(\bar{s}) \to w \in \mathbb{R}$, $w_{\hat{y}_{n_k}}(\bar{s}) \to \hat{w} \in \mathbb{R}$, and $\mu_C^{n_k} \circ \widetilde{w}_{y_{n_k}}^{-1}$ converges weakly to some distribution W, since the sequence of distributions of intermediate endowments $(\mu_C^n \circ \widetilde{w}_{y_{n_k}}^{-1})$ is tight, and hence is relatively weakly compact, by Prohorov's theorem (**?**, Theorem 16.3).

We prove that $\mu_C^{n_k} \circ \widetilde{w}_{\hat{y}_{n_k}}^{-1}$ converges also to W. Let $f : \mathcal{I} \times \mathbb{R}^L \to \mathbb{R}$ be Lipschitz (with Lipschitz constant K) and bounded (by M). By the Portmanteau theorem, it is enough to prove that $\int_{\mathcal{I}\times\mathbb{R}^L} f \, d\mu_C^{n_k} \circ \widetilde{w}_{\hat{y}_{n_k}}^{-1} \to \int_{\mathcal{I}\times\mathbb{R}^L} f \, dW$. Notice that

$$\begin{aligned}\left|\int_{\mathcal{I}\times\mathbb{R}^L} f \, d\mu_C^{n_k} \circ \widetilde{w}_{\hat{y}_{n_k}}^{-1} - \int_{\mathcal{I}\times\mathbb{R}^L} f \, dW\right| &\leq \int_{\Omega_C} \left|f(\widetilde{w}_{\hat{y}_{n_k}}) - f(\widetilde{w}_{y_{n_k}})\right| d\mu_C^{n_k} + \\ &+ \left|\int_{\mathcal{I}\times\mathbb{R}^L} f \, d\mu_C^{n_k} \circ \widetilde{w}_{y_{n_k}}^{-1} - \int_{\mathcal{I}\times\mathbb{R}^L} f \, dW\right|.\end{aligned}$$

Fix a $\delta > 0$. Since $\mu_C^{n_k} \circ \widetilde{w}_{y_{n_k}}^{-1}$ converges weakly to W, there exists $K_1 \in \mathbb{N}$ such that $\left|\int_{\mathcal{I}\times\mathbb{R}^L} f \, d\mu_C^{n_k} \circ \widetilde{w}_{y_{n_k}}^{-1} - \int_{\mathcal{I}\times\mathbb{R}^L} f \, dW\right| \leq \delta$ for all $k \geq K_1$. Denoting by $\|\cdot\|$

[16] An adherent point is the limit of a weakly converging subsequence of the sequence of intermediate endowment distributions.

the Euclidean norm in $\mathbb{R}^L$, we have

$$\int_{\Omega_C} \left| f(\widetilde{w}_{\hat{y}_{n_k}}) - f(\widetilde{w}_{y_{n_k}}) \right| d\mu_C^{n_k} \leq \int_{\Omega_C} \left| f(\widetilde{w}_{\hat{y}_{n_k}}) - f(\widetilde{w}_{y_{n_k}}) \right| \mathbf{1}_{\left\| \widetilde{w}_{\hat{y}_{n_k}} - \widetilde{w}_{y_{n_k}} \right\| \leq \delta} \, d\mu_C^{n_k} + $$
$$+ \int_{\Omega_C} \left| f(\widetilde{w}_{\hat{y}_{n_k}}) - f(\widetilde{w}_{y_{n_k}}) \right| \mathbf{1}_{\left\| \widetilde{w}_{\hat{y}_{n_k}} - \widetilde{w}_{y_{n_k}} \right\| > \delta} \, d\mu_C^{n_k}$$
$$\leq K\delta + 2M \cdot \mu_C^{n_k} \left(\left\| \widetilde{w}_{\hat{y}_{n_k}} - \widetilde{w}_{y_{n_k}} \right\| > \delta \right).$$

Let $\delta_k := \delta / \|\hat{y}_{n_k}(\bar{t}) - y_{n_k}(\bar{t})\|$. By the definition of intermediate endowments (4.1) and by (3.3),

$$\mu_C^{n_k} \left(\left\| \widetilde{w}_{\hat{y}_{n_k}} - \widetilde{w}_{y_{n_k}} \right\| > \delta \right) = \mu_C^{n_k} \left(\theta_{n_k}(\cdot, \mathcal{G}_{n_k}(\bar{t})) \cdot \|\hat{y}_{n_k}(\bar{t}) - y_{n_k}(\bar{t})\| > \delta \right)$$
$$= \int_{\Omega_C} \mathbf{1}_{\theta_{n_k}(\cdot, \mathcal{G}_n(\bar{t})) > \delta_k} d\mu_C^{n_k} \leq \frac{1}{\delta_k} \int_{\Omega_C} \theta_{n_k}(\cdot, \mathcal{G}_{n_k}(\bar{t})) d\mu_C^{n_k}$$
$$= \frac{\|\hat{y}_{n_k}(\bar{t}) - y_{n_k}(\bar{t})\|}{\delta} \mu_F^{n_k}(\mathcal{G}_{n_k}(\bar{t})).$$

As $\mu_F^{n_k}(\mathcal{G}_{n_k}(\bar{t})) \to 0$ and production plans belong to a compact, there exists $K_2 \in \mathbb{N}$ such that $\mu_C^{n_k} \left(\left\| \widetilde{w}_{\hat{y}_{n_k}} - \widetilde{w}_{y_{n_k}} \right\| > \delta \right) \leq \delta$ for all $k \geq K_2$. Therefore for any $k \geq \max\{K_1, K_2\}$,

$$\left| \int_{\mathcal{I} \times \mathbb{R}^L} f \, d\mu_C^{n_k} \circ \widetilde{w}_{\hat{y}_{n_k}}^{-1} - \int_{\mathcal{I} \times \mathbb{R}^L} f \, dW \right| \leq K\delta + 2M\delta + \delta,$$

and given that δ was arbitrary, we obtained the desired convergence of $\mu_C^{n_k} \circ \widetilde{w}_{\hat{y}_{n_k}}^{-1}$ to W.

Since $P(\cdot)$ has closed graph and is a singleton at W, then (along a subsequence) $p_{n_k}, \hat{p}_{n_k} \to p$, with $\{p\} = P(W)$. Taking limits as $k \to \infty$ in (5.2) and using the continuity of the indirect utility V^i, it follows that $V^i(p, \hat{w}) \leq V^i(p, w)$ and thus $p \cdot \hat{w} \leq p \cdot w$. However, (4.1) gives

$$p \left(w_{\hat{y}_{n_k}}(s_k) - w_{y_{n_k}}(s_k) \right) = p(\hat{y}_{n_k}(\bar{t}) - y_{n_k}(\bar{t})) \theta_{n_k}(s_k, \mathcal{G}_{n_k}(\bar{t})) \geq \eta \cdot \varepsilon, \quad (5.3)$$

and taking the limit with $k \to \infty$, we get $p \cdot \hat{w} \geq p \cdot w + \eta \cdot \varepsilon$. Hence a contradiction is obtained and the conclusion follows. □

Lemma 5.1 establishes the main result of **?** in our complete markets environment using an alternative, simpler proof, which does not require the "assumption that all consumers are typical" (his Assumption 8).[17] We illustrate below the restrictiveness of this condition imposed on the endogenous wealth distributions of the agents induced by a sequence of production plans.

[17]For a detailed discussion of **?** proof and its problems, and how we overcame them in our proof of Lemma 5.1, see **?**.

Example 5.1
We modify slightly the example of section 2, by removing the consumer that does not own shares in the prototype economy, and letting the endowments of the unique agent of the prototype economy be $(2, 2)$. Assume that in the finite n-fold replica $\mathcal{E}_n$, first firm is entirely owned by the first consumer, while the ownership of the other firms is distributed uniformly across the remaining consumers. This is exactly the example used by **?** to outline his results for the case with no uncertainty (the number of states of nature equals one).

Given a production plan $y = ((-\alpha_j, \alpha_j))_{j=1}^n$ in the n-fold replica $\mathcal{E}_n$, the resulting exchange equilibrium price vector, normalized to the unit simplex, is unique and depends only on the average production,

$$(p_1, p_2) = \left(\frac{2+\kappa(y)}{4}, \frac{2-\kappa(y)}{4}\right), \tag{5.4}$$

with $\kappa(y)$ defined as in (2.1).

Consider a sequence of production plans (y_n) associated to the replicas $(\mathcal{E}_n)$. In the economy $\mathcal{E}_n$, the wealth $W_{y_n}(l)$ of all agents $l > 1$ coincide. Fix an $l > 1$. Therefore the wealth distribution of agents in $\mathcal{E}_n$ is given by

$$\frac{1}{n}\delta_{W_{y_n}(1)} + (1 - \frac{1}{n})\delta_{W_{y_n}(l)},$$

which converges to $\delta_{\lim W_{y_n}(l)}$ whenever the average $\kappa(y_n)$ of y_n converges.[18]

The proof of **?** Theorem 1 rests on the assumption that "all consumers are typical" (Assumption 8), which in this context requires that if $\lim W_{y_n}(l)$ exists, then $\lim W_{y_n}(1)$ exists and equals $\lim W_{y_n}(l)$. Of course this is an extremely strong assumption which reduces the applicability of his theorem. In this example, it applies only to a sequence of production plans (y_n) with the property that on each subsequence where $\kappa(y_n)$ converges (that is, the average production converges), the profit of the first firm $p(y_n)y_n(1)$ converges to the same limit as the average profit $p(y_n)\sum_{j=1}^n y_n(j)/n$. With the numerical values here, this happens only if the production of the first firm converges to the limit of the average production, or the average production converges to zero.[19]

We show next that Cournot S-efficient production plans become arbitrarily close to profit maximization in large economies for firms having all their shareholders owning a fraction of the firm bounded away from zero. For finite economies $(\mathcal{E}_n)_{n\in\mathbb{N}}$, denote the set of firms that have only non-diversified

[18] δ_x is the Dirac measure at $x \in \mathbb{R}$.

[19] Notice that if $y_n = ((-\alpha_n(j), \alpha_n(j))_{j=1}^n$, then $W_{y_n}(1) = 2 + p(y_n)y_n(1) = 2 - \frac{1}{2}\kappa(y_n)\alpha_n(1)$, and

$$W_{y_n}(l) = 2 + p(y_n)\frac{\sum_{j=2}^n y_n(j)}{n-1} \to 2 - \frac{1}{2}(\lim \kappa(y_n))^2.$$

By contrast, our version of **?** result given in Lemma 5.1 holds without requiring that the production plan of firm 1 converges to the limit of the average production plans on subsequences where the latter exists.

shareholders and positive size infinitely often as

$$\Omega_F^{nd}((\mathcal{E}_n)) = \{t \in \Omega_F \mid \exists \varepsilon(t) > 0 \text{ s.t. for infinitely many } n \in \mathbb{N},$$
$$\mu_F^n(\mathcal{G}_n(t)) > 0 \text{ and } \theta_n(s, \mathcal{G}_n(t)) \geq \varepsilon(t), \text{ for } \gamma_n(t, \cdot)\text{-a.e. } s\}. \tag{5.5}$$

Theorem 5.2. *Consider a sequence of finite economies $(\mathcal{E}_n)$, and for each $n \in \mathbb{N}$, let $(\mathbf{p}_n, y_n)$ be a Cournot S-equilibrium of the economy $\mathcal{E}_n$. Assume that $P(\cdot)$ is a singleton at any adherent point of the sequence of intermediate endowment distributions induced by (y_n). Fix an $\varepsilon > 0$ and a firm $\bar{t} \in \Omega_F^{nd}((\mathcal{E}_n))$. For every $n \in \mathbb{N}$ consider a production plan $\hat{y}_n$ in $\mathcal{E}_n$ that differs from y_n only in the choice of firm $\bar{t}$. Then for every $\delta > 0$ there exists $N(\delta) \in \mathbb{N}$ such that $\mathbf{p}_n(y_n) \cdot y_n(\bar{t}) \geq \mathbf{p}_n(\hat{y}_n) \cdot \hat{y}_n(\bar{t}) - \delta$, for all $n > N(\delta)$.*

Proof. Suppose, by contradiction, that the conclusion is false. Then there exists $\delta > 0$ such that (along a subsequence) $p_n y_n(\bar{t}) + \eta < \hat{p}_n \hat{y}_n(\bar{t})$, where $p_n := \mathbf{p}_n(y_n)$ and $\hat{p}_n := \mathbf{p}_n(\hat{y}_n)$. Lemma 5.1 contradicts the Cournot S-efficiency of the plans (y_n) for all large n. □

Next we define a convergence notion on the space of private ownership economies, not restricted to replica economies (introduced at the end of section 3), and characterize the production plans of a limit economy that are the limit of a sequence of Cournot S-efficient equilibria of converging economies.

Definition 5.3. *A sequence $(\mathcal{E}_n)$ of finite economies converges to the economy $\mathcal{E}$ if the following hold:*

(i) $\mathcal{G}_n \subset \mathcal{G}$ for all n and for μ_F-a.e. $t \in \Omega_F$, $\mu_F(\mathcal{G}_n(t)) \to \mu_F(\mathcal{G}(t))$,

(ii) The ownership kernels θ_n converge to θ, in the sense that for each uniformly bounded sequence[20] (X_n) of random variables on Ω_F such that X_n is $\mathcal{G}_n$-measurable for all n and, for μ_C-a.e. $s \in \Omega_C$, $X_n \to X$ $\theta(s, \cdot)$-a.s., the following holds

$$\int_{\Omega_F} X_n(t)\theta_n(\cdot, dt) \to \int_{\Omega_F} X(t)\theta(\cdot, dt), \quad \mu_C\text{-a.s.}$$

(iii) μ_C^n has an extension to $\mathcal{F}$ that converges setwise to μ_C.[21]

Condition (i) requires the "size" of each firm along the sequence to approach the size of the firm in the limit. It is always satisfied if $(\mathcal{G}_n)$ is increasing ($\mathcal{G}_n \subset \mathcal{G}_{n+1}$ for all $n \in \mathbb{N}$) and asymptotically generates $\mathcal{G}$ ($\mathcal{G} = \sigma(\cup_{n \in \mathbb{N}} \mathcal{G}_n)$), requirements expressed by the notation $\mathcal{G}_n \nearrow \mathcal{G}$. Condition (ii) indicates that the ownership kernels θ_n in the finite economies "approach" the limit ownership

[20]The sequence (X_n), with $X_n : \Omega_F \to \mathbb{R}$, is uniformly bounded if $\sup_n |X_n| < M$ for some $M \in \mathbb{R}$.

[21]This means that there exist measures $\tilde{\mu}_C^n(\cdot)$ on $\mathcal{F}$ which coincide with μ_C^n when restricted to $\mathcal{F}_n$ (i.e., $\tilde{\mu}_C^n(\cdot)|_{\mathcal{F}_n} = \mu_C^n(\cdot)$), and $\tilde{\mu}_C^n(S) \to \mu_C(S)$ for all $S \in \mathcal{F}$. Lemma C.1 gives sufficient conditions for the existence of such extensions.

structure described by the kernel θ. It is satisfied if, for example, $\mathcal{G}_n \nearrow \mathcal{G}$ and for μ_C-almost all $s \in \Omega_C$, the kernel $\theta_n(s, \cdot)$ has an extension to $\mathcal{G}$ that converges setwise to $\theta(s, \cdot)$ (Lemma C.2). Sufficient conditions for the existence of such an extension are given in Lemma C.1. In this case, for almost all consumers, the number of shares they own in a group of firms in each finite economy converges to the number of shares held in the respective set of firms in the limit economy, which makes the "closeness" notion between θ_n and θ embedded in condition (ii) intuitive. In particular, for the concentrated and diffuse ownership replica economies discussed in Section 3, the kernels θ_n^c, θ_n^d defined in (3.5),(3.7) have extensions to $\mathcal{G}$, equal to θ^c, θ^d, and hence they trivially converge setwise. Finally, condition (iii) in conjunction with condition (ii) guarantees that the distribution of intermediate endowments of the finite economies converges to the distribution of intermediate endowments in the limit economy (Proposition 5.5). It simply means that the relative size of a set of consumers in each finite economy approaches its size in the limit economy.

The convergence notion for finite economies introduced in Definition 5.3 is flexible enough to allow any economy to be approximated by finite economies. The idea is to construct the ownership kernel of a finite economy as a "conditional expectation" of the ownership kernel in the limit economy, by averaging out the ownership structures of consumers belonging to the same atom of the finite economy. In particular the kernels θ_n^c, θ_n^d (see (3.5),(3.7)) have this property. Indeed, for any $T \in 2^{\mathcal{J}} \otimes \mathcal{H}_n$, $s \in \Omega_C$ and $l \in \{c, d\}$, $\theta_n^l(s, T) = E^{\mu_C}[\theta^l(\cdot, T)|2^{\mathcal{J}} \otimes \mathcal{H}_n](s)$.

Proposition 5.4. *Consider an economy $\mathcal{E} = ((\Omega_C, \mathcal{F}, \mu_C); (\Omega_F, \mathcal{G}); \theta)$ with $(\Omega_F, \mathcal{G})$ Polish space, and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$, respectively $(\mathcal{G}_n)_{n\in\mathbb{N}}$, be finite σ-algebras on Ω_C, respectively Ω_F, such that $\mathcal{F}_n \nearrow \mathcal{F}$ and $\mathcal{G}_n \nearrow \mathcal{G}$. There exists a sequence of finite economies $(\mathcal{E}_n)_{n\in\mathbb{N}}$ with $\mathcal{E}_n = ((\Omega_C, \mathcal{F}_n, \mu_C); (\Omega_F, \mathcal{G}_n); \theta_n))$, converging to $\mathcal{E}$ and satisfying*

$$\theta_n(s, T) = E^{\mu_C}[\theta(\cdot, T)|\mathcal{F}_n](s), \ \mu_C\text{-a.e. } s \in \Omega_C, \forall T \in \mathcal{G}. \tag{5.6}$$

Proof. Consider the probability space $(\Omega_C \times \Omega_F, \mathcal{F}_n \otimes \mathcal{G}, \Theta)$, with $\Theta := \mu_C \otimes \theta/\alpha)$, where $\mu_C \otimes \theta$ is defined in (2.1) and $\alpha := (\mu_C \otimes \theta)(\Omega_F \otimes \Omega_C)$. Let π_F, respectively π_C be the projections of $(\Omega_C \times \Omega_F, \mathcal{F}_n \otimes \mathcal{G})$ on $(\Omega_F, \mathcal{G})$, respectively on $(\Omega_C, \mathcal{F}_n)$. **?** guarantees the existence of a regular conditional distribution of π_F given π_C, denoted by η.[22] Thus η is a probability kernel from $(\Omega_C, \mathcal{F}_n)$ to $(\Omega_F, \mathcal{G})$, such that for all $T \in \mathcal{G}$, $\eta(\pi_C, T) = E\left(\mathbf{1}_{\pi_F \in T}|\pi_C\right) = E\left(\mathbf{1}_{\Omega_C \times T}|\pi_C\right)$. For all $s \in \Omega_C$ and $T \in \mathcal{G}$, let

$$\theta_n(s, T) := \eta(s, T) \cdot E^{\mu_C}[\theta(\cdot, \Omega_F)|\mathcal{F}_n](s).$$

By construction θ_n is a kernel from $(\Omega_C, \mathcal{F}_n)$ to $(\Omega_F, \mathcal{G})$. Notice that for any

[22]In Appendix B, we constructed in an identical fashion the kernel γ as a regular conditional distribution of π_C given π_F.

$S \in \mathcal{F}_n$ and $T \in \mathcal{G}$,

$$\begin{aligned}
E^{\mu_C} \mathbf{1}_S \cdot \theta_n(\cdot, T) &= E^{\mu_C} \mathbf{1}_S \cdot \eta(\cdot, T)\theta(\cdot, \Omega_F) = \int_S \int_{\Omega_F} \eta(s, T)\theta(s, dt) d\mu_C(s) \\
&= \int_{S\times\Omega_F} \eta(\pi_C, T) d\Theta = \int_{S\times\Omega_F} \mathbf{1}_{\Omega_C \times T} d\Theta = \int_S \mathbf{1}_{\Omega_C} \int_{\Omega_F} \mathbf{1}_T \theta(s, dt) d\mu_C(s) \\
&= E^{\mu_C} \mathbf{1}_S \cdot \theta(\cdot, T).
\end{aligned}$$

The first equality uses the $\mathcal{F}_n$ measurability of η in the first argument, and the law of standard expectations. The rest use equation (3.3) and the properties of the conditional distribution η.

Therefore θ_n satisfies (5.6). Consider a sequence of random variables $(X_n)_n$ with $X_n : \Omega_F \to \mathbb{R}$ being $\mathcal{G}_n$-measurable, and let $X, Y : \Omega_F \to \mathbb{R}$, $\mathcal{G}$-measurable, such that $|X_n| \leq Y$, $\int_{\Omega_F} Y d\mu_F < \infty$ and $X_n \to X$, μ_C-a.s. Starting with simple functions and then extending the argument using a monotone class theorem (**?**, Theorem 1.1), it follows that

$$W_n(s) := \int_{\Omega_F} X_n d\theta_n(s, \cdot) = E^{\mu_C} \left[\int_{\Omega_F} X_n d\theta(s', \cdot) | \mathcal{F}_n \right](s). \tag{5.7}$$

By Lebesgue's dominated convergence theorem, the sequence of functions $f_n(s') := \int_{\Omega_F} X_n d\theta(s', \cdot)$ converges pointwise to $W(s') := \int_{\Omega_F} X d\theta(s', \cdot)$. Using an extension of the martingale convergence theorem due to Hunt (**?**, Theorem 2.8.5), it follows that $W_n \to W$, μ_C-a.s. Thus we proved that indeed θ_n converges to θ in the sense of Definition 5.3. □

By the martingale convergence theorem, (5.6) implies that for each $T \in \mathcal{G}$, $\theta_n(\cdot, T) \to \theta(\cdot, T)$, μ_C-a.e. However the set of μ_C-measure zero where the convergence might fail depends on T, hence the set of s where $\theta_n(s, T)$ does not converge to $\theta(s, T)$ for some $T \in \mathcal{G}$ might be large. Thus it might not be true that, for μ_C-a.e. $s \in \Omega_C$, $\theta_n(s, \cdot)$ converges setwise to $\theta(s, \cdot)$. All that is guaranteed is that for μ_C-a.e. $s \in \Omega_C$, $\theta_n(s, T) \to \theta(s, T)$, for all $T \in \cup_n \mathcal{G}_n$, since $\cup_n \mathcal{G}_n$ is countable. Therefore Proposition 5.4 shows that condition (ii) in the Definition 5.3 is satisfied by a large class of sequences of economies and corresponding limit economies, which might not be the case if we were to replace it with the stronger requirement of θ_n having extensions to $\mathcal{G}$ converging setwise to θ.

For a general sequence of convergent economies as in Definition 5.3 and a sequence of convergent production plans, we show that agents' intermediate endowments converge almost surely (with respect to the measure of agents in the limit economy) and the distribution of intermediate endowments across types converges weakly.

Proposition 5.5. *Let $(\mathcal{E}_n)$ be a sequence of finite economies converging to an economy $\mathcal{E}$. Let y be a production plan in $\mathcal{E}$ and, for every $n \in \mathbb{N}$, let y_n be a production plan in $\mathcal{E}_n$. If $y_n \to y$, μ_F-a.s., where $\mu_F = \mu_C\theta$, then w_{y_n} converges to w_y μ_C-a.s. and $\mu_C^n \circ \widetilde{w}_{y_n}^{-1}$ converges weakly to $\mu_C \circ \widetilde{w}_y^{-1}$.*

Proof. Using equation (3.3), for any $T \in \mathcal{B}(\Omega_F)$, with $\mu_F(T) = 0$ it follows that $\theta(s, T) = 0$ for μ_C-a.e. $s \in \Omega_C$. Thus the fact that $y_n \to y$ μ_F-a.s. implies that, for μ_C-a.e. $s \in \Omega_C$, $y_n \to y$, $\theta(s, \cdot)$-a.s. Since θ_n converges to θ, it follows that $w_{y_n} \to w_y$, μ_C-a.s. (see Definition 5.3,(ii)). For any $g : \mathcal{I} \times \mathbb{R}^L \to \mathbb{R}$ continuous and bounded, Lemma C.2 applied to the sequence μ_C^n having extensions converging setwise to μ_C gives

$$\int_{\mathcal{I}\times\mathbb{R}^L} g \, d\mu_C^n \circ \widetilde{w}_{y_n}^{-1} = \int_{\Omega_C} g \circ \widetilde{w}_{y_n} \, d\mu_C^n \to \int_{\Omega_C} g \circ \widetilde{w}_y \, d\mu_C = \int_{\mathcal{I}\times\mathbb{R}^L} g \, d\mu_C \circ \widetilde{w}_y^{-1}. \tag{5.8}$$

Thus the convergence of the distribution of intermediate endowments across types is established. □

The following theorem shows that converging Cournot S-equilibrium production plans become profit maximizing in the limit for all the firms having only non-diversified shareholders.

Theorem 5.6. *Let $(\mathcal{E}_n)$ be a sequence of finite economies converging to $\mathcal{E}$, assumed atomless, and for each $n \in \mathbb{N}$, let y_n be a Cournot S-equilibrium of the economy $\mathcal{E}_n$, such that $y_n \to y$ in μ_F-measure. Assume that there exists a unique equilibrium price p associated with the production plan y in the limit economy $\mathcal{E}$ (that is $P(y)$ is a singleton). Then y is profit maximizing at prices p for μ_F-almost all firms belonging to $\Omega_F^{nd}((\mathcal{E}_n))$.*

Proof. Assume, by contradiction, that y is not profit maximizing for a μ_F-positive measure of firms belonging to $\Omega_F^{nd}((\mathcal{E}_n))$. Let $\bar{y}$ be a profit maximizing production plan at price p, such that firms of the same type have identical production plans, that is, $\bar{y}$ is selected to be $2^{\mathcal{J}} \times [0, 1]$-measurable and satisfies:

$$p \cdot \bar{y}(t) = \max_{z \in Y_{\jmath(t)}} p \cdot z.$$

Notice that $\bar{y}$ is also $\mathcal{G}_n$-measurable, for any $n \in \mathbb{N}$. The boundedness of the production sets implies that $\bar{y}$ is bounded as well. Construct the $\mathcal{G}$-measurable function $d : \Omega_F \to \mathbb{R}$ defined as $d(\cdot) := p \cdot \bar{y}(\cdot) - p \cdot y(\cdot) \geq 0$ and, for all n, let $d_n : \Omega_F \to \mathbb{R}$, $d_n(\cdot) := p \cdot \bar{y}(\cdot) - p \cdot y_n(\cdot) \geq 0$, which is $\mathcal{G}_n$-measurable.

Along a subsequence, $y_n \to y$ μ_F-a.s., since any sequence convergent in measure has a subsequence converging almost surely (**?**, Theorem 9.2.1). Thus there exists a set $T \in \mathcal{G}$ with $\mu_F(T) > 0$ such that

$$T \subset \{d > 0\} \cap \Omega_F^{nd}((\mathcal{E}_n)) \cap \{y_n \to y\} \cap \{\mu_F(\mathcal{G}_n(\cdot) \to \mu_F(\mathcal{G}(\cdot))\}.$$

Let $\bar{t} \in T$. As $d_n \to d$ on the set of full μ_F-measure where $y_n \to y$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that

$$d_n(\bar{t}) \geq \delta, \quad \forall n \geq N. \tag{5.9}$$

As $\bar{t} \in \Omega_F^{nd}((\mathcal{E}_n))$, along a subsequence, $\mu_F^n(\mathcal{G}_n(\bar{t})) > 0$ for all n and there exists $\varepsilon > 0$ such that

$$\theta_n(s, \mathcal{G}_n(\bar{t})) \geq \varepsilon, \text{ for } \gamma_n(\bar{t}, \cdot)\text{-a.e. } s \text{ and for all } n. \tag{5.10}$$

Define the alternative production plans

$$\hat{y}_n = y_n + (\bar{y} - y_n)\mathbf{1}_{\mathcal{G}_n(\bar{t})}.$$

Since the limit economy is atomless, $\mu_F(\mathcal{G}_n(\bar{t})) \to \mu_F(\mathcal{G}(\bar{t})) = 0$. This implies that $\hat{y}_n$ converges in measure to y, and hence it converges almost surely to y along a subsequence. Consider a price selection $\mathbf{p}_n$ associated to y_n such that $(\mathbf{p}_n, y_n)$ is Cournot S-equilibrium for $\mathcal{E}_n$. Let $p_n = \mathbf{p}_n(y_n)$ and $\hat{p}_n = \mathbf{p}_n(\hat{y}_n)$. By Proposition 5.5, $\mu_C^n \circ \widetilde{w}_{\hat{y}_n}^{-1} \to \mu_C \circ \widetilde{w}_y^{-1}$. Since the price correspondence $P(\cdot)$ has closed graph and $\hat{p}_n \in P\left(\mu_C^n \circ \widetilde{w}_{\hat{y}_n}^{-1}\right)$, it follows that, for any convergent subsequence $(\hat{p}_{n_r})$ of $(\hat{p}_n)$,

$$\lim_{r\to\infty} \hat{p}_{n_r} \in P\left(\lim_{r\to\infty} \mu_C^{n_r} \circ \widetilde{w}_{\hat{y}_{n_r}}^{-1}\right) = P(\mu_C \circ \widetilde{w}_y^{-1}) = \{p\}.$$

By repeating the reasoning for $(p_n)_n$, it follows that on a subsequence, $p_n, \hat{p}_n \to p$.

Equation (5.9) implies that for each $n \in \mathbb{N}$,

$$\begin{aligned} p_n \cdot y_n(\bar{t}) &\leq p \cdot y_n(\bar{t}) + \|p_n - p\| \cdot \|y_n(\bar{t})\| \leq p \cdot \hat{y}_n(\bar{t}) - \delta + \|p_n - p\| \cdot \|y_n(\bar{t})\| \\ &\leq \hat{p}_n \cdot \hat{y}_n(\bar{t}) + \|\hat{p}_n - p\| \cdot \|\hat{y}_n(\bar{t})\| - \delta + \|p_n - p\| \cdot \|y_n(\bar{t})\|. \end{aligned}$$

Since all production plans belong to a compact, it follows that there exists $\eta > 0$ and $N(\eta) \geq N$, such that for all $n \geq N(\eta)$,

$$\hat{p}_n \cdot \hat{y}_n(\bar{t}) \geq p_n \cdot y_n(\bar{t}) + \eta,$$

which, by Lemma 5.1 and equation (5.10), contradicts the Cournot S-efficiency of the production plans y_n, for all large n. □

The theorem relies heavily on the assumption that there is a unique equilibrium price corresponding to the limit production plan y in the atomless economy. This condition is needed to insure continuity of the price selection at the limit point. While we cannot dispense with it completely, the requirement can be considerably relaxed, with a construction as in **?**. That approach allows for multiplicity of equilibria at the limit point, but requires regularity of the limit equilibrium and thus its *local* uniqueness. Even so, it remains a strong condition since, as shown by Roberts himself, existence of critical equilibria is non-pathological.

? pointed out that this negative result is alleviated if, instead of simple price selections, one uses *randomized* price selections (i.e., selections from the correspondence *coP* instead of P; this amounts to saying that firms hold non-trivial beliefs over the possible market clearing prices). As opposed to the case of simple price selections, the existence of continuous randomized price selections is a generic result (**?**). Allen proves therefore that, if firms maximize their *expected* profits with respect to some non-trivial beliefs over prices, convergence of Cournot equilibria (in which firms maximize profits) to competitive equilibria does obtain generically. However, the problem is more complex here and Allen's approach cannot be directly applied. The reason is that, as opposed to the standard Cournot model in which the firms maximize profits, in our model S-efficiency requires firms to make pairwise comparisons between a status quo and an alternative. For that, a firm has to use its beliefs over two different equilibrium sets. To make this comparison meaningful, some global beliefs need to be defined. This was done in **?**. Whether allowing for firms' non-trivial global beliefs over prices does indeed improve the convergence result is an interesting question which remains open for now and will be subject of future research.

For the diffuse ownership economies $(\mathcal{E}_n^d)$ defined in (3.6),(3.7), $\Omega_F^{nd}((\mathcal{E}_n^d)) = \emptyset$, and Theorem 5.6 is devoid of implications. For the concentrated ownership case described in (3.4),(3.5), $\Omega_F^{nd}((\mathcal{E}_n^c)) = \Omega_F$ and thus the limit of Cournot S-efficient equilibria is always a Walrasian equilibrium. Moreover, for this particular case, Theorem 5.6 holds even under the weaker assumption that production plans converge weakly (i.e., in distribution). This was shown by **?**, using the fact that in this symmetric environment, the distribution of intermediate endowments associated to a production plan depends solely on the distribution of the production plan (both in the finite and continuum economies), and retracing the steps in the proof of Theorem 5.6. For completeness, we give an alternative proof of this result here, by showing that it follows from Theorem 5.6, through a Skorohod embedding argument. For a production plan $y : (\mathcal{J} \times [0,1], 2^{\mathcal{J}} \otimes \mathcal{B}[0,1], \lambda_{\mathcal{J}} \otimes \lambda) \to \mathbb{R}^L$, we let $\mathcal{L}(y)$ be the distribution of $(y(1,\cdot), \ldots, y(J,\cdot)) : [0,1] \to (\mathbb{R}^L)^J$ and refer to it as the *law* of y. Thus $\mathcal{L}(y) := \lambda \circ (y(1,\cdot), \ldots, y(J,\cdot))^{-1}$.

Theorem 5.7. *Let y be a production plan in $\mathcal{E}^c$ and for each $n \in \mathbb{N}$, let y_n be a Cournot S-equilibrium of the concentrated ownership economy $\mathcal{E}_n^c$, such that the laws of (y_n) converge weakly to the law of y. Assume that there exists a unique equilibrium price p associated with the production plan y in the limit economy $\mathcal{E}^c$ (i.e. $P(y)$ is a singleton). Then y is profit maximizing at prices p.*

Proof. By Skorohod's embedding theorem (**?**, Theorem 4.30), there are alternative production plans $\hat{y}$ and $(\hat{y}_n)$, defined on $(\mathcal{J} \times [0,1], 2^{\mathcal{J}} \otimes \mathcal{B}[0,1], \lambda_{\mathcal{J}} \otimes \lambda)$, such that $\mathcal{L}(\hat{y}) = \mathcal{L}(y)$, $\mathcal{L}(y_n) = \mathcal{L}(\hat{y}_n)$ for all $n \in \mathbb{N}$, and $\hat{y}_n \to \hat{y}$ μ_F-a.s., where $\mu_F := \lambda_{\mathcal{J}} \otimes \lambda$.

Fix a $j \in \mathcal{J}$. Let $z \in y_n(j \times [0,1]) \subset \mathbb{R}^L$, and $A := (\hat{y}_n(j,\cdot))^{-1}(z)$. Since $\lambda(A) = \lambda((y_n(j,\cdot))^{-1}(z))$, there exists $k \in \{1, \ldots, n\}$ such that $\lambda(A) = k/n$ and there exist $\{l_1, \ldots, l_k\} \subset \{1, \ldots, n\}$ such that $(y_n(j,\cdot))^{-1}(z) = \cup_{r=1}^k H_n^{l_r}$.[23] Since λ is atomless, by Lyapunov's convexity theorem (**?**, Theorem 13.33) we can construct disjoint sets $A_{l_1}, \ldots, A_{l_k} \subset [0,1]$ such that $A = \cup_{r=1}^k A_{l_r}$ and $\lambda(A_{l_r}) = 1/n$ for all $r \in \{1, \ldots, k\}$. Repeating this process for each $z \in y_n(j \times [0,1])$, we construct for each $j \in \mathcal{J}$ the partition $A_1^j, \ldots, A_n^j$ of $[0,1]$ with $\lambda(A_l^j) = 1/n$ and $\hat{y}_n(j \times A_l^j) = y_n(j \times H_n^l)$ for all $l \in \{1, \ldots, n\}, j \in \mathcal{J}$. Thus $\hat{y}_n$ is measurable with respect to the algebra $\hat{\mathcal{G}}_n := \sigma\left(\left\{j \times A_l^j \mid j \in \mathcal{J}, l \in \{1, \ldots, n\}\right\}\right)$ of $\mathcal{J} \times [0,1]$.

[23]As defined in section 3, $H_n^k = \left(\frac{k-1}{n}, \frac{k}{n}\right]$ for $k \in \{2, \ldots, n\}$ and $H_n^1 = [0, \frac{1}{n}]$.

Construct the economy $\hat{\mathcal{E}}_n := ((\Omega_C, \mathcal{F}^c_n, \lambda_{\mathcal{I}} \otimes \lambda), (\Omega_F, \hat{\mathcal{G}}_n), \hat{\theta}_n)$, in which each consumer (i, H^l_n) owns a share s_{ij} of the firm $j \times A^j_l$. Thus

$$\hat{\theta}_n((i, H^k_n), (j, A^j_l)) := s_{ij} \cdot \delta_{kl}, \quad \forall (i, a) \in \Omega_C, j \in \mathcal{J}, 1 \leq l, k \leq n,$$

and $\delta_{kl} = 1$ if $k = l$, while $\delta_{kl} = 0$ if $k \neq l$. Notice that $\hat{y}_n$ is a Cournot S-efficient production plan for $\hat{\mathcal{E}}_n$. Otherwise, a welfare improving deviation for the shareholders of firm $j \times A^j_l$ in economy $\hat{\mathcal{E}}_n$ would be welfare improving for the shareholders of firm $j \times H^l_n$ in economy $\mathcal{E}_n$, contradicting the Cournot S-efficiency of the plan y_n.

The intermediate endowment of an agent $(i, \cdot)$ given a production plan y' in either of the economies $\mathcal{E}^c_n, \hat{\mathcal{E}}_n$ or $\mathcal{E}^c$ is $w_{y'}(i, \cdot) = e^i + \sum_{j \in \mathcal{J}} s(i, j) y'(j, \cdot)$. If (y'_n) in $(\hat{\mathcal{E}}_n)$ converge in law to $\hat{y}$, by the continuous mapping theorem (**?**, Theorem 9.3.7), it follows that the distribution $\mu_C \circ \widetilde{w}^{-1}_{y'_n}$ (of intermediate endowments across types induced by y') converges to $\mu_C \circ \widetilde{w}^{-1}_{\hat{y}}$. We can apply Theorem 5.6 to $(\hat{\mathcal{E}}_n), \mathcal{E}^c$ and $(\hat{y}_n)$ converging to $\hat{y}$ μ_F-a.s to conclude that $\hat{y}$ is profit maximizing on $\Omega^{nd}_F((\hat{\mathcal{E}}_n)) = \Omega_F$.

Theorem 5.6 holds despite the sequence $(\hat{\mathcal{E}}_n)$ not converging to $\mathcal{E}^c$. Indeed, condition (ii) in definition 5.3 might fail. However its only use in the proof of Theorem 5.6 is to ensure the convergence of the distribution of intermediate endowments associated to convergent production plans which, in this symmetric environment, holds as shown above. Condition (iii) in definition 5.3 is trivially satisfied; condition (i) also holds, since for all $t \in \Omega_F$, $\mu_F(\hat{\mathcal{G}}_n)(t) = \frac{1}{nJ} \to 0$. $\square$

Convergence in distribution of production plans is too weak for Theorem 5.6, in which we allow for heterogeneity in the ownership of firms and consumers of identical type. Given an arbitrary production plan, one can permute the choices of identical type firms, resulting in two production plans with identical distributions; however, in the presence of asymmetric ownership the two plans induce different distributions over the space of intermediate endowments.

Given a sequence of economies $(\mathcal{E}_n)$ converging to an economy $\mathcal{E}$, as in Theorem 5.6, it is natural to ask what is the relationship between the set $\Omega^{max}_F(\mathcal{E})$ (see (4.3)) of firms that are profit maximizing in any Cournot S-efficient equilibrium of the atomless limit economy, and the set $\Omega^{nd}_F((\mathcal{E}_n))$ of firms that are profit maximizing in the limit of a convergent sequence of Cournot S-efficient equilibria. We already remarked that for the diffuse ownership case (see (3.6),(3.7)), $\Omega^{max}_F(\mathcal{E}^d) = \Omega^{nd}_F((\mathcal{E}^d_n)) = \emptyset$, while for the concentrated ownership case (see (3.4),(3.5)), $\Omega^{max}_F(\mathcal{E}^c) = \Omega^{nd}_F((\mathcal{E}^c_n)) = \Omega_F$. Example 5.2 below constructs economies $\mathcal{E}_n \to \mathcal{E}$ where $\Omega^{nd}_F((\mathcal{E}_n)) = \emptyset$, while $\Omega^{max}_F(\mathcal{E}) = \Omega_F$. In fact, in all the examples included in the paper, $\Omega^{nd}_F((\mathcal{E}_n)_{n \in \mathbb{N}}) \subset \Omega^{max}_F(\mathcal{E})$, since the kernels (γ_n) denoting the distribution of firms' shares across consumers in $(\mathcal{E}_n)$ have (for all $t \in \Omega_F$) extensions to $\mathcal{F}$ converging setwise to the kernel γ associated to $\mathcal{E}$.[24]

[24]Indeed, let $\bar{t} \in \Omega^{nd}_F((\mathcal{E}_n))$. Thus there exists $\varepsilon > 0$ such that (along a subse-

The examples may also lead us to conjecture that a sequence of Cournot S-equilibria of a converging sequence of finite economies approaches a Cournot S-efficient equilibrium of the limit economy. If true, this property would imply, according to Proposition 4.3, that convergent sequences of Cournot S-equilibria of converging finite economies approach a profit maximizing plan of the limit economy for firms in $\Omega_F^{max}(\mathcal{E})$. The following example shows that the conjecture is not true and thus sequences of Cournot S-equilibria do not necessarily converge to a Cournot S-equilibrium of the limit economy.

Example 5.2
We modify the ownership structures in the Example 5.1. In the finite n-fold replica $\mathcal{E}_n$ and in the continuum replica $\mathcal{E}_\infty$, half of each firm is owned exclusively by the agent with the same name and the rest is uniformly distributed across all agents (including the agent with the same name). We will refer to this way of assigning ownership of the firms in the replicas as the *hybrid ownership structure*.

Given a production plan $y = ((-\alpha_j, \alpha_j))_{j=1}^n$ in the n-fold replica $\mathcal{E}_n$, the unique exchange equilibrium price vector, normalized to the unit simplex, is given by (5.4), with $\kappa(y)$ defined as in (2.1). For the continuum replica, prices have the same expression, with $\kappa(y)$ defined in (2.2). The Walrasian equilibrium in the finite and continuum replica economies are associated with prices $(\frac{1}{2}, \frac{1}{2})$ and $\kappa(y) = 0$, hence all firms choose the production plan $(0, 0)$ in a Walrasian equilibrium.

We start by determining the Cournot S-efficient production plans in the n-fold replica economy. The wealth of a consumer that is a majority shareholder in a firm choosing $(-\alpha, \alpha)$ is

$$w(\kappa(y), \alpha) = 2 + \frac{1}{2}(p_2 - p_1)(\alpha + \kappa(y)) = 2 - \frac{1}{4}\kappa^2(y) - \frac{1}{4}\kappa(y)\alpha,$$

and its utility is $u(\kappa(y), \alpha) = 2\ln w(\kappa(y), \alpha) - \ln(2p_1) - \ln(2p_2)$. We let $\kappa := \kappa(y)$ for brevity. Notice that

$$\frac{\partial u(\kappa, \alpha)}{\partial \kappa} = \frac{2}{n} \frac{(\kappa^3 - 4\alpha)}{(\kappa\alpha + \kappa^2 - 8)(\kappa + 2)(\kappa - 2)}, \tag{5.11}$$

and it follows that the derivative of u with respect to α is negative:

$$\frac{du(\kappa, \alpha)}{d\alpha} = \frac{1}{n}\frac{\partial u(\kappa, \alpha)}{\partial \kappa} + \frac{\partial u(\kappa, \alpha)}{\partial \alpha} = \frac{2}{n} \frac{(\kappa^3 - 8n\kappa - 4\alpha + 2n\kappa^3)}{(\kappa\alpha + \kappa^2 - 8)(\kappa + 2)(\kappa - 2)} < 0.$$

Thus a firm that chooses $(-\alpha, \alpha)$ hurts its majority shareholder by switching to a production plan (α', α') with $\alpha' > \alpha$. Moreover, (5.11) shows that by switching to a production plan (α', α') with $\alpha' < \alpha$, the firm hurts a minority

quence), $1 = \gamma_n(\bar{t}, \{\theta_n(\cdot, \mathcal{G}_n(\bar{t})) \geq \varepsilon\})$. Since $\mathbf{1}_{\mathcal{G}_n(\bar{t})} \to \mathbf{1}_{\mathcal{G}(\bar{t})}$ μ_F-a.s. (along a subsequence), by Definition 5.3, (ii), it follows that $\theta_n(\cdot, \mathcal{G}_n(\bar{t})) \to \theta(\cdot, \mathcal{G}(\bar{t}))$, μ_C-a.s., and hence $\mathbf{1}_{\theta_n(\cdot, \mathcal{G}_n(\bar{t})) \geq \varepsilon} \to \mathbf{1}_{\theta(\cdot, \mathcal{G}(\bar{t})) \geq \varepsilon}$ (μ_C-a.s.). By Lemma C.2, we conclude that $\gamma_n(\bar{t}, \{\theta_n(\cdot, \mathcal{G}_n(\bar{t})) \geq \varepsilon\}) \to \gamma(\bar{t}, \{\theta(\cdot, \mathcal{G}(\bar{t})) \geq \varepsilon\})$, hence $\bar{t} \in \Omega_F^{max}(\mathcal{E})$.

shareholder that owns half of a firm that chose a production plan $(-\beta, \beta)$ satisfying $\left(\kappa - \frac{\alpha}{n}\right)^3 \geq 4\beta$. This discussion enables us to construct a multitude of Cournot S-equilibria. In particular, for any $k \in \{1, 2, \ldots, n-1\}$, a production plan with k firms choosing $(0, 0)$ and $n - k$ firms choosing $(-1, 1)$ is always a Cournot S-equilibrium.

The economies $(\mathcal{E}_n)$ and $\mathcal{E}_\infty$ can be embedded in the general framework of section 3 as discussed there, by letting $\theta_n = \theta_n^c/2 + \theta_n^d/2$ and $\theta = \theta^c/2 + \theta^d/2$ (see (3.4)-(3.9)). Notice that for the limit economy, $\Omega_F^{max}(\mathcal{E}_\infty) = \Omega_F = [0, 1]$ (since $\mathcal{I} = \mathcal{J} = \{1\}$, we identify $\{1\} \times [0, 1]$ with $[0, 1]$) and thus by Proposition 4.3, the only Cournot S-equilibrium allocation of the continuum economy $\mathcal{E}_\infty$ coincides with the Walrasian equilibrium and corresponds to all firms choosing $(0, 0)$. This can be seen directly, also, since in the absence of price effects, any firm that chose $(-\alpha, \alpha)$ with $\alpha > 0$ will increase the wealth and hence the utility of its majority shareholder by switching to $(0, 0)$, while its minority shareholders are unaffected.

Notice that for any $\varepsilon > 0$ and $k \in \mathbb{N}$, $\Omega_F^{nd}((\mathcal{E}_n)) = \emptyset$, hence Theorem 5.6 has no bite in this example, suggesting that a sequence of Cournot S-equilibrium plans does not converge necessarily to a profit maximization plan. Indeed, for an arbitrary $\eta \in [0, 1]$, consider the production plan y^η in the limit economy in which firms in $[0, \eta]$ choose the production plan $(-1, 1)$ and the firms in $(\eta, 1]$ choose $(0, 0)$. Let the production plan y_n^η in the economy $\mathcal{E}_n$ be such that the first $[n \cdot \eta]_*$ firms (i.e., firms in $[0, [n \cdot \eta]_*/n]$) choose $(-1, 1)$ and the the rest choose $(0, 0)$ ($[n \cdot \eta]_*$ denotes the largest integer smaller than $n \cdot \eta$). Clearly $y_n^\eta \to y^\eta$ almost surely and (y_n^η) is a sequence of Cournot S-equilibrium production plans, but y^η is not a profit maximizing plan unless $\eta = 0$. This shows that a convergent sequence of Cournot S-equilibrium production plans in converging economies does not have to approach a Cournot S-equilibrium in the limit.

6 Conclusions

This paper contributes to the literature on non-cooperative foundations of Walrasian equilibrium, by pointing out to the firms' ownership structure as a potential source of inefficiency in arbitrarily large economies. If (some) shareholders control a firm's production decisions, its objective is shaped by the interaction between the price and the income effects on those shareholders' welfare. Each of these effects, and therefore the dominance of one over the other, depends on the ownership structure.

The Cournotian foundations of Walrasian equilibrium require the introduction of a notion of finite economies being close (converging) to a continuum economy. As explained in the introduction, this is difficult to achieve when one needs to allow for arbitrary ownership structures (whose role needs to be explored). The paper contributes to the literature by defining a suitable topology on the space of production economies, which generalizes previous results and allows for full generality on the ownership structure. We give conditions on the ownership structures of a sequence of economies, and of a limit economy, such that agents' intermediate endowments generated by a converging sequence of production plans approach the intermediate endowments of the limit economy. We focus on the large economies behavior of Cournot S-efficient equilibria, in which firms choose production plans that are not Pareto dominated from the point of view of their shareholders. The classical result of **?** indicates that profit maximization (under a specific price normalization) is a justified objective for an oligopolistic firm in a large economy, since gains obtained by deviating to shareholders' welfare improving plans are modest. We derive this result in our framework removing the very restrictive assumption "that agents are typical" used by **?** and imposed on the agents' wealth (which is an endogenous variable). We then use this result to show that if each of the (controlling) shareholders of a firm owns a significant (i.e., bounded away from zero) fraction of the firm, then Cournot S-efficient equilibria of large economies consists of production plans that are approximatively profit maximizing. Moreover, Cournot S-equilibria of a converging sequence of finite economies approaches a Walrasian equilibrium of the limit economy. For arbitrary ownership structures, sequences of Cournot S-efficient equilibria may not converge to a Walrasian equilibrium or to a Cournot S-equilibrium of the limit economy.

Although we do not model trade in shares, we do allow for *arbitrary* (fixed) distributions of shares in each finite economy along the converging sequence and identify the class of those ownership structures that are conducive to competitive behavior. Our results bear implications even for richer environments in which share trading is allowed. It shows, for example, that perfectly competitive behavior will prevail in any large economy model of security trade in which the (post-trade) equilibrium distribution of shares is concentrated. On the other hand, perfect diversification of individual portfolios across firms (as predicted, for example, by mean-variance portfolio selection models) might lead to inefficiencies.

Appendix

A Atoms of a countably generated σ-algebra

Let $\mathcal{A}$ be a σ-algebra on Ω. Define a binary relation on Ω as: $x \sim y$ if and only if $x \in A, A \in \mathcal{A} \Rightarrow y \in A$. Equivalently, if for $x \in \Omega$, we define $\mathcal{A}(x) := \cap\{A \in \mathcal{A} : x \in A\}$, then $x \sim y$ if and only if $y \in \mathcal{A}(x)$. It is easy to see that "$\sim$" is an equivalence relation, and hence $\mathcal{A}(x)$ is the equivalence class containing x. One is tempted to call $\mathcal{A}(x)$ an *atom* of $\mathcal{A}$, in the sense of the definition in footnote 3. However, in general $\mathcal{A}(x) \notin \mathcal{A}$.

We show in what follows that if $\mathcal{A}$ is a *countably generated* σ-algebra, i.e., if $\mathcal{A}$ is generated by a countable subset of itself, then $\mathcal{A}(x) \in \mathcal{A}, \forall x \in \Omega$. This means that $\mathcal{A}(x)$ is an atom of $\mathcal{A}$, i.e., for all $B \in \mathcal{A}$, either $\mathcal{A}(x) \subset B$ or $\mathcal{A}(x) \cap B = \emptyset$. Let $\mathcal{C}$ be a countable subset generating $\mathcal{A}$, i.e. $\mathcal{A} = \sigma(\mathcal{C})$, and let $\bar{\mathcal{C}}$ be the algebra generated by $\mathcal{C}$, which consists exactly of all elements of $\mathcal{C}$ together with all sets obtainable from finite sequences of set theoretic operations on $\mathcal{C}$. Thus $\bar{\mathcal{C}}$ is also countable. Fix a point $x \in A$, and let $\bar{\mathcal{C}}(x) := \cap\{C \in \bar{\mathcal{C}} | x \in C\}$. Define

$$\mathcal{D}_x := \{A \in \mathcal{A} \mid x \notin A\} \cup \{A \in \mathcal{A} \mid x \in A, \bar{\mathcal{C}}(x) \subset A\}.$$

It is easy to check that $\mathcal{D}_x$ is a λ-system, which means that it contains Ω and is closed under proper differences and increasing limits. Moreover $\bar{\mathcal{C}} \subset \mathcal{D}_x$, and $\bar{\mathcal{C}}$ is a $\pi-$system (i.e. is closed under finite intersections). The monotone class theorem (**?**, Th. 1.1) implies that

$$\mathcal{A} = \sigma(\bar{\mathcal{C}}) \subset \mathcal{D}_x \subset \mathcal{A},$$

and thus $\mathcal{D}_x = \mathcal{A}$. It follows that $\bar{\mathcal{C}}(x) = \mathcal{A}(x)$, but $\bar{\mathcal{C}}(x) \in \mathcal{A}$ since $\bar{\mathcal{C}}$ is countable.

B Construction of the γ-kernel

Let $\alpha := (\mu_C \otimes \theta)(\Omega_C \times \Omega_F)$. The assumptions made guarantee that $0 < \alpha < \infty$ and thus we can write $\mu_C \otimes \theta = \alpha \cdot \Theta$, with Θ a probability on $\mathcal{F} \otimes \mathcal{G}$. Define π_C, π_F to be the projection functions of $(\Omega_C \times \Omega_F, \mathcal{F} \otimes \mathcal{G})$ on $(\Omega_C, \mathcal{F})$, respectively on $(\Omega_F, \mathcal{G})$. Since $(\Omega_C, \mathcal{F})$ is a Polish space, there exists a regular conditional distribution of π_C given π_F, which is probability kernel γ from $(\Omega_F, \mathcal{G})$ to $(\Omega_C, \mathcal{F})$ (**?**, **?**). The kernel γ is γ is unique μ_F-a.s., in the sense that if γ' has the above properties, then for μ_F-a.e. $t \in \Omega_F$, $\gamma(t, \cdot) = \gamma'(t, \cdot)$.

Let Θ_F be the marginal of Θ on Ω_F. By construction, and using the fact that $\mu_C \otimes \theta = \alpha \cdot \Theta$, it follows that for any $g : \Omega_C \times \Omega_F \to \mathbb{R}$, which is $\mathcal{F} \otimes \mathcal{G}$-measurable and $\mu_C \otimes \theta$-integrable,

$$\begin{aligned}\int_{\Omega_C} \left[\int_{\Omega_F} g(s,t)\theta(s,dt)\right] \mu_C(ds) &= \int_{\Omega_C \times \Omega_F} g(s,t)(\mu_C \otimes \theta)(ds,dt) \qquad \text{(B.1)}\\ &= \int_{\Omega_F} \left[\int_{\Omega_C} g(s,t)\gamma(t,ds)\right] (\alpha \cdot \Theta_F)(dt)\\ &= \int_{\Omega_F} \left[\int_{\Omega_C} g(s,t)\gamma(t,ds)\right] \mu_F(dt),\end{aligned}$$

and hence we obtained equation (3.3).

C Setwise convergence of measures on a filtration

For all $n \in \mathbb{N}$, let ν_n be a measure on $(\Omega, \mathcal{A}_n)$ where $\mathcal{A}_n$ is finite and $\mathcal{A}_n \nearrow \mathcal{A}$ (i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ and $\mathcal{A} = \sigma(\cup_n A_n)$), and let ν be a finite measure on $(\Omega, \mathcal{A})$. The next result provides sufficient conditions for the existence of extensions $(\tilde{\nu}_n)$ of the measures (ν_n) to $\mathcal{A}$ that converge setwise to ν. This means that, for all $n \in \mathbb{N}$, the restriction of $\tilde{\nu}_n$ to $\mathcal{A}_n$ coincides with ν_n (i.e. $\tilde{\nu}_n|_{\mathcal{A}_n} = \nu_n$) and

$\tilde{\nu}_n(A) \to \nu(A)$ for all $A \in \mathcal{A}$. Notice that if such extensions $(\tilde{\nu}_n)$ are to exist, then for any m and $A_m \in \mathcal{A}_m$, $\lim_{n\to\infty} \nu_n(A_m) = \nu(A_m)$. It turns out that this condition is also sufficient, in the presence of a uniform boundedness condition imposed on (ν_n).

Lemma C.1. *Assume that*

(i) For any $m \in \mathbb{N}$ and $A_m \in \mathcal{A}_m$, $\lim_{n\to\infty} \nu_n(A_m) = \nu(A_m)$,

(ii) There exists $L > 0$ such that $\nu_n \le L \cdot \nu$ for all $n \in \mathbb{N}$, that is,

$$\nu_n(A) \le L \cdot \nu(A), \quad \forall n \in \mathbb{N}, \forall A \in \mathcal{A}_n.$$

Then (ν_n) have extensions to $\mathcal{A}$ that converge setwise to ν.

Proof. For all n, label the atoms of $\mathcal{A}_n$ as $A_1^n, A_2^n, \ldots, A_{k(n)}^n$. Define

$$\tilde{\nu}_n(A) := \sum_{i=1}^{k(n)} \nu_n(A_i^n) \cdot \frac{\nu(A \cap A_i^n)}{\nu(A_i^n)}, \quad \forall A \in \mathcal{A}.$$

Thus $\tilde{\nu}_n$ is constructed by summing the measures obtained as the conditionals of ν with respect to each atom of $\mathcal{A}_n$, scaled so that the measure of each atom of $\mathcal{A}_n$ coincides under $\tilde{\nu}_n$ and ν_n. Clearly $\tilde{\nu}_n$ is a measure on $\mathcal{A}$ which is equal to ν_n when restricted to $\mathcal{A}_n$. Define

$$\mathcal{D} := \{A \in \mathcal{A} \mid \tilde{\nu}_n(A) \to \nu(A)\}.$$

Condition (i) implies that $\cup_n \mathcal{A}_n \subset \mathcal{D}$. In particular, $\Omega \in \mathcal{D}$. Moreover, $\mathcal{D}$ is closed under proper differences, since if $A, B \in \mathcal{D}$ with $A \subset B$, then

$$\tilde{\nu}_n(B \setminus A) = \tilde{\nu}_n(B) - \tilde{\nu}_n(A) \to \nu(B) - \nu(A) = \nu(B \setminus A).$$

We will show that $\mathcal{D}$ is closed under increasing limits. Let $A_1, A_2, \ldots$ disjoint sets in $\mathcal{D}$. Notice that $\tilde{\nu}_n(\cup_m A_m) = \sum_m \tilde{\nu}_n(A_m)$, since $\tilde{\nu}_n$ is a sum of a finite number of measures, and $\tilde{\nu}_n(A_m) \le L \cdot \nu(A_m)$, while $\sum_m \nu(A_m) = \nu(\cup_m A_m) < \infty$. Lebesgue's dominated convergence theorem implies

$$\lim_{n\to\infty} \tilde{\nu}_n(\cup_m A_m) = \sum_m \lim_{n\to\infty} \tilde{\nu}_n(A_m) = \sum_m \nu(A_m) = \nu(\cup_m A_m).$$

It follows that $\cup_m A_m \in \mathcal{D}$. We proved that $\mathcal{D}$ is a λ-system containing the algebra $\cup_n \mathcal{A}_n$ which is a π-system, being closed under finite intersections. The $\pi - \lambda$ theorem (**?**, Theorem 1.1) implies that $\mathcal{D} = \mathcal{A}$, and hence we proved that, indeed, $\tilde{\nu}_n \to \nu$ setwise on $\mathcal{A}$. $\square$

If a sequence of measures $(\tilde{\nu}_n)$ on $\mathcal{A}$ converges setwise to ν, then $E^{\tilde{\nu}_n}(f) \to E^{\nu}(f)$ for any bounded function $f : \Omega \to \mathbb{R}$ which is $\mathcal{A}$-measurable (**?**, p.335). This result is strengthened in the next Lemma.

Lemma C.2. *Assume that for all n, ν_n has an extension to $\mathcal{A}$ that converges setwise to ν. For all $n \in \mathbb{N}$, let $X_n : \Omega \to \mathbb{R}$ such that X_n is $\mathcal{A}_n$-measurable, $|X_n| < M$, and $X_n \to X$, ν-almost surely, where $X : \Omega \to \mathbb{R}$ is $\mathcal{A}$-measurable. Then* $\lim_{n\to\infty} \int_\Omega X_n d\nu_n = \int_\Omega X d\nu$.

Proof. Let $\tilde{\nu}_n$ be an extension of ν_n to $\mathcal{A}$ that converges setwise to ν. X_n is $\mathcal{A}_n$-measurable, therefore $\int_\Omega X_n d\nu_n = \int_\Omega X_n d\tilde{\nu}_n$. By the triangle inequality,

$$\left| \int_\Omega X_n d\tilde{\nu}_n - \int_\Omega X d\nu \right| \leq \int_\Omega |X_n - X| d\tilde{\nu}_n + \left| \int_\Omega X d\tilde{\nu}_n - \int_\Omega X d\nu \right|. \qquad \text{(C.1)}$$

Pick $\varepsilon > 0$ arbitrary. Notice that

$$\int_\Omega |X_n - X| d\tilde{\nu}_n \leq \varepsilon \cdot \tilde{\nu}_n(\Omega) + 2M \cdot \tilde{\nu}_n(\{|X_n - X| \geq \varepsilon\}). \qquad \text{(C.2)}$$

Define $A_m := \cup_{n \geq m}\{|X_n - X| \geq \varepsilon\}$. Since $X_n \to X$, ν-a.s., it follows that $A_m \searrow A$ with $\nu(A) = 0$. The triangle inequality implies

$$|\tilde{\nu}_n(A_n) - \nu(A)| \leq \tilde{\nu}_n(A_n \setminus A) + |\tilde{\nu}_n(A) - \nu(A)|.$$

Since $(A_n \backslash A) \searrow \emptyset$, by the Vitali-Hahn-Saks theorem (**?**, p.34), $\lim_{m\to\infty} \sup_n \tilde{\nu}_n(A_m \backslash A) \to 0$. As $\tilde{\nu}_n(A) \to \nu(A)$ and $\int_\Omega X d\tilde{\nu}_n \to \int_\Omega X d\nu$, we can choose $N_1(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N_1(\varepsilon)$, $\tilde{\nu}_n(\{|X_n - X| \geq \varepsilon\}) \leq \varepsilon$ and $\left|\int_\Omega X d\tilde{\nu}_n - \int_\Omega X d\nu\right| \leq \varepsilon$. By the setwise convergence of $\tilde{\nu}_n$ to ν, we can choose $N_2(\varepsilon) \in \mathbb{N}$ such that $\tilde{\nu}_n(\Omega) \leq \nu(\Omega) + \varepsilon$, for all $n \geq N_2(\varepsilon)$. Equations (C.1) and (C.2) imply that for all $n \geq \max\{N_1(\varepsilon), N_2(\varepsilon)\}$,

$$\left| \int_\Omega X_n d\tilde{\nu}_n - \int_\Omega X d\nu \right| \leq \varepsilon \cdot (\nu(\Omega) + \varepsilon) + 2M\varepsilon + \varepsilon.$$

Since ε can be chosen arbitrarily small, the conclusion follows. □

References

ALIPRANTIS, C. D., AND K. C. BORDER (2006): *Infinite Dimensional Analysis. A Hitchhiker's Guide*. Springer, 3rd edn.

ALIPRANTIS, C. D., D. J. BROWN, AND O. BURKINSHAW (1987): "Edgeworth Equilibria in Production Economies," *Journal of Economic Theory*, 43(2), 252–291.

ALLEN, B. (1994): "Randomization and the limit points of monopolistic competition," *Journal of Mathematical Economics*, 23, 205–218.

BEJAN, C. (2005): "Market Imperfections and the Objective of a Privately Owned Firm," Ph.D. thesis, University of Minnesota.

——— (2008): "The Objective of a Privately Owned Firm under Imperfect Competition," *Economic Theory*, 37(1), 99–118.

Bejan, C., and F. Bidian (2010): "Ownership structure and efficiency in large economies," mimeo.

Debreu, G., and H. Scarf (1963): "A Limit Theorem on the Core of an Economy," *International Economic Review*, 4, 235–246.

Demichelis, S., and K. Ritzberger (2009): "A general equilibrium analysis of corporate control and the stock market," *Economic Theory*, doi: 10.1007/s00199-009-0511-8.

Dierker, E., and B. Grodal (1999): "The Price Normalization Problem in Imperfect Competition and the Objective of the Firm," *Economic Theory*, 14(2), 257–284.

Dudley, R. (2002): *Real Analysis and Probability*. Cambridge University Press.

Florenzano, M., and L. D. Mercato (2004): "Edgeworth and Lindahl-Foley equilibria of a general equilibrium model with private provision of public goods," Cahiers de la MSE, Universite Paris Pantheon-Sorbonne, B04082.

Gabszewicz, J. J., and J. P. Vial (1972): "Oligopoly "a la Cournot" in a general equilibrium analysis," *Journal of Economic Theory*, 4(3), 381–400.

Hart, O. D. (1979): "On Shareholder Unanimity in Large Stock Market Economies," *Econometrica*, 47(5), 1057–1083.

Hildenbrand, W. (1970): "On Economies with many Agents," *Journal of Economic Theory*, 2(2), 161–188.

——— (1975): "Distributions of Agents' Characteristics," *Journal of Mathematical Economics*, 2(2), 129–138.

Hildenbrand, W., and J. F. Mertens (1972): "Upper Hemi-Continuity of the Equilibrium-Set Correspondence for Pure Exchange Economies," *Econometrica*, 40(1), 99–108.

Kallenberg, O. (2002): *Foundations of modern probability*. Springer, 2nd edn.

Kopp, P. (1984): *Martingales and stochastic integration*. Cambridge University Press.

Mas-Colell, A. (1980): "Noncooperative Approaches to the Theory of Perfect Competition: Presentation," *Journal of Economic Theory*, 22(2), 121–135.

——— (1982): "The Cournotian foundations of Walrasian equilibrium theory: an exposition of recent theory," in *Advances in Economic Theory*. Cambridge University Press.

Mas-Colell, A., and J. H. Nachbar (1991): "On the Finiteness of the Number of Critical Equilibria, with an Application to Random Selections," *Journal of Mathematical Economics*, 20(4), 397–409.

McKenzie, L. W. (1981): "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, 49(4), 819–841.

Roberts, K. W. S. (1980): "The Limit Points of Monopolistic Competition," *Journal of Economic Theory*, 22(2), 256–278.

Stokey, N., and R. Lucas (1989): *Recursive Methods in Economic Dynamics*. Harvard Press.

Xiong, S., and C. Z. Zheng (2007): "Core equivalence theorem with production," *Journal of Economic Theory*, 137(1), 246 – 270.