Martingale properties of self-enforcing debt

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Abstract

Not-too-tight (NTT) debt limits are endogenous restrictions on debt that prevent agents from defaulting and opting for a specified continuation utility, while allowing for maximal credit expansion. For an agent facing some fixed prices for the Arrow securities, we prove that discounted NTT debt limits must differ by a martingale. Discounted debt limits are submartingales (martingales) under an interdiction to trade (borrow). The martingale components in debt limits can be converted into asset price bubbles.

Keywords: endogenous debt limits, not-too-tight constraints, rational bubbles, limited enforcement

JEL classification: G11, G12, D53, E44

1 Introduction

Alvarez and Jermann (2000) construct a theory of endogenous debt constraints in complete markets economies with limited enforcement of financial contracts. Following Kehoe and Levine (1993), they assume that agents can default on debt at the

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cost of being excluded permanently from financial markets. At each date and state, an agent is allowed to borrow the maximum amount which is self-enforcing (making repayment individually rational). These endogenous bounds on debt are referred to as debt limits that are *not-too-tight (NTT)* for the respective agent.

One can envision the NTT debt limits as being set by competitive financial intermediaries, with agents unable to trade directly with each other. The intermediaries set debt limits such that default is prevented, but credit is not restricted unnecessarily, since competing intermediaries could relax them and increase their profits (see Ábrahám and Cárceles-Poveda (2010) for such a formalization).

Kocherlakota (2008) uncovered a defining characteristic of the set of NTT debt limits for an agent facing a fixed pricing kernel (or, equivalently, fixed prices of the one-period Arrow securities at each date and state) and penalty for default: adding a martingale to some discounted NTT debt limits results in bounds that are also NTT. The proof is immediate, and it is a consequence of agent's budget constraint being unchanged under the martingale-inflated bounds, if the initial value of the martingale is added to his initial wealth.

We prove the converse, which is considerably more involved. A pair of discounted debt limits that are NTT (for a given agent, pricing kernel and penalties for default) *must* differ by a martingale.[1] This theorem does not depend on equilibrium considerations and stems only from the optimizing behavior of the agent. We allow for general penalties for default specified by a continuation utility that can be date and state contingent, and can depend on endogenous variables such as asset prices. When the punishment for default is the *interdiction to borrow*, Hellwig and Lorenzoni (2009) proved that discounted NTT debt limits are martingales. With this outside option, zero bounds on debt are NTT. Thus their result is a particular case of our theorem.[2] Our proof is also simpler, due to the use of martingale techniques (Snell envelopes). The use of Snell envelopes, while familiar in the theory of pricing American options, is novel to macroeconomics applications. The theorem does not imply

[1] Let p and $\bar{\phi}, \phi$ be stochastic processes representing the pricing kernel and two (sequences of) NTT debt limits. Then $p \cdot (\phi - \bar{\phi})$ is a martingale.

[2] Set $\bar{\phi}$ identically equal to zero at all dates and states. Hence $\bar{\phi}$ is NTT, and the debt limits ϕ are NTT if and only if $p \cdot \phi$ $(= p(\phi - \bar{\phi}))$ is a martingale. In Appendix B, we also show that the result in Hellwig and Lorenzoni (2009) can be leveraged (with the benefit of hindsight), via simple arguments, to apply to general penalties for default. One can prove that $p(\phi - \bar{\phi})$ is a martingale, for NTT debt limits satisfying the additional assumption $\phi \leq \bar{\phi}$. This extension is of limited use as it applies only to situations where one of the debt bounds is uniformly tighter than other.

however that NTT debt limits themselves are martingales (except for an interdiction to borrow as penalty for default). In fact, when the punishment for default is the interdiction to trade, we prove that the discounted NTT debt limits of each agent are only *submartingales*.

As known from Bloise, Reichlin, and Tirelli (2013) and the examples in Hellwig and Lorenzoni (2009) and Bidian and Bejan (2012), there is typically a multiplicity of equilibria (in terms of real allocations and pricing kernels) in economies with enforcement limitations. Our theorem does not compare NTT debt limits across equilibria with different pricing kernels, but rather focuses on a given equilibrium and it characterizes completely the set of debt limits offered to an agent by a competitive financial intermediary that takes as given the market rates and the contractual (enforcement) limitations. It establishes that such an intermediary can only alter the debt limits of the agent by a discounted martingale. The theorem does not depend on equilibrium considerations (that is, market clearing conditions).

The property of NTT debt limits uncovered by Kocherlakota (2008) seems to suggest that associated to any equilibrium, there is a continuum of possible NTT debt limits for the agents, differing from each other by (arbitrary) discounted martingales with zero expected value. Our full characterization of the NTT debt limits (for an agent facing a given pricing kernel and penalties for default) can be used to establish their *uniqueness*, when the present value of agent's endowments is finite, that is with *high interest rates*. In this case, borrowing should be limited by the agent's ability to repay his debt out of his future endowments (Santos and Woodford 1997), or equivalently, by the present value of future endowments. The difference of two such nonpositive discounted NTT debt limits is therefore a uniformly integrable martingale converging to zero, and hence identically equal to zero. When the punishment for default is the *interdiction to trade*, Alvarez and Jermann (2000, Proposition 4.11) prove that nonpositive NTT debt limits bounded by the present value of debt must exist. Our result establishes that such debt limits are in fact unique. With an interdiction to borrow as punishment for default, debt limits identically equal to zero are NTT, hence uniqueness implies that debt is unsustainable in the presence of high interest rates. This confirms the conclusion reached earlier by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009).

The assumption of high interest rates, however, is extremely restrictive in models with limited enforcement. In these environments, *low interest rates* (making the present value of aggregate endowment infinite) arise in equilibrium as a way to induce agents not to renege on their debt (Hellwig and Lorenzoni 2009). As shown by Santos and Woodford (1997), low interest rates are necessary for the existence of asset price bubbles. The martingale property of NTT debt limits suggests a strong connection to bubbles. Bubbles grow on average at the same rate as the interest rates and therefore they are positive martingales when discounted by the pricing kernel. By not discarding low interest rates equilibria for ad hoc reasons, we are able to pursue this connection.

Kocherlakota (2008) shows that an arbitrary bubble can be injected in the price of an infinitely-lived asset, without altering agents' consumption. This can be accomplished by an upward adjustment of agents' debt limits proportional to the size of the bubble and their initial endowment of the asset, which leaves them NTT. The introduction of a bubble gives consumers a windfall proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits. He refers to this result as the "bubble equivalence theorem".

While an intriguing way to generate bubbles, it raises the question whether the tighter debt bounds needed to sustain the bubble can remain *nonpositive*, due to the bubble component they now contain. Positive debt limits force agents to save and it implies that financial intermediaries have access to coercive tools that seems unreasonable with enforcement limitations. Clearly arbitrarily large bubble injections can only be sustained by forcing agents to save arbitrarily large amounts. Moreover, with high interest rates, even initially infinitesimal bubbles explode quickly and make agents's debt limits positive and large.

It is therefore unclear whether bubble injections can occur at all with nonpositive debt limits. As an immediate consequence of our characterization of NTT debt limits, we show that bubble injections leading to nonpositive debt limits are possible when agents are still allowed to borrow some predetermined (possibly zero or arbitrarily small) amounts after default. Therefore our theorem, which as explained above is a converse to Kocherlakota's (2008) characterization of NTT debt limits, it also provides the missing link needed to show the existence of bubble injections with nonpositive debt limits. Bubbles enable agents to circumvent a reduction in the availability of credit, and to achieve identical allocations to those possible under more relaxed, but still self-enforcing debt limits. A tightening of the debt limits would result in a drop in interest rates due to precautionary saving and a reduction in output (Guerrieri and Lorenzoni 2011). A bubble satisfies the need for additional liquidity, preventing the drop in interest rates and output. In this sense, bubble injections are expansionary.

The paper is organized as follows. Section 2 introduces the model, and defines the notion of an Alvarez-Jermann equilibrium, which is a sequential equilibrium where agents are subject to NTT debt limits. In Section 3 we prove that discounted NTT bounds (for a given agent, pricing kernel and penalties for default) are determined only up to a martingale and show that an interdiction to trade/borrow results in discounted NTT debt limits that are submartingales/martingales. Section 4 contains applications, and shows that the characterization result of Section 3 can be used to establish the uniqueness of NTT bounds under additional assumption, and the existence of rational bubbles. Appendix A contains proofs to some ancillary results used in Section 3. Appendix B compares in detail the proof of Theorem 3.3 with the proof of the particular instance of this theorem in Hellwig and Lorenzoni (2009) (for an interdiction to borrow). Appendix C establishes the necessary and sufficient transversality conditions for an agent's optimization problem.

2 The model

We consider a stochastic, discrete-time, infinite horizon economy. The time periods are indexed by the set of natural numbers $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$, which is an increasing sequence of σ-algebras on the set of states of the world Ω generating $\mathcal{F}$, that is such that $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$. Each σ-algebra $\mathcal{F}_t$ is interpreted as the information available at period t and it is finite. There is no initial information, therefore $\mathcal{F}_0 = \{\emptyset, \Omega\}$. For any $t \in \mathbb{N}$ and $A \in \mathcal{F}_t$ with $A \neq \emptyset$, we assume that $P(A) \neq 0$.

A sequence $x = (x_t)_{t\in\mathbb{N}}$ of random variables ($\mathcal{F}$-measurable real-valued functions) is an *adapted stochastic process* ("process" henceforth) if for each $t \in \mathbb{N}$, x_t is $\mathcal{F}_t$-measurable.[3] We let X be the set of all stochastic processes, and denote by X_+ (X_{++}) the set of nonnegative (strictly positive) processes in X. Thus $x \in X_+$ ($x \in X_{++}$) if $x_t \geq 0$ ($x_t > 0$) P-almost surely ("a.s." henceforth) for all $t \in \mathbb{N}$. We write

[3]Notice that the process x is *integrable*, since for any $t \in \mathbb{N}$, x_t belongs to the space of integrable random variables $L^1 := L^1(\Omega, \mathcal{F}, P)$, as $\mathcal{F}_t$ is finite.

$x \geq 0$ if x is a nonnegative process, and $x = 0$ if $x_t = 0$ P-a.s. for all $t \in \mathbb{N}$. All statements, equalities, and inequalities involving random variables are assumed to hold only P-a.s., and we omit this qualifier in what follows.

There is a single consumption good and a finite number, I, of consumers. An agent $i \in \{1, 2, \ldots, I\}$ has endowments $e^i \in X_+$, and his preferences are represented by a utility $U^i : X_+ \to \mathbb{R}$ given by $U^i(c) = E \sum_{t=0}^{\infty} u_t^i(c_t)$, where $u_t^i(\cdot) = \beta_t^i u^i(\cdot)$ and $E(\cdot)$ is the expectation operator with respect to the probability P. We assume that $\beta^i \in X_+$ and satisfies $E \sum_{t \geq 0} \beta_t^i < \infty$, and that $u^i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, differentiable, satisfies standard Inada conditions and is bounded from above by $\bar{u}^i \in \mathbb{R}$ and from below by $\underline{u}^i \in \mathbb{R}$. The conditional expectation given the information available at t, $\mathcal{F}_t$, is denoted by $E_t(\cdot)$. Given the absence of information at period 0, $E_0(\cdot) = E(\cdot)$. Let $U_t^i(c) := E_t \sum_{s \geq t} u_s^i(c_s)$ be the continuation utility of agent i after t provided by a consumption stream $c \in X_+$.

Each consumer can trade at each date and state a complete set of one-period Arrow securities. Their prices determine uniquely the pricing kernel $p \in X_{++}$, and conversely, the pricing kernel p determines unambiguously the prices of the Arrow securities. Additionally, there is a finite number J of infinitely-lived, disposable securities. Asset $j \in \{1, 2, \ldots, J\}$ pays dividends $d^j \in X_+$, and has an ex-dividend price per share $q^j \in X_+$. The dividend and price vector processes are $d := (d^1, \ldots, d^J) \in X_+^J$ and $q := (q^1, \ldots, q^J) \in X_+^J$. Consumer i has an initial endowment $\theta_{-1}^i \in \mathbb{R}_+^J$ of the infinitely-lived securities, and $a_0^i \in \mathbb{R}$ additional wealth, and his trading strategy in the J securities is represented by a vector process $\theta^i = (\theta^{i,1}, \ldots, \theta^{i,J})' \in X^J$, while his trading strategy in the Arrow securities is given by $a \in X$.

Consumer i faces debt constraints requiring his beginning of period financial wealth to exceed some bounds $\phi^i \in X$, meant to prevent Ponzi schemes. Thus if consumer i starts period T with wealth ν_T ($\mathcal{F}_T$-measurable) and faces constraints ϕ^i and prices p, q, he solves the problem $\max_{(c,a,\theta) \in B_T^i(\nu_T, \phi^i, p, q)} U_T^i(c)$, denoted $P_T^i(\nu_T, \phi^i, p, q)$, where $B_T^i(\nu_T, \phi, p, q)$ is his budget constraint following T, defined as

$$\begin{aligned} B_T^i(\nu_T, &\phi^i, p, q) := \{(c, a, \theta) \in \Theta^T X_+ \times \Theta^{T+1} X \times \Theta^T X^J \mid \\ &c_T + E_T \frac{p_{T+1}}{p_T} a_{T+1} + q_T \theta_T \leq e_T^i + \nu_T, \quad a_s + (q_s + d_s)\theta_{s-1} \geq \phi_s^i, \\ &c_s + E_s \frac{p_{s+1}}{p_s} a_{s+1} + q_s \theta_s \leq e_s^i + a_s + (q_s + d_s)\theta_{s-1}, \forall s > T\}. \end{aligned} \tag{2.1}$$

In the above notation, Θ represents the shift operator, that is, given a process $x = (x_n)_{n=0}^{\infty} \in X$, $\Theta^T x := (x_{T+n})_{n=0}^{\infty}$. Similarly, if $A \subset X$, then $\Theta^T A := \{\Theta^T x \mid x \in A\}$. The indirect utility of the agent is given by

$$V_T^i(\nu_T, \phi^i, p, q) := \max_{(c,a,\theta) \in B_T^i(\nu_T, \phi^i, p, q)} U_T^i(c). \tag{2.2}$$

Consumer i can elect to default on his debt and receive a continuation utility described by a process $V^{i,d}$. Thus by defaulting at period t, agent i can guarantee for himself a continuation utility $V_t^{i,d}$ (which is $\mathcal{F}_t$-measurable) and can depend on exogenous variables such as agents' endowments, but also on prices p, q, and even future debt limits $\phi_{t+1}^i, \phi_{t+2}^i, \ldots$. When we need to emphasize the functional dependence of penalties on prices and debt limits we use the full notation $V^{i,d}(p, q, \phi^i)$, but in most instances we drop the arguments and do not make the dependence explicit. The debt constraints ϕ^i are determined endogenously to reflect the maximal amount of debt agents can hold without defaulting. We say that the debt limits ϕ^i are *self-enforcing* for agent i at prices p, q given penalties $V^{i,d}$ if $B_t(\phi_t, \phi, p, q) \neq \emptyset$ for all $t \in \mathbb{N}$ and the agent prefers *not* to default, $V_t^i(\phi_t, \phi, p, q) \geq V_t^{i,d}, \forall t \in \mathbb{N}$. The debt limits ϕ^i are *not-too-tight (NTT)* for agent i (at prices p, q) given penalties $V^{i,d}$ if and only if

$$V_t^i(\phi_t, \phi, p, q) = V_t^{i,d}, \forall t \in \mathbb{N}. \tag{2.3}$$

Thus NTT debt limits are self-enforcing bounds that do not restrict credit unnecessarily. Alvarez and Jermann (2000), building on the work of Kehoe and Levine (1993), assume that the agents are banned from trading following default, that is

$$V_t^{i,d} := U_t^i(e^i), \forall t \in \mathbb{N}. \tag{2.4}$$

Hellwig and Lorenzoni (2009), following Bulow and Rogoff (1989), allow agents to continue to lend, but not to borrow, upon default. Hence agents can renege on their debt and be required to hold nonnegative wealth thereafter, resulting in a continuation utility that depends on prices,

$$V_t^{i,d} := V_t^i(0, 0, p, q), \forall t \in \mathbb{N}, \tag{2.5}$$

where the second argument in $V_t(0, 0, p, q)$ denotes the process equal to zero at any date and state.

A vector $\big(p, q, (c^i)_{i=1}^I, (a^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\phi^i)_{i=1}^I, (V^{i,d})_{i=1}^I\big)$ consisting of a pricing kernel p, prices q for the infinitely-lived securities, consumption (c^i), trading strategies (a^i) (in Arrow securities) and (θ^i) (in the infinitely-lived securities), debt constraints (ϕ^i) and penalties for default $(V^{i,d})$ is an *AJ-equilibrium* with initial securities holdings $(\theta^i_{-1})_{i=1}^I$ and initial additional wealth $(a^i_0)_{i=1}^I$ if

i. Consumption and portfolios of each agent i are feasible and optimal: $(c^i, a^i, \theta^i) \in B_0^i(a_0^i + (q_0 + d_0)\theta^i_{-1}, \phi^i, p, q)$ and $U(c^i) = V_0^i(a_0^i + (q_0 + d_0)\theta^i_{-1}, \phi^i, p, q)$.

ii. Markets clear: $\sum_{i=1}^I c_t^i = \sum_{i=1}^I e_t^i, \sum_{i=1}^I \theta_t^i = \sum_{i=1}^I \theta^i_{-1}, \sum_{i=1}^I a_t^i = 0, \forall t \geq 0$.

iii. For each i, ϕ^i is NTT given $V^{i,d}$: $V_t^i(\phi_t^i, \phi^i, p, q) = V_t^{i,d}$, for all $t \geq 0$.

A pricing kernel p and security prices q under which the problem of an agent admits a solution have to exclude arbitrage opportunities, which implies that (see for example Bidian 2011, Chapter 2)

$$q_t = E_t \frac{p_{t+1}}{p_t}(q_{t+1} + d_{t+1}), \forall t \geq 0. \tag{2.6}$$

3 Characterization of not-too-tight debt limits

There is an intimate connection between NTT debt limits and martingales,[4] which will be explored here. Throughout this section we fix an agent i facing a given pricing kernel p, prices q for the infinitely-lived securities, and penalties for default $V^{i,d}$. We assume that prices p, q exclude arbitrage opportunities, that is they satisfy (2.6). If $(c^i, a^i, \theta^i) \in B_T^i(\nu_T, \phi^i, p, q)$, then $(c^i, a') \in B_T^i(\nu_T, \phi^i, p)$, where for all $s > T$, $a'_s := a_s^i + (q_s + d_s)\theta^i_{s-1}$ (that is, a'_s is the beginning of period s wealth of the agent), and

$$\begin{aligned} B_T^i(\nu_T, \phi^i, p) :=& \{(c, a) \in \Theta^T X_+ \times \Theta^T X \,|a_T = \nu_T, \\ & c_{T+t} + E_{T+t}\frac{p_{T+t+1}}{p_{T+t}} a_{T+t+1} \leq e^i_{T+t} + a_{T+t},\ a_{T+t+1} \geq \phi^i_{T+t+1}, \forall t \geq 0\}. \end{aligned} \tag{3.1}$$

[4] A process $m \in X$ is a *martingale* if $m_t = E_t m_{t+1}$, for all $t \geq 0$, while m is a *submartingale* (respectively *supermartingale*) if $m_t \leq E_t m_{t+1}$ (respectively $m_t \geq E_t m_{t+1}$) for all $t \geq 0$.

Therefore we can focus on the simpler budgets of the form (3.1), in which we can imagine that the agent is choosing directly the (beginning of period) wealth holdings. We denote the problem $\max_{(c,a)\in B^i_T(\nu_T,\phi^i,p)} U^i_T(c)$ by $P^i_T(\nu_T,\phi^i,p)$. Its optimal solution is $C^i_T(\nu_T,\phi^i,p)$, and the maximum continuation utility attainable by the agent is $V^i_T(\nu_T,\phi^i,p)$:

$$C^i_T(\nu_T,\phi^i,p) := \operatorname{argmax}_{(c,a)\in B^i_T(\nu_T,\phi^i,p)} U^i_T(c), \tag{3.2}$$

$$V^i_T(\nu_T,\phi^i,p) := \max_{(c,a)\in B^i_T(\nu_T,\phi^i,p)} U^i_T(c). \tag{3.3}$$

As a consequence of the equivalence of the budgets $B^i_T(\nu_T,\phi^i,p,q)$ and $B^i_T(\nu_T,\phi^i,p)$ (from the point of view of consumption), the consumption component in $C^i_T(\nu_T,\phi^i,p,q)$ and $C^i_T(\nu_T,\phi^i,p)$ coincide, and

$$V^i_T(\nu_T,\phi^i,p,q) = V^i_T(\nu_T,\phi^i,p). \tag{3.4}$$

We henceforth drop the last argument (q) in the indirect utility of the agent, as arbitrage opportunities are absent in an equilibrium.

For the rest of the section we drop the agent-specific superscript i as we focus on a single agent, and we fix a pair of debt limits $\bar{\phi},\phi$ such that $\bar{\phi}$ is NTT (for the chosen agent, at prices p and penalties V^d). The characterization results of NTT bounds that follow require:

Assumption 3.1. *Debt limits $\bar{\phi},\phi$ satisfy*

$$V^d(p,q,\phi) = V^d(p,q,\bar{\phi}).$$

Assumption (3.1) makes the continuation utilities after default under the two debt limits $\bar{\phi},\phi$ equal. It is clearly satisfied for penalties such as (2.4) and (2.5) since continuation utilities do not depend on agent's debt limits.[5] Set

$$M := p(\phi - \bar{\phi}). \tag{3.5}$$

We show next that discounted NTT constraints satisfying Assumption 3.1 are de-

[5]Bidian and Bejan (2012) analyze an example where the agents are subject to a temporary interdiction to trade after default and the continuation utilities depend on debt limits, but Assumption 3.1 holds nevertheless.

termined only up to a martingale, that is we prove that ϕ are NTT (for the given agent at prices p and penalties V^d) if and only if M is a martingale. The "if" part (sufficiency) is immediate, and was shown by Kocherlakota (2008) (for less general penalties for default).

Proposition 3.1. *If $\bar{\phi}$ are NTT and M is a martingale, then $V_t(\bar{\phi}_t, \bar{\phi}, p) = V_t(\phi_t, \phi, p)$ for all $t \in \mathbb{N}$ and therefore ϕ are NTT if Assumption 3.1 holds.*

Proof. It is immediate to check that $(c, a) \in B_t(\bar{\phi}_t, \bar{\phi}, p)$ if and only if $(c, a + \phi - \bar{\phi}) \in B_t(\phi_t, \phi, p)$. Thus for all $t \in \mathbb{N}$, $V_t(\bar{\phi}_t, \bar{\phi}, p) = V_t(\phi_t, \phi, p) = V_t^d(p, q, \phi)$, and equal also to $V_t^d(p, q, \bar{\phi})$ under Assumption 3.1, thus ϕ are NTT. □

The next result is related.

Proposition 3.2. *If M is a supermartingale, then for any $t \geq 0$, $V_t(\phi_t, \phi, p) \geq V_t(\bar{\phi}_t, \bar{\phi}, p)$ with strict inequality on the set $\{M_t > E_t M_{t+1}\}$.*

Proof. It is immediate to check that if $(c, a) \in B_t(\bar{\phi}_t, \bar{\phi}, p)$, then $(\tilde{c}, a + \phi - \bar{\phi}) \in B_t(\phi_t, \phi, p)$, where $\tilde{c}_s := c_s + E_s (M_s - M_{s+1}) / p_s \geq c_s$, for all $s \geq t$. Since $\tilde{c}_t > c_t$ on $\{M_t > E_t M_{t+1}\}$, the conclusion follows. □

The rest of this section is dedicated to proving the converse to Proposition 3.1: the difference of discounted NTT debt limits is a martingale. In the proof we will use intensively truncations of an agent's problem between "periods" where the debt limits bind, and analyze and perturb the agent's optimal asset and consumption paths between two such "periods". In stochastic economies, the first time when debt limits bind represents a "cut" in the event tree, and therefore we need to introduce the notion of a *stopping time*, which is a function $T : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\{T = n\} \in \mathcal{F}_n$, for all $n \in \mathbb{N}$. A stopping time T is said to be *finite* if $T < \infty$, and *bounded* if there exists $n \in \mathbb{N}$ such that $T < n$.

A stopping time T induces the σ-algebra $\mathcal{F}_T$ of events known at T,

$$\mathcal{F}_T := \{A \in \mathcal{F} \mid A \cap \{T = n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}.$$

The operator $E_T(\cdot)$ denotes the conditional expectation with respect to $\mathcal{F}_T$. Let $x = (x_n) \in X$ and T be a finite stopping time. The random variable x_T is defined as $x_T(\omega) := x_{T(\omega)}(\omega)$, for all $\omega \in \Omega$. The process x *starting at* T is defined as the sequence of random variables $(x_{T+n})_{n=0}^{\infty}$, which we denote also by $\Theta^T x$ (where Θ is the shift operator introduced before). By extension, if $A \subset X$, then $\Theta^T A := \{\Theta^T x \mid x \in A\}$. Let S be another stopping time, not necessarily finite, such that $T \leq S$. The process x *stopped at* S and starting at T is defined as the sequence of random variables $(x_{(T+n)\wedge S})_{n=0}^{\infty}$, where $(T+n) \wedge S$ is an abbreviated notation for $\min\{T+n, S\}$. We use also the alternative notation $(x_n)_{n=T}^{S}$ for the process x stopped at S and starting at T. The notations for an agent's budget, indirect utility and solution to his optimization problem introduced in (2.1) and (3.1)-(3.3) apply also when T is a finite stopping time rather than a deterministic time (period). We denote the indicator function of a set $A \in \mathcal{F}$ by $\mathbf{1}_A$. Thus $\mathbf{1}_A : \Omega \to \mathbb{R}$, $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, while $\mathbf{1}_A(\omega) = 0$ if $\omega \notin A$.

Let T be an arbitrary stopping time. Define $\alpha(T)$ to be the first time the bounds ϕ bind after T, when the agent starts with wealth ϕ_T at T and faces bounds ϕ. Concretely, for each $\omega \in \{T < \infty\}$,

$$\alpha(T)(\omega) := \inf\left\{t \mid t \in \mathbb{N}, t > T(\omega), a_t(\omega) = \phi_t(\omega), (c, a) \in C_T(\phi_T, \phi, p)\right\}, \qquad (3.6)$$

and for $\omega \in \{T = \infty\}$, $\alpha(T)(\omega) := \infty$. Notice that $\alpha(T)$ is well-defined, as the set $C_T(\phi_T, \phi, p)$ contains a unique element. Indeed, strict concavity of the period utilities (u_t) imply that if if $(c, a), (c', a') \in C_T(\phi_T, \phi, p)$, then $c = c'$, otherwise $((c + c')/2, (a + a')/2) \in B_T(\phi_T, \phi, p)$ would be strictly preferred by the agent to both (c, a) and (c', a'). But then for any $s \geq T$, $V_s(a_s, \phi_s, p) = U_s(c) = U_s(c') = V_s(a'_s, \phi, p)$, hence $a_s = a'_s$ (V_s is strictly increasing in initial wealth), and therefore $(c, a) = (c', a')$. With multiple optimal paths (without strict concavity), our arguments would go through, but we would have to be explicit about which optimal path is selected in the definition of $\alpha(T)$. We also set $\alpha^0(T) := T$ and for $k \geq 1$, we define $\alpha^k(T)$ recursively as $\alpha^k(T) := \alpha(\alpha^{k-1}(T))$.

We impose the following consistency condition between $\bar{\phi}$ and ϕ:

Assumption 3.2. *For each $t \in \mathbb{N}$, the process $\left(p \cdot \bar{\phi}\right)_{s=t}^{\alpha(t)}$ is uniformly integrable.*[6]

This assumption is used in Proposition A.2 to show that the process $(M_s)_{s=t}^{\alpha(t)}$ converges a.s. and in L^1. It is a consistency condition between $\bar{\phi}$ and ϕ, requiring that $\bar{\phi}$ does not grow unboundedly large (in discounted terms) between two periods in

[6] A process $x = (x_n)_n$ is uniformly integrable if and only if $\lim_{a\to\infty} \sup_n E(|x_n| \cdot \mathbf{1}_{|x_n|>a}) = 0$.

which the debt limits ϕ bind. It is an extremely mild assumption, since it is imposed piecewise on time intervals $[t, \alpha(t)]$, rather than on the whole horizon. Therefore, if debt limits ϕ bind in bounded time, Assumption 3.2 is automatically satisfied. Moreover, Proposition A.2 shows that $(p \cdot \phi)_{s=t}^{\alpha(t)}$ is uniformly integrable, and therefore Assumption 3.2 is satisfied if on each time interval $[t, \alpha(t)]$, $\bar{\phi}$ are bounded from below by an arbitrarily large multiple of ϕ (that is, if there exists $K(t) > 0$ such that $\bar{\phi}_s \geq K(t)\phi_s$, for all $s \in [t, \alpha(t) + 1)$. Clearly Assumption 3.2 is also trivially satisfied when the penalties are the interdiction to borrow (2.5) analyzed in Hellwig and Lorenzoni (2009), since $\bar{\phi} = 0$ are NTT in this case.

We can prove now the converse to Proposition 3.1, which completes the characterization of NTT debt limits. The proof uses the property that the optimal asset holdings of the agent are nondecreasing in initial wealth (see Lemma A.1):

$$(c, a) \in C_t(\nu, \phi, p), (c', a') \in C_t(\nu', \phi, p) \quad \Rightarrow \quad a'_s \geq a_s, \forall s \geq t. \tag{3.7}$$

Theorem 3.3. *If $\bar{\phi}, \phi \leq 0$ are NTT (given p, q, V^d) and Assumptions 3.1 and 3.2 hold, then the process $M := p(\phi - \bar{\phi})$ is a martingale.*

Proof. Fix a natural number t.

STEP 1. We show that

$$M_t \geq E_t M_{\alpha(t)}, \tag{3.8}$$

where $M_{\alpha(t)} := \lim_{n\to\infty} M_{\alpha(t)\wedge n}$. By Proposition A.2, the limit is well-defined (a.s. and in L^1), and $(M_s)_{s=t}^{\alpha(t)}$ has a (lower) Snell envelope $(\hat{M}_s)_{s=t}^{\alpha(t)}$, which is the largest submartingale dominated from above by M (that is $\hat{M} \leq M$), and it satisfies $\hat{M}_s = M_s \wedge E_s\hat{M}_{s+1}$ and

$$\hat{M}_{\alpha(t)} = M_{\alpha(t)}. \tag{3.9}$$

We prove that $(\hat{M}_s)_{s=t}^{\alpha(t)}$ is in fact a martingale, rather than just a submartingale. Assume, by contradiction, that there exists $n \in \mathbb{N}$ such that $\{t \leq n < \alpha(t)\} \cap \{\hat{M}_n < E_n\hat{M}_{n+1}\}$ has positive probability. Until we reach a contradiction, all statements below are restricted to the set $\{t \leq n < \alpha(t)\} \cap \{\hat{M}_n < E_n\hat{M}_{n+1}\}$ (which is $\mathcal{F}_n$-measurable). Notice that $\hat{M}_n = M_n$, since $\hat{M}_n = M_n \wedge E_n\hat{M}_{n+1}$ and $\hat{M}_n < E_n\hat{M}_{n+1}$.

Let $(c,a) \in C_n(\phi_n, \phi, p)$. Define $(\tilde{a}_s)_{s=n+1}^{\alpha(t)}$ by

$$\tilde{a}_s := a_s - \frac{\hat{M}_s}{p_s} \geq \phi_s - \frac{M_s}{p_s} = \phi_s - (\phi_s - \bar{\phi}_s) = \bar{\phi}_s,$$

and let $\tilde{a}_n = \bar{\phi}_n$. Let $(\tilde{c}_s)_{s=n}^{\alpha(t)-1}$ be the consumption supported by asset holdings $\tilde{a}$, thus $p_s(\tilde{c}_s - c_s) = p_s(\tilde{a}_s - a_s) - E_s p_{s+1}(\tilde{a}_{s+1} - a_{s+1})$. Hence

$$\begin{aligned} p_n(\tilde{c}_n - c_n) &= -M_n + E_n \hat{M}_{n+1} = -\hat{M}_n + E_n \hat{M}_{n+1} > 0, \\ p_s(\tilde{c}_s - c_s) &= -\hat{M}_s + E_s \hat{M}_{s+1} \geq 0, \quad n+1 \leq s < \alpha(t). \end{aligned}$$

We reached a contradiction, since

$$\begin{aligned} V_n^d = V_n(\bar{\phi}_n, \bar{\phi}, p) &\geq E_n \left(\sum_{s=n}^{\alpha(t)-1} u_s(\tilde{c}_s) + V_{\alpha(t)}^d \mathbf{1}_{\alpha(t)<\infty} \right) \\ > E_n \left(\sum_{s=n}^{\alpha(t)-1} u_s(c_s) + V_{\alpha(t)}^d \mathbf{1}_{\alpha(t)<\infty} \right) &= V_n(\phi_n, \phi, p) = V_n^d. \end{aligned}$$

Having established that $\hat{M}$ is a martingale, (3.8) follows now from (3.9).

STEP 2. We show that

$$M_t = E_t M_{\alpha(t)}. \tag{3.10}$$

For each $k \in \mathbb{N}$, repeat the construction in STEP 1 for $\alpha^k(t)$ instead of t, on the set where $\{\alpha^k(t) < \infty\}$, and obtain the martingale $(\hat{M}_s)_{s=\alpha^k(t)+1}^{\alpha^{k+1}(t)}$ (the lower Snell envelope of $(M_s)_{s=\alpha^k(t)+1}^{\alpha^{k+1}(t)}$), dominated from above by M and such that $\hat{M}_{\alpha^{k+1}(t)} = M_{\alpha^{k+1}(t)}$.[7] We let also $\hat{M}_t := E_t \hat{M}_{\alpha(t)}$. By (3.8), the resulting process $(\hat{M}_s)_{s=t}^{\infty}$ is a supermartingale, $\hat{M} \leq M$, and for all $k \geq 1$, $\hat{M}_{\alpha^k(t)} = M_{\alpha^k(t)}$.

Construct the process $(\hat{\phi})_{s=t}^{\infty}$ defined by $\hat{\phi}_s := \bar{\phi}_s + \hat{M}_s/p_s$. It follows that $\phi_s \geq \hat{\phi}_s$ for all $s \geq t$, and $\hat{\phi}_{\alpha^k(t)} = \phi_{\alpha^k(t)}$ for all $k \geq 1$. Let $(\bar{c}, \bar{a}) \in C_t(\phi_t, \phi, p)$. We claim that $(\bar{c}, \bar{a})$ is also an optimal solution for the problem $P_t(\phi_t, \hat{\phi}, p)$ with relaxed debt limits, that is we show that $(\bar{c}, \bar{a}) \in C_t(\phi_t, \hat{\phi}, p)$.[8] Let $(c,a) \in B_t(\phi_t, \hat{\phi}, p)$ and $\eta_n := \alpha^k(t) \wedge n$,

[7] Equivalently, for $\alpha^k(t) + 1 \leq s < \alpha^{k+1}(t) + 1$ define $\hat{M}_s := E_s M_{\alpha^{k+1}(t)}$ and use repeatedly the property (3.7) and (3.8) to show that $\hat{M} \leq M$.

[8] Since $\bar{a}$ binds at the same dates and states under the ϕ and $\hat{\phi}$ bounds, $(\bar{c}, \bar{a})$ satisfies the Kuhn-

for $k \geq 1$ and $n \geq t$. By (C.1) and (C.6),

$$E_t \sum_{s=t}^{\eta_n - 1} (u_s(c_s) - u_s(\bar{c}_s)) \leq E_t u'_{\eta_n}(\bar{c}_{\eta_n})(\bar{a}_{\eta_n} - \hat{\phi}_{\eta_n}) \leq$$

$$\leq E_t u'_{\eta_n}(\bar{c}_{\eta_n})(\bar{a}_{\eta_n} - \phi_{\eta_n}) + \frac{u'_t(\bar{c}_t)}{p_t} E_t p_{\eta_n}(\phi_{\eta_n} - \hat{\phi}_{\eta_n}) =$$

$$= E_t u'_n(\bar{c}_n)(\bar{a}_n - \phi_n)\mathbf{1}_{n \leq \alpha^k(t)} + \frac{u'_t(\bar{c}_t)}{p_t} E_t(M_{\eta_n} - \hat{M}_{\eta_n}), \tag{3.11}$$

as $(\bar{a}_{\eta_n} - \hat{\phi}_{\eta_n})\mathbf{1}_{n > \alpha^k(t)} = 0$. Using the necessary transversality condition (C.3),

$$\lim_{n \to \infty} E_t u'_n(\bar{c}_n)(\bar{a}_n - \phi_n)\mathbf{1}_{n \leq \alpha^k(t)} \leq \lim_{n \to \infty} E_t u'_n(\bar{c}_n)(\bar{a}_n - \phi_n) = 0,$$

and $\lim_{n\to\infty} E_t(M_{\eta_n} - \hat{M}_{\eta_n}) = 0$ since $M_{\eta_n} \to M_{\alpha^k(t)}$, $\hat{M}_{\eta_n} \to \hat{M}_{\alpha^k(t)}$ (a.s. and in L^1) and $\hat{M}_{\alpha^k(t)} = M_{\alpha^k(t)}$. Making $n \to \infty$ in (3.11), $E_t \sum_{s=t}^{\alpha^k(t)-1} (u_s(c_s) - u_s(\bar{c}_s)) \leq 0$. By letting $k \to \infty$, it follows that that $U_t(\bar{c}) \geq U_t(c)$. Since (c, a) was an arbitrary feasible path in $B_t(\phi_t, \hat{\phi}, p)$, we conclude that $(\bar{c}, \bar{a}) \in C_t(\phi_t, \hat{\phi}, p)$.

Therefore $V_t(\phi_t, \hat{\phi}, p) = V_t(\phi_t, \phi, p) = V_t^d$, and

$$V_t^d = V_t(\phi_t, \hat{\phi}, p) \geq V_t(\hat{\phi}_t, \hat{\phi}, p) \geq V_t^d.$$

The first inequality above is strict if $\phi_t > \hat{\phi}_t$ and the second one is strict if $\hat{M}$ is not a martingale, but rather only a supermartingale, by Proposition 3.2. Thus $\hat{M}$ is a martingale and $\phi_t = \hat{\phi}_t$. Thus $M_t = \hat{M}_t = E_t M_{\alpha(t)}$, and (3.10) obtains.

STEP 3. We show that

$$M_t = E_t M_{t+1}. \tag{3.12}$$

It is enough to prove that

$$M_{t+1} = E_{t+1} M_{\alpha(t)}, \tag{3.13}$$

since then $M_t = E_t M_{\alpha(t)} = E_t E_{t+1} M_{\alpha(t)} = E_t M_{t+1}$, as desired. Let $\eta^0 := t+1$ and for $m \geq 0$, $\eta^{m+1} := \alpha(\eta^m) \wedge \alpha(t)$. Thus $\eta^m \nearrow \alpha(t)$. Fix $l \in \mathbb{N}$. We show first that $M_{\eta^l} = E_{\eta^l} M_{\eta^{l+1}}$. On the set $\{\eta^l < \alpha(t)\}$, the monotonicity property (3.7) implies

Tucker conditions for the problem $P_t(\phi_t, \hat{\phi}, p)$. However it is unclear whether it satisfies also the sufficient transversality condition.

that $\alpha(\eta^l) \leq \alpha(t)$, thus $\eta^{l+1} = \alpha(\eta^l)$. By (3.10),

$$\mathbf{1}_{\eta^l<\alpha(t)} \cdot E_{\eta^l} M_{\eta^{l+1}} = \mathbf{1}_{\eta^l<\alpha(t)} \cdot E_{\eta^l} M_{\alpha(\eta^l)} = \mathbf{1}_{\eta^l<\alpha(t)} \cdot M_{\eta^l}.$$

On the set $\{\eta^l = \alpha(t)\}$, $\eta^{l+1} = \eta^l = \alpha(t)$. Therefore

$$E_{\eta^l} M_{\eta^{l+1}} = \mathbf{1}_{\eta^l<\alpha(t)} \cdot E_{\eta^l} M_{\eta^{l+1}} + \mathbf{1}_{\eta^l=\alpha(t)} \cdot E_{\eta^l} M_{\eta^l} = \mathbf{1}_{\eta^l<\alpha(t)} \cdot M_{\eta^l} + \mathbf{1}_{\eta^l=\alpha(t)} \cdot M_{\eta^l} = M_{\eta^l}.$$

Using the law of iterated expectations,

$$M_{t+1} = M_{\eta^0} = E_{\eta^0} M_{\eta^1} = \ldots = E_{\eta^0} M_{\eta^l} = E_{t+1} M_{\eta^l}, \forall l \in \mathbb{N}.$$

By the uniform integrability of (M_{η^l}), which is a consequence of Proposition 3.1, part 2,

$$M_{t+1} = \lim_{l\to\infty} E_{t+1} M_{\eta^l} = E_{t+1} \lim_{l\to\infty} M_{\eta^l} = E_{t+1} M_{\alpha(t)}.$$

Therefore (3.13) holds and hence (3.12) is true, thus M is a martingale. $\square$

The idea of the proof is depicted in Figure 1, for the deterministic case, which is more transparent. Without uncertainty, submartingales, respectively martingales, respectively supermartingales are increasing, respectively constant, respectively decreasing sequences. The solid line represents the process $M := p(\phi - \bar{\phi})$, while the dotted line represents $\hat{M}$, whose construction is explained below.

In Step 1 we fix an arbitrary period of time t, and denote by $\alpha(t)$ the first period when agent's debt limits bind after t, if he starts with wealth ϕ_t at t and faces debt limits ϕ. We construct the Snell envelope $\hat{M}$ of M on the interval $[t, \alpha(t)]$ (the largest submartingale smaller than M, or in this context, the largest increasing function lying below M on the respective interval), and show that it has to be in fact a martingale (otherwise the agent will default when faced with debt limits ϕ). It follows that the process M *sampled* at t and $\alpha(t)$ is a supermartingale.

In Step 2, we construct in a similar fashion the Snell Envelope $\hat{M}$ for the process M on the intervals $[t, \alpha(t)], (\alpha(t), \alpha^2(t)], (\alpha^2(t), \alpha^3(t)], \ldots$, where $\alpha^k(t)$ represents the k-th time debt limits ϕ bind after t, in the problem $P_t(\phi_t, \phi, p)$. By Step 1, $\hat{M}$ is a supermartingale. Using $\hat{M}$, we construct the relaxed bounds $\hat{\phi} := \bar{\phi} + \hat{M}/p \leq \phi$, which coincide with ϕ at $\alpha(t), \alpha^2(t), \ldots$, that is whenever ϕ are binding in the problem $P_t(\phi_t, \phi, p)$. Therefore the optimal solution for $P_t(\phi_t, \phi, p)$ is also a solution of the relaxed problem (with larger feasible set) $P_t(\phi_t, \hat{\phi}, p)$. By Proposition 3.2, we conclude that $\phi_t = \hat{\phi}_t$, and therefore the process M sampled at t and $\alpha(t)$ is a martingale (rather than just a supermartingale, as shown in Step 1).

Finally, in Step 3 we show that M must be a martingale. Fix an arbitrary period t. It is enough to show that $M_t = M_{t+1}$ (we are in a deterministic world here). By the previous two steps, we know that $M_s = M_{\alpha(s)}$, for all s. If $\alpha(t) = t+1$, we are done. If this is not the case, the monotonicity property (3.7) guarantees that $\alpha(t+1) \leq \alpha(t)$. Indeed, the debt limits of the agent do not bind at $t+1$ in the problem $P_t(\phi_t, \phi, p)$, as they bind for the first time only at $\alpha(t) > t+1$. Therefore the wealth of the agent at $t+1$, along the optimal path for problem $P_t(\phi_t, \phi, p)$ strictly exceeds ϕ_{t+1}, and therefore the debt limits in the problem $P_{t+1}(\phi_{t+1}, \phi, p)$ must bind before or at the latest at $\alpha(t)$, by (3.7). If $\alpha(t+1) = \alpha(t)$, then $M_t = M_{\alpha(t)} = M_{\alpha(t+1)} = M_{t+1}$, as desired. Otherwise, by an identical reasoning we have $\alpha^2(t+1) \leq \alpha(t)$. We can continue this iterative process, which stops as soon as $\alpha^k(t+1) = \alpha(t)$, as $M_t = M_{\alpha(t)} = M_{\alpha^k(t+1)} = M_{t+1}$. A finite number of iterations is needed if $\alpha(t)$ is finite, otherwise one takes the limit as $k \to \infty$ to get $\lim_{k\to\infty} M_{\alpha^k(t+1)} = M_{\alpha(t)}$ and reach the conclusion.

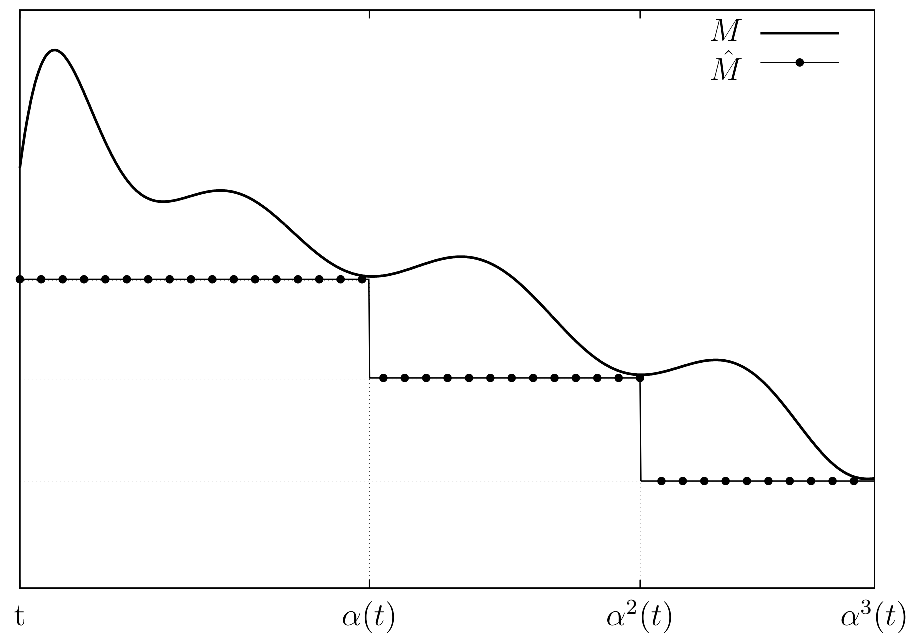

Figure 1: Illustration of the proof of Theorem 3.3.

When the penalty for default is the interdiction to borrow (2.5), Theorem 3.3 implies that any NTT debt limits ϕ are discounted martingales and therefore the result of Hellwig and Lorenzoni (2009) obtains as a particular case of our theorem - a detailed comparison is offered in Appendix B.

Theorem 3.3 allows for almost arbitrary debt limits (satisfying only the mild Assumption 3.2), in which one of them does not have to dominate the other. Therefore Theorem 3.3 delivers a general characterization of NTT debt limits, with a simpler proof. Appendix B contains a detailed comparison between the proof of Theorem 3.3 and the proof of Hellwig and Lorenzoni (2009).

The result in Theorem 3.3 should not be interpreted as saying that NTT debt limits are discounted martingales. This is true when the penalty for default is the interdiction to borrow (2.5) (as seen before), but not in general. For the other canonical case when the penalty for default is the interdiction to trade (2.4), we can in fact show that the NTT debt limits are discounted submartingales.

Proposition 3.4. *Assume that ϕ are NTT debt limits when the agent faces penalties* (2.4) *(no trading after default). Then $p \cdot \phi$ is a submartingale converging a.s.*

Proof. The agent will default at period t, when starting with wealth ϕ_t at period t, on the set $\{p_t\phi_t > E_t p_{t+1}\phi_{t+1}\}$. Indeed, let $(c, a) \in C_t(\phi_t, \phi, p)$. Construct $(c', a') \in B_t(\phi_t, \phi, p)$ (see (3.1)) given by $c'_t := e_t + (p_t\phi_t - E_t p_{t+1}\phi_{t+1})/p_t$, $a'_t := \phi_t$, and $(c', a') \in C_{t+1}(\phi_{t+1}, \phi, p)$ (hence $a'_{t+1} := \phi_{t+1}$). On the set $\{p_t\phi_t > E_t p_{t+1}\phi_{t+1}\}$, $c'_t > e_t$, and

$$U_t(c') = u_t(c'_t) + E_t V^d_{t+1} > u_t(e_t) + E_t V^d_{t+1} = u_t(e_t) + E_t U_{t+1}(e) = V^d_t.$$

It follows that $U_t(c') > U_t(c) = V^d_t$ on the set $\{p_t\phi_t > E_t p_{t+1}\phi_{t+1}\}$, contradicting the optimality of the path c. Hence $p_t\phi_t \leq E_t p_{t+1}\phi_{t+1}$ for all t and therefore $p \cdot \phi$ is a submartingale. Since $\phi \leq 0$, the martingale convergence theorem (Kopp 1984, Theorem 2.6.1) applies, and $(p_t\phi_t)$ converges a.s. to an integrable variable. □

4 Applications

The property of NTT debt limits uncovered by Kocherlakota (2008) and stated in Proposition 3.1 suggests that associated to any equilibrium allocations and prices, there is a continuum of possible NTT debt limits for the agents, differing from each other by (arbitrary) discounted martingales with zero expected value, and preserving the total amount of credit in the economy. Indeed, consider an AJ-equilibrium $\left(p, q, (c^i)_{i=1}^I, (\bar{a}^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\bar{\phi}^i)_{i=1}^I, (V^{i,d})_{i=1}^I\right)$. For each $i \in \{1, \ldots, I-1\}$, let $\varepsilon^i \in X$ such that $p \cdot \varepsilon$ is a martingale and such that $\varepsilon_0^i = 0$ (thus $p \cdot \varepsilon^i$ is a zero mean martingale). Set $\varepsilon^I := -\sum_{i=1}^{I-1} \varepsilon^i$. Then $\left(p, q, (c^i)_{i=1}^I, (a^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\phi^i)_{i=1}^I, (V^{i,d})_{i=1}^I\right)$ is also an AJ-equilibrium, where $a^i := \bar{a}^i + \varepsilon^i$ and $\phi^i := \bar{\phi}^i + \varepsilon^i$. This is an immediate consequence of the equivalence of agents' budget constraints established in Proposition 3.1.

We show that Theorem 3.3 can remove the multiplicity of NTT debt limits outlined above, under some additional assumptions on debt limits. In this sense, the theorem can be viewed as a uniqueness result. Concretely, nonpositive NTT debt limits that are bounded by the present value of agent's future endowments (assumed finite) are unique.

Proposition 4.1. *For each $t \in \mathbb{N}$, let $Y_t := \frac{1}{p_t} E_t \sum_{s \geq t} p_s e_s$ and assume $Y_0 < \infty$. Let $\phi, \bar{\phi}$ be NTT given V^d and satisfying Assumption 3.1. If $0 \geq \phi, \bar{\phi} \geq -Y$, then $\phi = \bar{\phi}$.*

Proof. Notice that the process $p \cdot Y$ is a uniformly integrable positive supermartingale converging to zero a.s. and in L^1. Thus Assumption 3.2 is satisfied (with $\alpha(t)$ replaced by ∞), and the conclusion follows by Theorem 3.3. $\square$

Therefore with *high interest rates* (that is, with a finite discounted present value of endowment) and borrowing limited by the agent's ability to repay his debt out of his future endowments (Santos and Woodford 1997), nonpositive NTT debt limits are unique (for a given agent, pricing kernel and penalties for default). Proposition 4.1 fills some gaps and gives a unified view of results obtained for various penalties for default. When the punishment for default is the interdiction to trade, Alvarez and Jermann (2000, Proposition 4.11) prove that given any sequential equilibrium with NTT debt limits and high interest rates, one can construct an equivalent equilibrium with identical pricing kernel and consumption, but with nonpositive NTT debt limits bounded by the present value of aggregate endowment. Proposition 4.1 shows that such debt limits are in fact unique. Moreover, when the punishment for default is the loss of borrowing privileges, nonpositive NTT debt limits restricted by the present value of future endowments must be identically equal to zero, and therefore no borrowing can be sustained in an equilibrium, as pointed out before by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009).

The assumption of high interest rates is ad-hoc and extremely restrictive in models with limited enforcement. In these environments, low interest rates arise in equilibrium as a way to induce agents not to default. As shown in Santos and Woodford (1997), rational bubbles are discounted martingales and they require low interest rates. The martingale characterization of NTT debts in Proposition 3.1 and Theorem 3.3 can be used to show that robust bubbles can arise in limited enforcement economies. Such bubbles enable agents to circumvent credit crunches (tight debt limits), and therefore are expansionary.

We introduce first the definition of a bubble and then pursue the connection between bubbles and self-enforcing debt. By (2.6), asset prices satisfy

$$q_t = \frac{1}{p_t} E_t \sum_{s>t} p_s d_s + \lim_{n\to\infty} \frac{1}{p_t} E_t p_n q_n.$$

Let $f_t(p,d) := \frac{1}{p_t} E_t \sum_{s>t} p_s d_s$ denote the discounted present value at t of future dividends d, that is the *fundamental value* of d at period t. It follows that

$$b_t(p,q) := \frac{1}{p_t} \lim_{n\to\infty} E_t p_n q_n \tag{4.1}$$

is well-defined and nonnegative, and $q_t = f_t(p,d) + b_t(p,q)$. The process $b(p,q)$ represents the part of asset prices in excess of fundamental values, and it is called the *bubble* component in the asset prices q. Notice that for all $t \in \mathbb{N}$, $p_t b_t(p,q) = E_t p_{t+1} b_{t+1}(p,q)$. Hence $p \cdot b(p,q)$ is a nonnegative martingale and $b(p,q) = 0$ if and only if $0 = b_0(p,q)$ $(= \frac{1}{p_0} \lim_{t\to\infty} E p_t q_t)$.

We compare pairs of AJ-equilibria, therefore to avoid lengthy notation, we set $\mathcal{E} := \big(p, q, (c^i)_{i=1}^I, (a^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\phi^i)_{i=1}^I, (V^{i,d})_{i=1}^I\big)$, while $\bar{\mathcal{E}}, \tilde{\mathcal{E}}, \hat{\mathcal{E}}$ denote similar vectors, with all variables barred, tilded, respectively hatted. We say that the AJ-equilibria $\mathcal{E}, \hat{\mathcal{E}}$ are *equivalent* if pricing kernels, consumptions and penalties for default coincide: $\hat{p} = p$, $\hat{c}^i = c^i$, $\hat{V}^{i,d} = V^{i,d}$, for all agents i. Notice that this equivalence notion allows for a redistribution of initial asset holdings among agents.

We analyze in what follows a class of default penalties described by some exogenous nonpositive "penalty" debt restrictions $\bar{\phi}^i \le 0$ for each agent i. If an agent i

subject to some debt limits ϕ^i less tight than $\bar{\phi}^i$ ($\phi^i \leq \bar{\phi}^i$) defaults at t, the agent has his debt discharged, in exchange for a "fee" $|\bar{\phi}^i_t|$ at t and tighter future debt limits $\bar{\phi}^i$ (and repayment of debt is strictly enforced after a default). The debt limits after default $\bar{\phi}^i \leq 0$ can be arbitrarily small in absolute value, or even zero, in which case we have an interdiction to borrow upon default, (2.5). For example, we can take $\bar{\phi}^i := -k^i e^i$, where $k^i \geq 0$ are some predetermined loan-to-income ratios. The set of nonnegative discounted martingales associated to the pricing kernel p is denoted by

$$M(p) := \{m \in X_+ \mid p \cdot m \text{ is a martingale}\}.$$

Proposition 4.2. *Let $(\bar{\phi}^1, \ldots, \bar{\phi}^I) \in -X^I_+$. Consider an AJ-equilibrium $\mathcal{E}$ with debt limits ϕ^i with $\phi^i \leq \bar{\phi}^i$, and penalties for default given by $V^{i,d}_t := V^i_t(\bar{\phi}^i_t, \bar{\phi}^i, p, q)$, for all i, t. Assume $\sum_i \phi^i_0 \neq \sum_i \bar{\phi}^i_0$. Fix an arbitrary asset j in unit supply (without loss of generality). Then $\bar{\phi}^i - \phi^i \in M(p)$ for all i and the following hold:*

a. Let $\varepsilon \in M(p)$ such that $\theta^{i,j}_{-1} \cdot \varepsilon \leq \bar{\phi}^i - \phi^i$, for all i. There exists an equivalent AJ-equilibrium that has a bubble ε in asset j, identical initial endowments of the assets for the agents, and tighter debt limit $(\hat{\phi}^i)^I_{i=1}$, with $\hat{\phi}^i := \phi^i + \theta^{i,j}_{-1} \cdot \varepsilon$ $(\leq \bar{\phi}^i)$.

b. Let $\varepsilon := \sum^I_{i=1}(\bar{\phi}^i - \phi^i)$. There exists an equivalent AJ-equilibrium that has a bubble ε in asset j and tighter debt limits $(\bar{\phi}^i)^I_{i=1}$.

Proof. $\phi^i, \bar{\phi}^i$ satisfy Assumption 3.1 by construction. They also satisfy Assumption 3.2, since $\bar{\phi}^i$ is bounded from below by ϕ^i. Indeed, Proposition A.2 shows that for any t, $(p \cdot \phi)^{\alpha(t)}_{s=t}$ is uniformly integrable, which coupled with $\phi \leq \bar{\phi} \leq 0$ guarantees that $\left(p \cdot \bar{\phi}\right)^{\alpha(t)}_{s=t}$ is uniformly integrable (see also the discussion after Assumption 3.2). Theorem 3.3 ensures that for each agent i, $\bar{\phi}^i - \phi^i$ are nonnegative discounted martingales.

For any $\varepsilon \in M(p)$, the "bubble equivalence theorem" (Kocherlakota 2008, Theorem 4) implies that $\mathcal{E}$ is equivalent to an equilibrium $\hat{\mathcal{E}}$ having a bubble ε in asset j, where for each period $t \geq 0$, $\hat{q}^j = q^j + \varepsilon$, $\hat{q}^k = q^k$ for $k \neq j$, $\hat{\theta}^i_{t-1} = \theta^i_{t-1}$, $\hat{a}^i_t := a_t + \varepsilon_t(\theta^{i,j}_{-1} - \theta^{i,j}_{t-1})$ and $\hat{\phi}^i_t := \phi^i_t + \varepsilon_t \cdot \theta^{i,j}_{-1}$. The proof is immediate and relies on the equality of agents' budgets constraints in $\mathcal{E}$ and $\hat{\mathcal{E}}$. Market clearing conditions are clearly satisfied. Bounds $\hat{\phi}^i$ remain NTT by Proposition 3.1, as

$$V^i_t(\hat{\phi}^i_t, \hat{\phi}^i, p, \hat{q}) = V^i_t(\hat{\phi}^i_t, \hat{\phi}^i, p) = V^i_t(\phi^i_t, \phi^i, p) = V^i_t(\phi^i_t, \phi^i, p, q) = V^i_t(\bar{\phi}^i_t, \bar{\phi}^i, p, q).$$

The first part of the Proposition now follows, as the perturbed debt limits $\hat{\phi}^i$ are less than $\bar{\phi}^i$ whenever $\theta_{-1}^{i,j} \cdot \varepsilon \leq \bar{\phi}^i - \phi^i$, for all i.

For the second part, we construct first an equilibrium $\bar{\mathcal{E}}$ equivalent to $\mathcal{E}$, which differs only in terms of the debt limits and Arrow security holdings. Every agent i except the first has $\bar{\phi}^i$ debt limits, agent 1 has debt limits $\bar{\phi}^1 - \varepsilon$, and the Arrow securities holdings are $\bar{a}^1 := a^1 - \varepsilon$ and $\bar{a}^i := a^i + \bar{\phi}^i - \phi^i$ for $i \neq 1$. Market clearing conditions hold, debt constraints are satisfied, while agents' budgets coincide, as shown in (the proof of) Proposition 3.1.

In turn, $\bar{\mathcal{E}}$ is equivalent to an equilibrium $\tilde{\mathcal{E}}$ with identical debt limits for the agents, in which agent 1 is the sole owner of asset j: $\tilde{\theta}_{t-1}^{1,j} := 1$, $\tilde{\theta}_{t-1}^{i,j} := 0$, for each $t \geq 0$ and agent $i \neq 1$. This can be accomplished by setting, for each period t, agent i and asset $k \neq j$, $\tilde{\theta}_{t-1}^{i,k} := \theta_{t-1}^{i,k}$, $\tilde{a}_t^i := \bar{a}_t^i + (q_t + d_t)(\theta_{t-1}^i - \tilde{\theta}_{t-1}^i)$. Showing that $\tilde{\mathcal{E}}$ is an AJ-equilibrium equivalent to $\bar{\mathcal{E}}$ is immediate, since agents have identical wealth levels at all times, and only the allocation of their wealth between Arrow securities and infinitely-lived assets is changed. Applying the bubble equivalence theorem to $\tilde{\mathcal{E}}$ (instead of $\mathcal{E}$) produces an equivalent equilibrium $\hat{\mathcal{E}}$ with the desired properties. $\square$

Proposition 4.2 is an instance of the "bubble equivalence theorem" of Kocherlakota (2008), who showed that an arbitrary bubble can be injected in an infinitely-lived asset, while leaving agents' budget constraints (hence consumption) unchanged, as long as the debt constraints of the agents are allowed to be adjusted upwards by their initial endowment of the asset multiplied by the bubble term. The introduction of a bubble gives consumers a windfall proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits.

Another interpretation of Proposition 4.2, in the light of our Theorem 3.3, is as follows. Assume that the competitive financial intermediaries that set the NTT debt limits for the agents (see the discussion in the introduction and the model in Ábrahám and Cárceles-Poveda (2010)) decide to tighten them. Being competitive, they take the interest rates (pricing kernel) as given, and therefore, by Theorem 3.3, they will choose new debt limits that are tighter by a (positive) discounted martingale. Without a bubble, this reduction in available credit would lead to lower interest rates (due to precautionary saving) and to a recession, for realistic calibrations (Guerrieri and Lorenzoni 2011). The total reduction in credit is a discounted martingale, and a bubble in an asset in unit supply equal to this reduction in credit would completely compensate for the reduced liquidity in the economy.

The bubble equivalence theorem did not receive the attention it deserves, since it was usually assumed that the new (tighter) debt bounds required to sustain the bubble injection in a positive supply asset must eventually become positive, due to the bubble component they now contain.[9] Forced saving, however, seems implausible (especially with enforcement limitations). Proposition 4.2 showcases the power of Theorem 3.3, and points out that debt limits can remain nonpositive (and NTT) after a bubble injection, if the penalties for default are sufficiently mild - agents are still allowed to borrow some predetermined (possibly zero or arbitrarily small) amounts after default. Keeping the agents' initial endowments of assets fixed guarantees the existence of an equivalent bubbly equilibrium only if the initial endowments of long-lived securities and agents' excess debt limits over the penalty levels satisfy an additional consistency condition (always satisfied for example if only one agent owns the asset j initially, and his debt limits are not identically equal to the penalty levels). If we allow initial transfers (of stocks and Arrow securities) among agents, then a bubble of size up to the (absolute value of) total debt limits in excess of penalty levels can be injected in an asset in unit supply.

5 Conclusion

We consider an infinite horizon, complete markets economy, in which agents have the option to default on debt at any period in exchange for a continuation utility that can be date and state contingent, and can depend on the pricing kernel. For an agent facing a given pricing kernel and penalty for default, we characterize the set of debt limits that allow for maximum credit expansion while preventing default, à la Alvarez and Jermann (2000), known as "not-too-tight" (NTT) debt limits. We show that two discounted NTT debt limits for an agent facing a given pricing kernel must differ by a martingale.

[9]Hellwig and Lorenzoni (2009) is an exception, as they show that with low interest rates arising under the penalty 2.5, debt limits can contain bubble (martingale) components that remain bounded. However they do not connect this observation with the possibility of bubbles in asset prices using Kocherlakota's (2008) mechanism of transferring bubbles from debt limits into asset prices. Moreover, the interdiction to borrow penalty (see 3.5) analyzed by Hellwig and Lorenzoni (2009) is just a member the family of penalties allowed in Proposition 4.2.

Our characterization is crucial for establishing the uniqueness of NTT debt limits bounded (in absolute value) by the present value of future endowments. Moreover, it can be used to show that the tighter bounds resulting from the injection of a bubble using Kocherlakota's (2008) mechanism can remain nonpositive, despite the bubble component they contain. If agents are still allowed to borrow predetermined fixed fractions (arbitrarily small and possibly zero) of their endowments upon default, an equilibrium can sustain bubbles (on assets in unit supply) equal to the total debt limits in excess of the penalty levels.

Thus economies with endogenous (NTT) debt limits provide robust examples of bubbles, in the presence of rational, forward looking agents. These bubbles satisfy a need for liquidity triggered by credit crunches.

A Omitted proofs in Section 3

Lemma A.1. *Given any $t \in \mathbb{N}$ and $\mathcal{F}_t$-measurable random variables $\nu' \geq \nu$,*

$$(c, a) \in C_t(\nu, \phi, p), (c', a') \in C_t(\nu', \phi, p) \quad \Rightarrow \quad a'_s \geq a_s, \forall s \geq t.$$

Proof. It is enough to prove that $a'_{t+1} \geq a_{t+1}$ and the conclusion follows by iteration. If $c'_t < c_t$, then on $\{a'_{t+1} > \phi_{t+1}\}$ it must be that $a'_{t+1} \leq a_{t+1}$, as V_{t+1} is strictly concave by standard arguments and the first order conditions are (we drop the fixed arguments p, ϕ in the indirect utility function)

$$\frac{u'_t(c'_t)}{V'_{t+1}(a'_{t+1})} = \frac{p_t}{p_{t+1}} \leq \frac{u'_t(c_t)}{V'_{t+1}(a_{t+1})}.$$

Moreover, on $\{a'_{t+1} = \phi_{t+1}\}$, $\phi_{t+1} = a'_{t+1} \leq a_{t+1}$, thus $a'_{t+1} \leq a_{t+1}$. This contradicts $a_t \leq a'_t$, as

$$a_t = c_t + E_t \frac{p_{t+1}}{p_t} a_{t+1} - e_t > c'_t + E_t \frac{p_{t+1}}{p_t} a'_{t+1} - e_t = a'_t.$$

We proved that $c'_t \geq c_t$. Clearly $a'_{t+1} \geq a_{t+1}$ on the set $\{a_{t+1} = \phi_{t+1}\}$. On $\{a_{t+1} > \phi_{t=1}\}$, agent's first order conditions are

$$\frac{u'_t(c_t)}{V'_{t+1}(a_{t+1})} = \frac{p_t}{p_{t+1}} \leq \frac{u'_t(c'_t)}{V'_{t+1}(a'_{t+1})},$$

implying that $a'_{t+1} \geq a_{t+1}$, as required. $\square$

Proposition A.2. *Let $t \in \mathbb{N}$ and $\phi, \bar{\phi} \leq 0$ debt limits such that $B_s(\phi_s, \phi, p) \neq \emptyset$, $B_s(\bar{\phi}_s, \bar{\phi}, p) \neq \emptyset$, for all $s \geq t$. For each $n \geq t$ natural, let $\eta_n := \alpha(t) \wedge n$. Then*

1. *$(p_s\phi_s)_{s=t}^{\alpha(t)}$ converges a.s. and in L^1, and it is greater than the uniformly integrable submartingale $(\underline{Z}_s)_{s=t}^{\alpha(t)} := \left(E_s p_{\alpha(t)}\phi_{\alpha(t)} - E_s \sum_{\tau=s}^{\alpha(t)-1} p_\tau e_\tau\right)_{s=t}^{\alpha(t)}$, where $p_{\alpha(t)}\phi_{\alpha(t)} := \lim_{n\to\infty} p_{\eta_n}\phi_{\eta_n}$.*

2. *If Assumption 3.2 holds, then $(M_s)_{s=t}^{\alpha(t)}$ converges a.s. and in L^1, and it is bounded from below by $\underline{Z}$ and from above by the uniformly integrable submartingale $(\bar{Z}_s)_{s=t}^{\alpha(t)} := \left(-p_s\bar{\phi}_s + \sum_{\tau=t}^{s-1} p_\tau e_\tau\right)_{s=t}^{\alpha(t)}$.*

3. *If Assumption 3.2 holds, then $(M_s)_{s=t}^{\alpha(t)}$ admits a (lower) Snell envelope $(\hat{M})_{s=t}^{\alpha(t)}$, which is the largest submartingale dominated from above by M (that is $\hat{M} \leq M$). The Snell envelope $\hat{M}$ satisfies:*

   (i) $E_t(\hat{M}_s) = \inf_{s \leq T < \alpha(t)+1} E_t M_T$, for all $t \leq s < \alpha(t)$.

   (ii) $\hat{M}_s = M_s \wedge E_s\hat{M}_{s+1}$, for all $t \leq s < \alpha(t)$.

   (iii) $\hat{M}_{\alpha(t)} = M_{\alpha(t)}$, where $\hat{M}_{\alpha(t)} := \lim_{n\to\infty} \hat{M}_{\eta_n}$, $M_{\alpha(t)} := \lim_{n\to\infty} M_{\eta_n}$.

Proof. Let $(c, a) \in C_t(\phi_t, \phi, p)$. Aggregation of agent's budget constraints gives

$$E_t \sum_{s=t}^{\eta_n - 1} p_s c_s = E_t \sum_{s=t}^{\eta_n - 1} p_s e_s + p_t\phi_t - E_t p_{\eta_n}\phi_{\eta_n} - E_t p_{\eta_n}\left(a_{\eta_n} - \phi_{\eta_n}\right). \tag{A.1}$$

The inequality $u'(x)x \leq u(x) - u(0) \leq \bar{u} - \underline{u}$ and the first order conditions for the problem $P_t(\phi_t, \phi, p)$ give

$$0 < E_t \sum_{s=t}^{\eta_n - 1} p_s c_s = \frac{p_t}{\beta_t u'(c_t)} \cdot E_t \sum_{s=t}^{\eta_n - 1} \beta_s u'(c_s)c_s \leq \bar{U} < \infty, \tag{A.2}$$

where $\bar{U} := p_t(\beta_t u'(c_t))^{-1}\left(\bar{u} - \underline{u}\right) E_t \sum_{s \geq t} \beta_s$. Since $(c, a) \in C_t(\phi_t, \phi, p)$, by the transversality condition (Lemma 1.1 in the supplement to Bidian and Bejan 2012),

$$\lim_{n\to\infty} E_t p_{\eta_n}\left(a_{\eta_n} - \phi_{\eta_n}\right) = \lim_{n\to\infty} E_t p_n \left(a_n - \phi_n\right) \mathbf{1}_{n<\alpha(t)} \tag{A.3}$$

$$= \lim_{n\to\infty} \frac{p_t}{u'_t(c_t)} E_t u'_n(c_n)(a_n - \phi_n)\mathbf{1}_{n<\alpha(t)} \leq \lim_{n\to\infty} \frac{p_t}{u'_t(c_t)} E_t u'_n(c_n)(a_n - \phi_n) = 0.$$

From (A.1)-(A.3),

$$\lim_{n\to\infty}\left(E_t\sum_{s=t}^{\eta_n-1}p_se_s-E_tp_{\eta_n}\phi_{\eta_n}\right)=-p_t\phi_t+\lim_{n\to\infty}E_t\sum_{s=t}^{\eta_n-1}p_sc_s\le\bar U-p_t\phi_t. \tag{A.4}$$

As $\phi\le 0$, $-E_tp_{\eta_n}\phi_{\eta_n}\ge 0$. Let $\sum_{s=t}^{\alpha(t)-1}p_se_s:=\lim_{n\to\infty}\sum_{s=t}^{\eta_n-1}p_se_s$. Using the monotone convergence theorem in (A.4),

$$E_t\sum_{s=t}^{\alpha(t)-1}p_se_s=\lim_{n\to\infty}E_t\sum_{s=t}^{\eta_n-1}p_se_s\le\lim_{n\to\infty}\left(E_t\sum_{s=t}^{\eta_n-1}p_se_s-E_tp_{\eta_n}\phi_{\eta_n}\right)\le\bar U-p_t\phi_t<\infty.$$

Therefore $\sum_{s=t}^{\alpha(t)-1}p_se_s$ is integrable and $(p_{\eta_n}\phi_{\eta_n})_n$ is L^1-bounded,

$$\sup_{n\ge t}-E_t\left|p_{\eta_n}\phi_{\eta_n}\right|=\sup_{n\ge t}E_t\left|p_{\eta_n}\phi_{\eta_n}\right|<\infty. \tag{A.5}$$

At any period $s\in\mathbb{N}$, since $B_s(\phi_s,\phi,p)\ne\emptyset$, the agent can consume at least 0 if his beginning of period s wealth is ϕ_s and he faces the bounds ϕ. Thus

$$p_s\phi_s+p_se_s\ge E_sp_{s+1}\phi_{s+1}. \tag{A.6}$$

It follows that $\left(p_s\phi_s-\sum_{\tau=t}^{s-1}p_\tau e_\tau\right)_{s=t}^{\alpha(t)}$ is a supermartingale, which converges by (A.5) (Kopp 1984, Corollary 2.6.2). We infer that $(p_{\eta_n}\phi_{\eta_n})_n$ converges a.s., and hence converges also in L^1, since $(-E_tp_{\eta_n}\phi_{\eta_n}=E_t|p_{\eta_n}\phi_{\eta_n}|)_n$ converges by (A.4) (Kallenberg 2002, Lemma 1.32). As the supermartingale $\left(p_s\phi_s-\sum_{\tau=t}^{s-1}p_\tau e_\tau\right)_{s=t}^{\alpha(t)}$ converges a.s. and in L^1, $p_s\phi_s-\sum_{\tau=t}^{s-1}p_\tau e_\tau\ge E_s\lim_{n\to\infty}p_{\eta_n}\phi_{\eta_n}-E_s\lim_{n\to\infty}\sum_{\tau=t}^{\eta_n-1}p_\tau e_\tau$, thus[10]

$$p_s\phi_s\ge E_sp_{\alpha(t)}\phi_{\alpha(t)}-E_s\sum_{\tau=s}^{\alpha(t)-1}p_\tau e_\tau. \tag{A.7}$$

We conclude that $(p_s\phi_s)_{s=t}^{\alpha(t)}$ is bounded from below by the uniformly integrable submartingale $\left(E_sp_{\alpha(t)}\phi_{\alpha(t)}-E_s\sum_{\tau=s}^{\alpha(t)-1}p_\tau e_\tau\right)_{s=t}^{\alpha(t)}$.

2. Similarly, (A.6) with ϕ replaced by $\bar\phi$ shows that $\left(p_s\bar\phi_s-\sum_{\tau=t}^{s-1}p_\tau e_\tau\right)_{s=t}^{\alpha(t)}$ is a supermartingale. Thus $(\bar Z_s)_{s=t}^{\alpha(t)}:=\left(-p_s\bar\phi_s+\sum_{\tau=t}^{s-1}p_\tau e_\tau\right)_{s=t}^{\alpha(t)}$ is a positive submartin-

[10] Alternatively, we can use (A.1) written at an arbitrary period $s\in[t,\alpha(t))$ rather than t and take the limit $n\to\infty$.

gale dominating M, as $\bar{Z} \geq -p \cdot \bar{\phi} \geq p(\phi - \bar{\phi}) = M$. Using Assumption 3.2, we infer that $\bar{Z}$ is a uniformly integrable submartingale, and therefore $(p_{\eta_n} \bar{\phi}_{\eta_n})_n$ converges a.s. and in L^1. Hence $(M_s)_{s=t}^{\alpha(t)}$ is uniformly integrable, converges a.s. and in L^1, and it is bounded from below, respectively from above, by the uniformly integrable submartingale $\underline{Z}$, respectively $\bar{Z}$.

3. The (lower) Snell envelope $(\hat{M}_s)_{s=t}^{\alpha(t)}$ is constructed as $\hat{M}_s := \inf_{s \leq T < \alpha(t)+1} E_s M_T$, for $t \leq s < \alpha(t) + 1$ (Kopp 1984, Theorem 2.11.3). The inf in the definition of $\hat{M}_s$ refers to the *essential infimum* over all finite stopping times T greater than s and smaller or equal to $\alpha(t)$, which can be an uncountable family (Kopp 1984, Proposition 2.11.1). Kopp (1984, Theorem 2.11.3) assumes the integrability of $\sup_n |M_{\eta_n}|$, but the boundedness conditions on M established in part 2 are enough for the existence of $\inf_{s \leq T < \alpha(t)+1} E_s M_T$. Indeed, as the family (M_T) with T running over the finite stopping times $s \leq T \leq \alpha(s)$ is downward filtering,[11] there exists a sequence of stopping times (T_n) with $s \leq T_n < \alpha(t) + 1$ such that $\inf_{s \leq T < \alpha(t)+1} M_T = \lim_{n \to \infty} M_{T_n}$. Since $\underline{Z}_{T_n} \leq M_{T_n} \leq \bar{Z}_{T_n}$ and $(\underline{Z}_{T_n})_n, (\bar{Z}_{T_n})_n$ are uniformly integrable by the optional sampling theorem (Kopp 1984, Theorem 2.10.4), it follows that $(M_{T_n})_n$ is uniformly integrable. Therefore $\hat{M}_s = \lim_{n \to \infty} E_s M_{T_n}$ and is well defined.

Parts (i) and (ii) are shown in Kopp (1984, Theorem 2.11.3). By (i), there exists an increasing sequence of stopping times $(T_n)_{n \geq 1}$ such that $T_1 = t$, $T_n \nearrow \alpha(t)$, and for $n \geq 1$, $T_{n+1} \geq (T_n + 1) \wedge \alpha(t)$ and $E_t \hat{M}_{(T_n+1) \wedge \alpha(t)} \geq E_t M_{T_{n+1}} - \frac{1}{n}$. Taking the limit with $n \to \infty$, $E_t \hat{M}_{\alpha(t)} \geq E_t M_{\alpha(t)}$. As $\hat{M} \leq M$, we conclude that $\hat{M}_{\alpha(t)} = M_{\alpha(t)}$. $\square$

B Comparison with Hellwig and Lorenzoni (2009)

Our Theorem 3.3 is stronger than the main result in Hellwig and Lorenzoni (2009), with a simpler proof which also fixes some oversights in their proof. Hellwig and Lorenzoni's (2009) result is a particular case of our Theorem 3.3. They focus on the case when the penalty for default is the interdiction to borrow 2.5 and therefore the sequence of debt limits identical equal to zero ($\bar{\phi} = 0$) is NTT.

In this section, we show first that Hellwig and Lorenzoni's (2009) result can be extended to general penalties for default, and used to obtain a *weaker* form of our

[11] This means that for any two such stopping times T_1, T_2, there is another stopping time T such that $M_T \leq M_{T_1} \wedge M_{T_2}$. The property is immediate and established in Kopp (1984, Theorem 2.11.3).

Theorem 3.3,[12] restricted to pairs of debt limits in which one is uniformly tighter (greater everywhere) than the other, $\phi \leq \bar{\phi}$. However their result does not imply ours, even after this generalization. Then we compare their proof with ours. The main departure and simplification is that we bypass the complicated construction in their Lemma 2 by using Snell envelopes. The other steps in the proof are similar, once technical shortcomings (related mainly to transversality conditions and switching the order expectations and limits) are taken care of.

Let $\phi, \bar{\phi}$ be NTT for an agent given some penalties for default V^d. Assume that $\phi, \bar{\phi}$ satisfy Assumption 3.1 and that $\phi \leq \bar{\phi}$. Consider a different (fictitious) agent with perturbed endowments $e'_t := e_t + \bar{\phi}_t - \frac{1}{p_t} E_t p_{t+1} \bar{\phi}_{t+1}$. Notice that $e' \geq 0$ as $B_t(\bar{\phi}_t, \bar{\phi}, p) \neq \emptyset$, for all t. Denote by $B'_t(\nu_t, \phi', p)$ and $V'_t(\nu_t, \phi', p)$ the budget and indirect utilities at t of the agent with perturbed endowments and faced with some arbitrary debt limits ϕ' and initial wealth ν_t. It is immediate to check that

$$(c, a) \in B_t(\phi'_t, \phi', p) \Leftrightarrow (c, a - \bar{\phi}) \in B'_t(\phi'_t - \bar{\phi}_t, \phi' - \bar{\phi}, p). \tag{B.1}$$

Taking $\phi' = \bar{\phi}$ in (B.1), we infer that $V_t(\bar{\phi}_t, \bar{\phi}, p) = V^d_t = V'_t(0, 0, p)$. Therefore the continuation utilities after default of the initial agent with endowments e and subject to general punishments V^d_t coincide with the continuation utilities after default of the agent with endowments e' and subject to the interdiction to borrow after default (2.5). Choosing $\phi' = \phi$ in (B.1), we get $V_t(\phi_t, \phi, p) = V^d_t = V'_t(\phi_t - \bar{\phi}_t, \phi - \bar{\phi}, p)$, and therefore $\phi - \bar{\phi}$ are NTT for the agent with endowments e' and subject to penalties (2.5). Now the crucial ordering assumption $\phi \leq \bar{\phi}$ imposed here on the pair of debt limits $\phi, \bar{\phi}$ implies that $\phi - \bar{\phi} \leq 0$ and therefore we can apply the result in Hellwig and Lorenzoni (2009) to the agent with endowments e', debt limits $\phi - \bar{\phi}$ and subject to penalties (2.5) to conclude that $p(\phi - \bar{\phi})$ is a martingale.

Note that this reasoning cannot be applied to arbitrary NTT debt limits $\phi, \bar{\phi}$, since Hellwig and Lorenzoni's (2009) proof crucially requires that an agent's debt limits must be nonpositive, hence $\phi - \bar{\phi} \leq 0$ is needed. While we impose $\phi, \bar{\phi} \leq 0$ in our proof, we do not require that $\phi \leq \bar{\phi}$ (that is, we do not require that one of the debt limits is always tighter than the other). Therefore in some dates and states ϕ can be larger than $\bar{\phi}$, while in other dates and states, $\bar{\phi}$ is larger than ϕ. Such non-ordered debt limits are exactly the ones occuring in our discussion of

[12] We thank an anonymous referee for pointing this out to us.

the multiplicity of equilibria resulting from perturbations of agents' debt limits by zero mean discounted martingales (see the discussion at the beginning of Section 4). Therefore the partial characterization of NTT debt limits afforded by the extension of Hellwig and Lorenzoni's (2009) outlined above is of limited use and cannot be applied, for example, to establish the uniqueness result of Section 4.

In the remaining of this section, we compare and contrast Hellwig and Lorenzoni's (2009) proof to ours. We first make the parallel between their "event tree" notation and ours (which uses the standard language of stochastic processes), and then all the discussion will be transcribed in our notation (for simplicity). For $\omega \in \Omega$ and $t \in \mathbb{N}$, the date t "node" containing state ("leaf") ω represents the set of states that are known to be possible at t if the true state is ω, $\mathcal{F}_t(\omega) := \cap\{A \in \mathcal{F}_t \mid \omega \in A\}$. For arbitrary $\tau \geq 0$ and $\bar{\omega} \in \mathcal{F}_t(\omega)$, $\mathcal{F}_{t+\tau}(\bar{\omega})$ is a date $t+\tau$ "successor" node of $\mathcal{F}_t(\omega)$. Hellwig and Lorenzoni (2009) use s^t for a period t node $\mathcal{F}_t(\omega)$, and $s^{t+\tau}$ for a successor $\mathcal{F}_{t+\tau}(\bar{\omega})$ (with $\bar{\omega} \in \mathcal{F}_t(\omega)$). For fixed $t, \tau \geq 0$, node s^t, and successor $s^{t+\tau}$, they let (see page 1158)

$$\begin{aligned}
N_\tau(s^t) &= \{\mathcal{F}_{t+\tau}(\omega') \mid t+\tau < \alpha(t)(\omega'), \omega' \in \mathcal{F}_t(\omega)\}, \\
B_\tau(s^t) &= \{\mathcal{F}_{t+\tau}(\omega') \mid t+\tau = \alpha(t)(\omega'), \omega' \in \mathcal{F}_t(\omega)\}, \\
N(s^t) &= \{\mathcal{F}_s(\omega') \mid t+\tau < \alpha(t)(\omega'), \omega' \in \mathcal{F}_t(\omega)\}, \\
B(s^t) &= \{\mathcal{F}_s(\omega') \mid s = \alpha(t)(\omega'), \omega' \in \mathcal{F}_t(\omega)\}, \\
N(s^{t+\tau}; s^t) &= \{\mathcal{F}_s(\omega') \mid t+\tau \leq s < \alpha(t)(\omega'), \omega' \in \mathcal{F}_{t+\tau}(\bar{\omega})\}, \\
B(s^{t+\tau}; s^t) &= \{\mathcal{F}_s(\omega') \mid t+\tau \leq s = \alpha(t)(\omega'), \omega' \in \mathcal{F}_{t+\tau}(\bar{\omega})\}.
\end{aligned} \tag{B.2}$$

Thus our use of the stopping time $\alpha(t)$ (as the first time the constraints ϕ bind after t along an optimal path when agent starts with wealth ϕ_t at t) makes all the above (rather complicated) notation introduced by Hellwig and Lorenzoni (2009) redundant. They also define $w(s^{t+\tau}; s^t) := \frac{1}{p_{t+\tau}} E_{t+\tau} \sum_{s=t+\tau}^{\alpha(t)-1} p_s e_s$ on $\mathcal{F}_t(\omega)$, and on $\mathcal{F}_{t+\tau}(\bar{\omega})$, they set

$$\hat{\phi}(s^{t+\tau}; s^t) := \sum_{\mathcal{F}_s(\omega') \in B(s^{t+\tau}; s^t)} \frac{P(\mathcal{F}_s(\omega')) \cdot p_s(\omega')}{P(\mathcal{F}_{t+\tau}(\bar{\omega})) \cdot p_{t+\tau}(\bar{\omega})} \phi_s(\omega').$$

Notice that $\hat{\phi}$ is not well defined when $\alpha(t)$ is infinite. Based on our read of their proof, we believe they intended $\hat{\phi}(s^{t+\tau}; s^t) := \lim_{n\to\infty} \frac{1}{p_{t+\tau}} E_{t+\tau} p_{\eta_n} \phi_{\eta_n}$, for $t+\tau < \alpha(t)$, where $\eta_n := n \wedge \alpha(t)$. This limit can be shown to be finite by the arguments in the first part of our Proposition A.2, or using their Lemma 1, once some minor typos there are fixed. Specifically, last term in the first formula in Lemma 1 should be $\lim_{n\to\infty} \frac{1}{p_{t+\tau}} E_{t+\tau} p_{\eta_n} (a^*_{\eta_n} - \phi_{\eta_n})$ instead of $\lim_{n\to\infty} \frac{1}{p_{t+\tau}} E_{t+\tau} p_n a^*_n \mathbf{1}_{n<\alpha(t)}$, and the transversality condition should be $\lim_{n\to\infty} \frac{1}{p_{t+\tau}} E_{t+\tau} p_{\eta_n} (a^*_{\eta_n} - \phi_{\eta_n}) = 0$ rather than $\lim_{n\to\infty} \frac{1}{p_{t+\tau}} E_{t+\tau} p_n a^*_n \mathbf{1}_{n<\alpha(t)} = 0$ (where a^* are the optimal asset holdings in the problem $P_t(\phi_t, \phi, p)$). In Lemma 1 they also establish that $w(s^{t+\tau}; s^t) < \infty$ and $\phi_{t+\tau} + w(s^{t+\tau}; s^t) > \hat{\phi}(s^{t+\tau}; s^t) > -\infty$ (on $\mathcal{F}_{t+\tau}(\bar{\omega})$). These conclusions follow also from our Proposition A.2, which additionally proves that the process $(p_s\phi_s)_{s=t}^{\alpha(t)}$ converge a.s. and in L^1.

The crucial step in their proof is Lemma 2, where they show the existence of some "auxiliary" debt limits $(\tilde{\phi}_s)_{s\geq t}$ such that

$$\tilde{\phi}_s = \begin{cases} \phi_s, & \text{if } s = \alpha^k(t) \text{ for } k \geq 0, \\ \frac{1}{p_s} E_s \min\{p_{s+1}\phi_{s+1}, p_{s+1}\tilde{\phi}_{s+1}\} & \text{otherwise.} \end{cases}$$

The bounds $\tilde{\phi}$ are the limit of a nondecreasing sequence of debt bounds $\phi^{(n)}$ (thus $\phi^{(n)} \nearrow \tilde{\phi}$), obtained iteratively. Condition $\phi \leq 0$ is essential to guarantee boundedness from above of $(\phi^{(n)})_n$ and the existence of the limit $\tilde{\phi}$. Lemma 3 shows, furthermore, that $\tilde{\phi} \leq \phi$.

Our proof also uses some auxiliary debt limits $\hat{M}/p$, where $\hat{M}$ is the "piecewise" Snell envelope of $p \cdot (\phi - \bar{\phi})$ ($= p \cdot \phi$ for $\bar{\phi} = 0$) on intervals $(\alpha^k(t), \alpha^{k+1}(t)]$. Therefore we bypass entirely their Lemma 2 and the need for $\phi - \bar{\phi}$ to be nonpositive. This is the major simplification in our proof and what enables us to obtain the general Theorem 3.3.

Finally Lemma 4, respectively Lemma 5 of Hellwig and Lorenzoni (2009) are similar to our Step 2, respectively Step 3 in the proof of Theorem 3.3. They don't check that the transversality condition holds in the relaxed problem in their Lemma 4 (we dealt with this in our Step 2), which is rather delicate. We also had to use repeatedly in these two steps the a.s and L^1 convergence of the process $(p_s\phi_s)_{s=t}^{\alpha(t)}$ (shown in our Proposition A.2) in order to exchange the order of limits and expectations. This property is used implicitly by Hellwig and Lorenzoni (2009), disguised by the notation (B.2).

C Transversality conditions

We analyze the problem $P_t(\hat{a}_t, \phi, p)$ of a consumer that faces debt bounds ϕ, pricing kernel p and starts with wealth $\hat{a}_t$ ($\mathcal{F}_t$-measurable) at period t (see Section 3 in the main text). Let $(\bar{c}, \bar{a}) \in C_t(\hat{a}_t, \phi, p)$ be the optimal consumption (assumed strictly positive) and asset holdings for the agent. Familiar variational arguments show that $(\bar{c}, \bar{a})$ satisfies the following Kuhn-Tucker necessary conditions, for all $s \geq t$:

$$u'_s(\bar{c}_s) - u'_{s+1}(\bar{c}_{s+1})\frac{p_s}{p_{s+1}} \geq 0, \tag{C.1}$$

$$\left(u'_s(\bar{c}_s) - u'_{s+1}(\bar{c}_{s+1})\frac{p_s}{p_{s+1}}\right)(\bar{a}_{s+1} - \phi_{s+1}) = 0. \tag{C.2}$$

Let $\bar{e}_s := e_s + \phi_s - E_s \frac{p_{s+1}}{p_s}\phi_{s+1}$, for all $s \geq t$. Adapting the arguments of Forno and Montrucchio (2003), we obtain the following necessary transversality condition:[13]

Lemma C.1 (Necessary transversality condition)**.** *The optimal path* $(\bar{c}, \bar{a})$ *satisfies*

$$\lim_{s\to\infty} E_t u'_s(\bar{c}_s)(\bar{a}_s - \phi_s) = 0. \tag{C.3}$$

Proof. Fix an $\bar{\varepsilon} > 0$ a period $s > t$. Concavity implies that for any $0 < \varepsilon < \bar{\varepsilon}$ and $n \geq t$,

$$u_n(\bar{c}_n) - u_n(\bar{c}_n + \varepsilon(\bar{e}_n - \bar{c}_n)) \leq \frac{\varepsilon}{\bar{\varepsilon}}\left(u_n(\bar{c}_n) - u_n(\bar{c}_n + \bar{\varepsilon}(\bar{e}_n - \bar{c}_n))\right).$$

We construct the alternative asset holdings process $(a_n(\varepsilon))_{n=t}^{\infty}$ where $a_n(\varepsilon) = \bar{a}_n$ if $t \leq n \leq s$, and $a_n(\varepsilon) = (1-\varepsilon)\bar{a}_n + \varepsilon\phi_n$ if $n \geq s+1$. It sustains the feasible consumption process $(c_n(\varepsilon))_{n=s}^{\infty}$ defined by $c_n(\varepsilon) = \bar{c}_n$ if $t \leq n < s$, $c_s(\varepsilon) = \bar{c}_s + E_s\frac{p_{s+1}}{p_s}(\bar{a}_{s+1} - \phi_{s+1})$, and $c_n(\varepsilon) = \bar{c}_n + \varepsilon(\bar{e}_n - \bar{c}_n)$ for $n > s$. Optimality of $\bar{c}$ implies that

$$0 \leq E_t\left(u_s(\bar{c}_s) - u_s(c_s(\varepsilon))\right) + \limsup_{T\to\infty} E_t \sum_{n=s+1}^{T}\left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon))\right). \tag{C.4}$$

[13] The proof works for general period utilities $u_t(\cdot)$, not necessarily of the discounted and bounded variety assumed in the text, if one uses a weak optimality criterion (Forno and Montrucchio 2003) and if there exists $\bar{\varepsilon} > 0$ such that $E\sum_{s=t}^{\infty}\left(u_s(\bar{c}_s) - u_s(\bar{c}_s + \bar{\varepsilon}(\bar{e}_s - \bar{c}_s))\right)^+ < \infty$.

Notice that

$$\sum_{n=s+1}^{T} \frac{1}{\varepsilon} \left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon)) \right) \leq \sum_{n=s+1}^{\infty} \frac{1}{\bar{\varepsilon}} \left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon})) \right)^+ ,$$

and the term $\sum_{n=s+1}^{\infty} \frac{1}{\bar{\varepsilon}} \left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon})) \right)^+$ is integrable, by hypothesis. Fatou's lemma gives

$$\begin{aligned} \limsup_{T\to\infty} E_t \sum_{n=s+1}^{T} \frac{1}{\varepsilon} \left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon)) \right) &\leq E_t \limsup_{T\to\infty} \sum_{n=s+1}^{T} \frac{1}{\varepsilon} \left(u_n(\bar{c}_n) - u_n(c_n(\varepsilon)) \right) \\ &\leq E_t \sum_{n=s+1}^{\infty} \frac{1}{\bar{\varepsilon}} \left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon})) \right)^+ . \qquad \text{(C.5)} \end{aligned}$$

Dividing both sides of (C.4) by ε and using (C.5),

$$-E_t \frac{1}{\varepsilon} \left(u_s(\bar{c}_s) - u_s(c_s(\varepsilon)) \right) \leq E_t \sum_{n=s+1}^{\infty} \frac{1}{\bar{\varepsilon}} \left(u_n(\bar{c}_n) - u_n(c_n(\bar{\varepsilon})) \right)^+ < \infty.$$

By the monotone convergence theorem, when $\varepsilon \searrow 0$, the left hand side of the above equation converges to $E_t u_s'(\bar{c}_s) \frac{p_{s+1}}{p_s} (\bar{a}_{s+1} - \phi_{s+1})$, which equals $E_t u_{s+1}'(\bar{c}_{s+1})(\bar{a}_{s+1} - \phi_{s+1})$, due to the Kuhn-Tucker equations (C.1),(C.2). The conclusion follows by letting $s \to \infty$. $\square$

We include for completeness the standard proof of sufficiency of the Kuhn-Tucker and transversality conditions for the optimality of a path.

Lemma C.2 (Sufficient transversality condition)**.** *If a feasible path $(\bar{c}, \bar{a}) \in B_t(\hat{a}_t, \phi, p)$ satisfies the Kuhn-Tucker conditions* (C.1) *and* (C.2)*, then for any other feasible path $(c, a) \in B_t(\hat{a}_t, \phi, p)$ and any bounded stopping time $T \geq t$,*

$$E_t \sum_{s=t}^{T} (u_s(c_s) - u_s(\bar{c}_s)) \leq E_t u_{T+1}'(\bar{c}_{T+1})(\bar{a}_{T+1} - \phi_{T+1}). \qquad \text{(C.6)}$$

Thus a sufficient condition for $(\bar{c}, \bar{a})$ to be optimal for problem $P_t(\hat{a}_t, \phi, p)$ is

$$\liminf_{s\to\infty} E_t u_s'(\bar{c}_s)(\bar{a}_s - \phi_s) = 0, \qquad \text{(C.7)}$$

Proof. Let $\mu_{s+1} := u'_s(\bar{c}_s) - u'_{s+1}(\bar{c}_{s+1})\frac{p_s}{p_{s+1}}$. Consider an arbitrary feasible path $(c, a) \in B_t(\hat{a}_t, \phi, p)$. Using concavity of $u_s(\cdot)$ and the budget constraints,

$$
\begin{aligned}
E_t \sum_{s=t}^{T} (u_s(c_s) - u_s(\bar{c}_s)) \leq E_t \sum_{s=t}^{T} u'_s(\bar{c}_s)(c_s - \bar{c}_s) = \\
= E_t \sum_{s=t}^{T} u'_s(\bar{c}_s) \left(a_s - \phi_s - E_s \frac{p_{s+1}}{p_s}(a_{s+1} - \phi_{s+1}) \right) - \\
-E_t \sum_{s=t}^{T} u'_s(\bar{c}_s) \left(\bar{a}_s - \phi_s - E_s \frac{p_{s+1}}{p_s}(\bar{a}_{s+1} - \phi_{s+1}) \right).
\end{aligned}
$$

We analyze separately the last two terms. Using the Kuhn-Tucker conditions (C.1) and (C.2), which show that $\mu_{s+1} \geq 0$ for all $s \geq t$, it follows that

$$
\begin{aligned}
& E_t \sum_{s=t}^{T} u'_s(\bar{c}_s) \left((a_s - \phi_s) - E_s \frac{p_{s+1}}{p_s}(a_{s+1} - \phi_{s+1}) \right) \\
& = E_t \sum_{s=t}^{T} \left(u'_s(\bar{c}_s)(a_s - \phi_s) - \left(u'_{s+1}(\bar{c}_{s+1}) + \frac{p_{s+1}}{p_s}\mu_{s+1} \right)(a_{s+1} - \phi_{s+1}) \right) \\
& \leq u'_t(\bar{c}_t)(a_t - \phi_t) - E_t u'_{T+1}(\bar{c}_{T+1})(a_{T+1} - \phi_{T+1}) \leq E_t u'_t(\bar{c}_t)(a_t - \phi_t).
\end{aligned}
$$

Similarly, using both (C.1) and (C.2),

$$
\begin{aligned}
& E_t \sum_{s=t}^{T} u'_s(\bar{c}_s) \left((a_s - \phi_s) - E_s \frac{p_{s+1}}{p_s}(a_{s+1} - \phi_{s+1}) \right) \\
& = u'_t(\bar{c}_t)(\bar{a}_t - \phi_t) - E_t u'_{T+1}(\bar{c}_{T+1})(\bar{a}_{T+1} - \phi_{T+1}).
\end{aligned}
$$

Moreover $a_t = \bar{a}_t$ since they equal the initial period t wealth of the consumer, $\hat{a}_t$. Thus

$$
\liminf_{T\to\infty} E_t \sum_{s=t}^{T} (u_s(c_s) - u_s(\bar{c}_s)) \leq \liminf_{T\to\infty} E_t u'_{T+1}(\bar{c}_{T+1})(\bar{a}_{T+1} - \phi_{T+1}) = 0,
$$

and therefore $(\bar{c}, \bar{a})$ is optimal for $P_t(\hat{a}_t, \phi_t, p)$. $\square$

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