Generalizing the Axiomatization of the Core

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Abstract

By definition, the core assumes the formation of the grand coalition. In this setting Peleg (1986) axiomatizes the core on the class of balanced transferable utility (TU) games. We generalize Peleg's results by relaxing the usual feasibility condition, thus allowing proper coalitions to arise. Using non-emptiness, individual rationality, and adapted versions of Peleg's reduced game property (consistency) and superadditivity, we generalize his axiomatization of the core to larger families of games. Our axioms characterize the C-core (Guesnerie and Oddou (1979), Sun, Trockel, and Yang (2008)) and the aspiration core (Bennett (1983), Cross (1967)), two core extensions previously studied in the literature. The second result generalizes Peleg's axiomatization to the entire family of TU-games.

Key words: core extensions; axiomatization; aspiration core; C-core; consistency

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1 Introduction

Cooperative game theory is ideally equipped to deal with issues regarding coalition formation. Nevertheless, its two main solution concepts, the core (Gillies 1959) and the Shapley value (Shapley 1953), assume that all players will work together in a single group. Perhaps not surprinsingly, the axiomatization literature typically restricts attention to solution concepts that select a way to distribute the worth of the *grand coalition* among its members. Any payoff vector exceeding such amount is discarded as unfeasible (Peleg 1985, Peleg 1986, Peleg 1989, Tadenuma 1992, Hwang and Sudhölter 2001, Voorneveld and Van Den Nouweland 1998). In this paper we investigate the role of the feasibility condition in the axiomatization of the core, and show that the axioms that uniquely characterize the core on the domain of balanced games, when coupled with a relaxed feasibility condition, characterize the aspiration core (a non-empty core extension) on the domain of all transferable utility games.

Peleg (1986) proved that, on the domain of balanced games, the core is the only solution concept that satisfies non-emptiness, individual rationality, super-additivity and consistency. In his definition, a solution concepts must select payoff vectors which are feasible for the grand coalition. We modify the feasibility restriction so that Peleg's (1986) ideas can be applicable to settings where no coalition structure is assumed.[1] In fact, we adapt his axioms to arbitrary transferable utility (TU) games. As a result we obtain axiomatizations of the C-stable solution (Guesnerie and Oddou 1979) (a.k.a. C-core (Sun, Trockel, and Yang 2008)) and the aspiration core (Bennett 1983) (Cross 1967) (a.k.a. balanced aspiration solution). Both solution concepts coincide with the core when the latter is non-empty, but are well-defined for a larger family of games. Peleg's (1986) core axiomatization then becomes a particular case of our more general results. Furthermore, both c-core and aspiration core payoff vectors are associated with a set of coalitions that will form to make them feasible.

Two of the original axioms used by Peleg (1986), superadditivity and consistency, implicitly depend on grand coaliton feasibility. We replace them with similar properties that are well-known in the literature and do not rely on feasibility for the grand coalition. First, traditional reduced games (Davis and Maschler 1965) make an exception in their definition to ensure that payoff vectors "add up" to the worth of the grand coalition. We use a more

[1] In the last section, Peleg (1986) allows for other coalition structures to appear. However, coalitions that form are determined exogenously.

general version of consistency (Moldovanu and Winter 1994), one that treats all coalitions in the same way. Second, following the lines of Aumann (1985) and Hart (1985), we impose a feasibility requirement on superadditivity. In both instances, when applied to the family of balanced games, the axioms coincide with those used by Peleg (1986). In this sense, we generalize his results to arbitrary TU-games.

The paper is organized as follows. Notation and basic definitions are introduced in Section 2 and axioms are listed in Section 3. The main results are given in Section 4 and Section 5 concludes.

2 Definitions and notation

2.1 TU-games

Given a set of agents $\mathcal{U}$, a *cooperative TU-game* is an ordered pair (N, v) where N is a finite non-empty subset of $\mathcal{U}$ and $v : 2^N \longrightarrow \mathbb{R}$ is a function such that $v(\emptyset) = 0$. Γ denotes the space of all cooperative TU-games. Let $\mathcal{N} = \{S \subseteq N \mid S \neq \emptyset\}$ be the set of *coalitions* of (N, v). For every $S \in \mathcal{N}$, we call $v(S)$ the *worth* of coalition S. Possible outcomes of a game (N, v) are described by vectors $x \in \mathbb{R}^N$ that assign a *payoff* x_i to every $i \in N$. For every $S \in \mathcal{N}$ and $x \in \mathbb{R}^N$, define $x(S) = \sum_{i \in S} x_i$ and let $x^S \in \mathbb{R}^S$ be such that $x_i^S = x_i$ for every $i \in S$. The *generating collection* of $x \in \mathbb{R}^N$ is defined as $\mathcal{GC}(x) = \{S \in \mathcal{N} \mid x(S) = v(S)\}$. A payoff vector x is an *aspiration* of the game (N, v) if $x(S) \geq v(S)$ for every $S \in \mathcal{N}$ and $\bigcup_{S \in \mathcal{GC}(x)} S = N$.

2.2 Feasibility

We define feasibility by taking into account all possible arrangements of agents devoting fractions of their time to different coalitions, not just the grand coalition. Let (N, v) be an arbitrary TU-game. Define a *production plan* for N as a mapping $\lambda : \mathcal{N} \longrightarrow [0, 1]$ such that $\sum_{S \ni i} \lambda_S = 1$ for every $i \in N$. We interpret λ_T as the fraction of time during which coalition T is active. The requirement that $\sum_{S \ni i} \lambda_S = 1$ is a time-feasibility condition, under the assumption that every agent is endowed with one unit of time. Let $\Lambda(N)$ denote the set of all production plans for N.[2] Define the worth of any production plan $\lambda \in \Lambda(N)$ as

$$v(\lambda) = \sum_{S \in \mathcal{N}} \lambda_S v(S).$$

[2]Elements of $\Lambda(N)$ are known in the literature as balancing weights. For every $\lambda \in \Lambda(N)$, the set $\{S \in \mathcal{N} \mid \lambda_S > 0\}$ is a *(strictly) balanced family of coalitions.*

Definition 2.1 *The set of* **feasible** *payoff vectors of* (N, v) *is*

$$X^*_\Lambda(N, v) = \{x \in \mathbb{R}^N \mid x(N) \le v(\lambda) \text{ for some } \lambda \in \Lambda(N)\}.$$

Classical axiomatization literature (Peleg 1986, Peleg 1989, Tadenuma 1992, Hwang and Sudhölter 2001, Voorneveld and Van Den Nouweland 1998) only works with the set of payoff vectors

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \le v(N)\}.$$

The set $X^*(N, v)$ contains payoff vectors that are feasible when only the grand coalition can form. Clearly, $X^*(N, v) \subseteq X^*_\Lambda(N, v)$.

The subset of $X^*_\Lambda(N, v)$ constructed below contains payoff vectors that are feasible when agents cannot divide their time among various coalitions, and thus only disjoint coalitions can form.

A family of coalitions $\pi \subseteq \mathcal{N}$ is a *partition of* N if $\cup_{P \in \pi} P = N$ and for every $P, Q \in \pi$ such that $P \neq Q$, $P \cap Q = \emptyset$. Let $\Pi(N)$ denote the family of all partitions of N. For every partition $\pi \in \Pi(N)$ define its worth as

$$v(\pi) = \sum_{P \in \pi} v(P).$$

For every TU-game (N, v) let

$$X^*_\Pi(N, v) = \{x \in \mathbb{R}^N \mid x(N) \le v(\pi) \text{ for some } \pi \in \Pi(N)\}.$$

Remark 2.2 *Notice that every partition* $\pi \in \Pi(N)$ *(in particular* $\{N\} \in \Pi(N)$*) can be naturally identified with the production plan* $\lambda^\pi \in \Lambda(N)$ *defined as* $\lambda^\pi_S = 1$ *if* $S \in \pi$ *and* $\lambda^\pi_S = 0$ *otherwise. Thus, for every* $(N, v) \in \Gamma$,

$$X^*(N, v) \subseteq X^*_\Pi(N, v) \subseteq X^*_\Lambda(N, v).$$

2.3 Efficiency

The set of *efficient* payoff vectors for every $(N, v) \in \Gamma$ is defined as

$$X_\Lambda(N, v) = \arg\max\{x(N) \mid x \in X^*_\Lambda(N, v)\}.$$

A production plan $\hat{\lambda} \in \Lambda(N)$ is *efficient* if $v(\hat{\lambda}) = \max\{v(\lambda) \mid \lambda \in \Lambda(N)\}$.

This definition of efficiency differs from the one typically used in the literature, which implicitly assumes that forming the grand coalition is Pareto-optimal. Peleg (1986), for example, defines the set of efficient payoff vectors of a TU-game (N, v) as

$$X(N, v) = \{x \in X^*(N, v) \mid x(N) = v(N)\} = \arg\max\{x(N) \mid x \in X^*(N, v)\}.$$

2.4 Solution concepts

Fix a family of games $\Gamma_0 \subseteq \Gamma$. A *solution concept* on Γ_0 is a mapping σ that assigns to every game $(N, v) \in \Gamma_0$ a (possibly empty) set $\sigma(N, v) \subseteq X^*_\Lambda(N, v)$.

The following are the definitions of the solution concepts that are our main object of study.

The *core* (Gillies 1959) is defined as

$$C(N, v) = \{x \in X^*(N, v) \mid x(S) \geq v(S)\ \forall S \in \mathcal{N}\}.$$

The subdomain of *balanced* TU-games is denoted by

$$\Gamma_c = \{(N, v) \in \Gamma \mid C(N, v) \neq \emptyset\}.$$

Bondareva (1963) and Shapley (1967) showed that $(N, v) \in \Gamma_c$ if and only if devoting the entire time to the grand coalition is an efficient production plan.

Changing the definition of the core by using the sets $X^*_\Pi(N, v)$ and $X^*_\Lambda(N, v)$ instead of $X^*(N, v)$ generates two different solution concepts.

The *c-core* (Sun, Trockel, and Yang 2008) or *c-stable set* (Guesnerie and Oddou 1979) is defined as

$$cC(N, v) = \{x \in X^*_\Pi(N, v) \mid x(S) \geq v(S)\ \forall S \in \mathcal{N}\}.$$

This definition leads to a new family of games, those with a non-empty c-core. The subdomain of *c-balanced* TU games is denoted by

$$\Gamma_{cc} = \{(N, v) \in \Gamma \mid cC(N, v) \neq \emptyset\}.$$

The *aspiration core* or *balanced aspiration set* (Bennett 1983) (see also (Cross 1967) and (Albers 1979)) is defined as[3]

$$AC(N, v) = \{x \in X^*_\Lambda(N, v) \mid x(S) \geq v(S)\ \forall S \in \mathcal{N}\}.$$

Remark 2.3 *Bennett (1983) shows that for every $(N, v) \in \Gamma$, $AC(N, v) \neq \emptyset$.*

Remark 2.4 *Notice that Remark 2.2 and the previous definitions imply that for every $(N, v) \in \Gamma$*

$$C(N, v) \subseteq cC(N, v) \subseteq AC(N, v).$$

[3]Bennett (1983) originally defines the aspiration core as the set of minimal sum aspirations and goes on to show the equivalence with the definition above.

Proposition 2.5 *If $(N, v) \in \Gamma_c$, then $X(N, v) = X_\Pi(N, v) = X_\Lambda(N, v)$. Also, if $(N, v) \in \Gamma_{cc}$, then $X_\Pi(N, v) = X_\Lambda(N, v)$.*

The proof of this proposition uses standard techniques and it is left to the reader.

Remark 2.6 *Applying Proposition 2.5 to the definition of the solution concepts implies that whenever the c-core is not empty, it coincides with the aspiration core. Similarly, whenever the core is not empty, it coincides with the aspiration core. From these observations and Remark 2.3 we also conclude that the aspiration core is a non-empty core extension.*

3 The axioms

Let Γ_0 be an arbitrary subset of Γ. The following are the axioms relevant to our results:

Non-emptiness (NE): A solution σ on Γ_0 satisfies *NE* if for every $(N, v) \in \Gamma_0$, $\sigma(N, v) \neq \emptyset$.

Individual rationality (IR): A solution σ on Γ_0 satisfies *IR* if for every $(N, v) \in \Gamma_0$, every $x \in \sigma(N, v)$, and every $i \in N$, $x_i \geq v(\{i\})$.

We now present two versions of reduced games and their corresponding consistency axioms. Fix $(N, v) \in \Gamma$, $S \in \mathcal{N}$, and $x \in \mathbb{R}^N$. Define the *DM-reduced game* (Davis and Maschler 1965) of (N, v) with respect to S and x as $(S, v^x) \in \Gamma$ such that

$$v^x(T) = \begin{cases} 0 & \text{if } T = \emptyset \\ v(N) - x(N \backslash S) & \text{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \backslash S\} & \text{otherwise} \end{cases}$$

DM-consistency (DM-CON): A solution σ on Γ_0 satisfies *DM-CON* if for every $(N, v) \in \Gamma_0$, every $S \in \mathcal{N}$, and every $x \in \sigma(N, v)$, it is true that $(S, v^x) \in \Gamma_0$ and $x^S \in \sigma(S, v^x)$.

The *MW-reduced game* (Moldovanu and Winter 1994) of (N, v) with respect to S and x is the game $(S, v_*^x) \in \Gamma$ such that

$$v_*^x(T) = \begin{cases} 0 & \text{if } T = \emptyset \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \backslash S\} & \text{otherwise} \end{cases}$$

MW-consistency (MW-CON): A solution σ on Γ_0 satisfies *MW-CON* if for every $(N, v) \in \Gamma_0$, every $S \in \mathcal{N}$, and every $x \in \sigma(N, v)$, it is true that $(S, v_*^x) \in \Gamma_0$ and $x^S \in \sigma(S, v_*^x)$.

Remark 3.1 *Note that if $v \in \Gamma_c$ and $x \in C(N, v)$ then the two versions of reduced game coincide. Indeed, for every $S \in \mathcal{N}$, the games (S, v^x) and (S, v_*^x) differ at most on the worth assigned to S. To show that $v^x(S) = v_*^x(S)$, notice that $v^x(S) = v(S \cup (N \setminus S)) - x(N \setminus S) \leq \max\{v(S \cup Q) - x(Q) \mid Q \subseteq N \backslash S\} = v_*^x(S)$. Conversely, as $x \in C(N, v)$, for every $Q \subseteq N \backslash S$ we have $v^x(S) = x(S) \geq v(S \cup Q) - x(Q)$, so $v^x(S) \geq v_*^x(S)$. We conclude that the core satisfies MW-CON on Γ_c because, as Peleg (1986) shows, the core satisfies DM-CON on Γ_c.*

Superadditivity (SUPA): A solution σ on Γ_0 satisfies *SUPA* if for every pair of games $(N, v_A), (N, v_B) \in \Gamma_0$, every $x_A \in \sigma(N, v_A)$ and every $x_B \in \sigma(N, v_B)$, it is true that $x_A + x_B \in \sigma(N, v_A + v_B)$ whenever $(N, v_A + v_B) \in \Gamma_0$.

Conditional Superadditivity (C-SUPA): A solution σ on Γ_0 satisfies **C-SUPA** if for every pair of games $(N, v_A), (N, v_B) \in \Gamma_0$, every $x_A \in \sigma(N, v_A)$ and every $x_B \in \sigma(N, v_B)$, it is true that $x_A + x_B \in \sigma(N, v_A + v_B)$ whenever $(N, v_A + v_B) \in \Gamma_0$ and $x_A + x_B$ is feasible for $(N, v_A + v_B)$.

Remark 3.2 *Notice that consistency axioms require the corresponding reduced game to lie in the domain of games where the solution is defined. There is no such requirement for superadditivity axioms. Therefore, if a solution σ on $\Gamma_1 \subseteq \Gamma$ satisfies C-SUPA (or SUPA), the axiom is immediately inherited by σ when defined on any subdomain $\Gamma_0 \subseteq \Gamma_1$.*

Remark 3.3 *(Peleg 1986) shows that the core satisfies SUPA on Γ_c. Note that by Proposition 2.5, feasibility of $x_A + x_B$ is trivial on the domain of balanced games. Therefore, as C-SUPA coincides with SUPA on Γ_c, the core satisfies C-SUPA on Γ_c.*

4 Axiomatizations

Proposition 4.1 *The aspiration core satisfies NE, IR, MW-CON, and C-SUPA on* Γ.

Proof. *NE* is satisfied by Remark 2.3, *IR* is satisfied by definition, and Hokari and Kibris (2003) proved that the aspiration core satisfies *MW-CON* on Γ, so it only remains to show *C-SUPA* is also satisfied. Let $(N, v_A), (N, v_B) \in \Gamma$, $x_A \in AC(N, v_A)$, $x_B \in AC(N, v_B)$, be such that $x_A + x_B$ is feasible for $(N, v_A + v_B) \in \Gamma$. Then, for every $S \in \mathcal{N}$, $(x_A + x_B)(S) = x_A(S) + x_B(S) \geq v_A(S) + v_B(S) = (v_A + v_B)(S)$. Thus, $x_A + x_B \in AC(N, v_A + v_B)$ ■

Proposition 4.2 *Let* σ *be a solution concept defined on* $\Gamma_0 \subseteq \Gamma$ *satisfying IR and MW-CON. If* $(N, v) \in \Gamma_0$ *and* $x \in \sigma(N, v)$, *then* $x(S) \geq v(S)$ *for every* $S \in \mathcal{N}$.

Proof. Let σ be a solution concept on Γ_0 satisfying *IR* and *MW-CON*. Let $x \in \sigma(N, v)$, $S \in \mathcal{N}$ and choose any $i \in S$. By *MW-CON*, $x_i \in \sigma(\{i\}, v_*^x)$, so *IR* implies

$$x_i \geq v_*^x(\{i\}) = \max\{v(Q \cup \{i\}) - x(Q) \mid Q \subseteq N \setminus \{i\}\} \geq v(S) - x(S \setminus \{i\}).$$

This means that $x(S) \geq v(S)$, as desired. ■

Proposition 4.3 *If* σ *is a solution concept defined on* $\Gamma_0 \subseteq \Gamma$ *that satisfies IR and MW-CON then, for every* $(N, v) \in \Gamma_0$, *every payoff vector in* $\sigma(N, v)$ *must be efficient.*

Proof. Let $\hat{\lambda}$ be an efficient production plan for $(N, v) \in \Gamma_0$. If $x \in \sigma(N, v)$ we want to show $x(N) = v(\hat{\lambda})$. By feasibility we know $x(N) \leq v(\hat{\lambda})$. Conversely, Proposition 4.2 implies that

$$x(N) = \sum_{R \in \mathcal{N}} \hat{\lambda}(R) x(R) \geq \sum_{R \in \mathcal{N}} \hat{\lambda}(R) v(R) = v(\hat{\lambda}),$$

as we wanted. ■

Proposition 4.4 *If the solution concept* σ *defined on* $\Gamma_0 \subseteq \Gamma$ *satisfies IR and MW-CON, then* $\sigma(N, v) \subseteq AC(N, v)$ *for every* $(N, v) \in \Gamma_0$.

Proof. This is an immediate consequence of Proposition 4.2 and feasibility. ■

Proposition 4.5 *Let $\mathcal{U}$ have at least three elements. If a solution concept σ defined on Γ satisfies NE, IR, MW-CON and C-SUPA, then $AC(N,v) \subseteq \sigma(N,v)$ for every $(N,v) \in \Gamma$.*

Proof. Let $x \in AC(N,v)$.

<u>**Case** $|N| \geq 3$</u>: Define $(N,w) \in \Gamma_c$ as

$$w(S) = \begin{cases} x(S) & \text{if } |S| \geq 2 \\ v(S) & \text{if } |S| = 1 \end{cases} \tag{1}$$

Note that $C(N,w) = \{x\}$. Then, by Proposition 4.4 and Remark 2.6, $\sigma(N,w) \subseteq AC(N,w) = C(N,w) = \{x\}$. *NE* then implies $x \in \sigma(N,w)$.

Consider now the game $(N,z) \in \Gamma$ defined as

$$z(S) = v(S) - w(S) \text{ for every } S \in \mathcal{N} \tag{2}$$

The vector $\mathbf{0} \in \mathbb{R}^N$ is in $AC(N,z)$ because, by definition of (N,z), every $S \in \mathcal{N}$ satisfies $0 \geq z(S)$, and the production plan associated with partition $\{\{i\} \mid i \in N\}$ makes $\mathbf{0}$ feasible in (N,z). Furthermore, given $\mathbf{0} \in AC(N,z)$, Proposition 4.3 implies $y(N) = 0$ for every $y \in AC(N,z)$. Then, as the aspiration core is individually rational and $z(\{i\}) = 0$ for every $i \in N$, $AC(N,z) = \{\mathbf{0}\}$. Again, Proposition 4.4 implies $\sigma(N,z) \subseteq AC(N,z) = \{\mathbf{0}\}$, so *NE* implies $\mathbf{0} \in \sigma(N,z)$.

Notice that x is feasible for (N,v) as $x \in AC(N,v)$, so $C - SUPA$ implies $x \in AC(N, w+z) = AC(N,v)$ as we wanted.

<u>**Case** $|N| = 2$ **and** $|AC(N,v)| > 1$</u>: In this case $\sum_{|S|=1} v(S) < v(N)$. Let $x = (x_1, x_2) \in AC(N,v)$ and define $\tilde{x} = (x, 0) \in \mathbb{R}^3$. Let $d \in \mathcal{U} \setminus N$, a non-empty set because $|\mathcal{U}| \geq 3$. Consider the game $(N \cup \{d\}, \tilde{v}) \in \Gamma_c$ defined by

$$\tilde{v}(S) = \begin{cases} v(S \setminus \{d\}) & \text{if } |S| \leq 2 \text{ and } S \neq N \\ \sum_{i \in N} v(\{i\}) & \text{if } S = N \\ v(N) & \text{if } S = N \cup \{d\} \end{cases} \tag{3}$$

Using the case $|N| \geq 3$ and Proposition 2.6, conclude that $\tilde{x} \in C(N \cup \{d\}, \tilde{v}) = AC(N \cup \{d\}, \tilde{v}) = \sigma(N \cup \{d\}, \tilde{v})$. It is simple to verify that $(N, \tilde{v}^{\tilde{x}}_*) = (N,v)$. then, use *MW-CON* to conclude that $x = \tilde{x}_N \in \sigma(N, \tilde{v}^{\tilde{x}}_*) = \sigma(N,v)$ as we wanted.

Case $|N| \leq 2$ and $|AC(N,v)| = 1$: By Proposition 4.4, $\sigma(N,v) \subseteq AC(N,v) = \{x\}$, so *NE* implies $x \in \sigma(N,v)$. ■

We are now ready to state our main results.

Theorem 4.6 *Let $\mathcal{U}$ have at least three elements. The aspiration core is the only solution concept on Γ that satisfies NE, IR, MW-CON, and C-SUPA.*

Proof. Combine Propositions 4.1, 4.4, and 4.5. ■

Theorem 4.7 *Let $\mathcal{U}$ have at least three elements. The core is the unique solution concept defined on Γ_c that satisfies NE, IR, MW-CON, and C-SUPA.*

Proof. By definition the core satisfies *NE* and *IR*. By Remark 3.1 the core satisfies *MW-CON*. By Proposition 4.1 the aspiration core satisfies *C-SUPA* on Γ, so Remarks 2.6 and 3.2 imply the core satisfies *C-SUPA* on Γ_c. Now, let a solution σ on Γ_c satisfy the axioms and fix a game $(N,v) \in \Gamma_c$. Then Proposition 4.4 and Remark 2.6 imply $\sigma(N,v) \subseteq AC(N,v) = C(N,v)$. On the other hand, in the proof of Theorem 4.6, $(N,v) \in \Gamma_c$ implies the games defined in (1), (2), and (3) are also in Γ_c. Hence, the proof remains valid on the domain of balanced games and $C(N,v) = AC(N,v) \subseteq \sigma(N,v)$. Thus, $\sigma(N,v) = C(N,v)$. ■

Remark 4.8 *Remarks 3.1 and 3.3 also imply Theorem 4.7 is, in fact, equivalent to Peleg's (1986) axiomatization.*

Theorem 4.6 can also be used to obtain a characterization of the *c*-core on the domain Γ_{cc} as follows. To the best of our knowledge, this is the first axiomatization of the c-core in the literature.

Theorem 4.9 *Let $\mathcal{U}$ have at least three elements. The c-core is the unique solution concept defined on Γ^{cc} that satisfies NE, IR, MW-CON, and C-SUPA.*

Proof. By definition the c-core satisfies *NE* and *IR*. Reasoning as in the previous result, Proposition 4.1 and Remarks 2.6 and 3.2 imply the c-core satisfies *C-SUPA* on Γ_{cc}. We now show that the c-core satisfies *MW-CON* on Γ_{cc}. Let $(N,v) \in \Gamma_{cc}$, $x \in cC(N,v)$ and $S \in \mathcal{N}$. By definition, there must exist $\pi \in \Pi(N)$ such that $x(N) \leq v(\pi)$. However, as $x \in cC(N,v)$,

$x(N) = \sum_{P\in\pi} x(P) \geq \sum_{P\in\pi} v(P) = v(\pi)$. Hence, $x(N) = v(\pi)$ and $x(P) = v(P)$ for every $P \in \pi$. Let $\bar{\pi} \in \Pi(S)$ be defined by

$$\bar{\pi} = \{\bar{P} \subseteq S \mid \bar{P} = P \cap S \text{ for some } P \in \pi\}.$$

Then, for every $\bar{P} = P \cap S \in \bar{\pi}$ we have

$$x(\bar{P}) = v(\bar{P} \cup (P \setminus S)) - x(P \setminus S) \leq v_*^x(\bar{P}),$$

and

$$x(S) = \sum_{\bar{P}\in\bar{\pi}} x(\bar{P}) \leq \sum_{\bar{P}\in\bar{\pi}} v_*^x(\bar{P}) = v_*^x(\bar{\pi}).$$

Hence, $x \in X_\Pi(S, v_*^x)$. By Proposition 4.1 the aspiration core satisfies *MW-CON* on Γ and thus $x(T) \geq v_*^x(T)$ for every $T \subseteq S$. It follows that $x \in cC(S, v_*^x)$.

As in the proof of Theorem 4.7, Propositions 4.4 and 4.5 are adaptable to work on Γ_{cc}, so every solution satisfying the axioms on this subdomain must coincide with the c-core. ■

5 Independence of the axioms

The following examples show that no axiom in our main result is implied by the others.

Example 5.1 *Consider the solution concept σ_1 on Γ such that $\sigma_1(N, v) = \emptyset$ for every $(N, v) \in \Gamma$. σ_1 violates NE but vacuously satisfies IR, MW-CON, and C-SUPA. Therefore NE is independent of the other axioms.*

Example 5.2 *Consider the solution concept σ_2 on Γ such that $\sigma_2(N, v) = X^*(N, v)$ for every $(N, v) \in \Gamma$. It satisfies NE because $AC(N, v) \subseteq X^*(N, v)$ is non-empty by Proposition 4.1. It satisfies C-SUPA by definition. We now show that it satisfies MW-CON. For every $(N, v) \in \Gamma$, every $S \in \mathcal{N}$ and every $x \in X^*(N, v)$, there exists $\lambda \in \Lambda(N)$ such that $x(N) \leq v(\lambda)$. Consider the function $\bar{\lambda} : \mathcal{N} \longrightarrow \mathbb{R}_+$ defined for every $\emptyset \neq T \subseteq S$ as*

$$\bar{\lambda}(T) = \sum_{\substack{R\subseteq N \\ R\cap S = T}} \lambda(R).$$

Then $\bar{\lambda} \in \lambda_S$ as

$$\sum_{\substack{T\subseteq S \\ T\ni i}} \bar{\lambda}_T = \sum_{\substack{T\subseteq S \\ T\ni i}} \sum_{\substack{R\subseteq N \\ R\cap S = T}} \lambda(R) = \sum_{\substack{R\subseteq N \\ R\ni i}} \lambda(R) = 1.$$

Additionally, $x_S \in X^*(S, v_*^x)$ *because*

$$\begin{aligned}
x(S) &= \sum_{T \subseteq S} \bar{\lambda}(T)x(T) = \sum_{T \subseteq S} \sum_{\substack{R \in \mathcal{N} \\ R \cap S = T}} \lambda(R)x(T) \\
&= \sum_{R \in \mathcal{N}} \lambda(R)x(R \cap S) + \sum_{R \in \mathcal{N}} \lambda(R)x(R \setminus S) - \sum_{R \in \mathcal{N}} \lambda(R)x(R \setminus S) \\
&= \sum_{R \in \mathcal{N}} \lambda(R)x(R) - \sum_{R \in \mathcal{N}} \lambda(R)x(R \setminus S) = x(N) - \sum_{R \in \mathcal{N}} \lambda(R)x(R \setminus S) \\
&\leq v(\lambda) - \sum_{R \in \mathcal{N}} \lambda(R)x(R \setminus S) = \sum_{R \in \mathcal{N}} \lambda(R)[v(R) - x(R \setminus S)] \\
&\leq \sum_{R \in \mathcal{N}} \lambda(R)v_*^x(R \cap S) = \sum_{T \subseteq S} \sum_{\substack{R \in \mathcal{N} \\ R \cap S = T}} \lambda(R)v_*^x(T) \\
&= \sum_{T \subseteq S} \bar{\lambda}(T)v_*^x(T) = v_*^x(\bar{\lambda}).
\end{aligned}$$

It is also clear that σ_2 *is not individually rational, so IR is independent of the other axioms.*

Example 5.3 *Consider the solution concept* σ_3 *on* Γ *such that* $\sigma_3(N, v) = \{x \in X^*(N, v) \mid x_i \geq v(\{i\}) \ \forall i \in N\}$ *for every* $(N, v) \in \Gamma$. σ_3 *clearly satisfies NE, IR, and C-SUPA. Therefore our results imply that* σ_3 *does not comply with MW-CON.*

Example 5.4 *Following Schmeidler's (1969) procedure but using the set of aspirations, Bennett (1981) defines the aspiration nucleolus. She shows that the concept satisfies NE, Hokari and Kibris (2003) show that it complies with MW-CON. The aspiration nucleolus also satisfies IR as Sharkey (1993) shows it is a subsolution of the aspiration core. Hence, our axiomatization implies that the aspiration nucleolus is not conditionally superadditive.*

6 Final comments and related literature

It is of particular importance that our aspiration core axiomatization holds on the entire domain of TU-games, Γ. Hwang and Sudhölter (2001) solved an important technical problem by providing an axiomatic characterization of the core on the entire domain of TU-games, but their axioms characterize the empty solution outside the domain of balanced games. Closer to our work is Orshan and Sudhölter's (2010) axiomatization of the *positive core*, a non-empty core extension. However, their work still assumes the grand coalition forms. Unlike the concepts we study, if a game is not balanced, every vector in the positive core can be improved upon by some coalition. Modifying the feasibility constraint allows us to characterize a natural extension of the core to non-balanced games while also suggesting a family of coalitions that are likely to form.

Keiding (2006) gives another axiomatization of the aspiration core. We share with his work the use of *MW-CON*. However, he adds a class of auxiliary non-transferable utility games to the domain of TU-games, while our results hold within the family Γ of TU-games.

Among the first core axiomatizations are (Peleg 1985) (for NTU games), (Peleg 1986, Peleg 1989, Tadenuma 1992, Voorneveld and Van Den Nouweland 1998). These papers worked with the family of balanced games Γ_c, so there is some circularity in their characterizations, as they use the core to define the core. Our c-core axiomatization is subject to the same type of criticism, but we also provide a characterisation of a solution concept that extends the c-core outside its natural domain, Γ_{cc}.

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