Beyond the Grand Tour: The Aspiration Core Approach to the Infinite-Period Traveling Salesman Problem

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Abstract

We introduce the Infinite Period Traveling Salesman Problem (∞-TSP), a generalization of the classical TSP, in which a single service provider repeatedly visits the same set of customers. Our approach leverages the aspiration core, a cooperative game-theoretic concept that extends the classical core to cases where it may be empty while allowing the formation of proper coalitions. The aspiration core provides robust cost allocation rules and identifies the subsets of customers who should be visited together in specific tours. When the core of the one-shot Traveling Salesman Game is empty, the suggested tour schedule leads to a strictly lower average cost per visit than repeating a single grand tour. We further refine our framework using the aspiration nucleolus, a unique and "fair" cost allocation selected from the aspiration core. Our solutions preserve stability and are applicable to general graphs that may be incomplete and may violate the triangle inequality.

Keywords: Traveling Salesman Problem; empty core; aspiration core; aspiration nucleolus

1 Introduction

The typical motivation for the classical traveling salesman problem (TSP) is the following. A service provider is hired by a group of customers to render a specific service at their different locations. The provider starts from the

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company's hub or home city, visits each customer once, and returns to the hub. The problem has two main objectives: finding the route that minimizes total cost, and ensuring a "fair" distribution of this cost among the serviced customers. The cost allocation goal is, typically, *coalitional rationality*, meaning that no subgroup of customers has an incentive to negotiate separately with another service provider. Such cost allocations are said to be in the *core* of the corresponding cooperative game.

We address the challenges faced by agents/service providers who travel to the same set of locations frequently. Each location has to be visited repeatedly, over an arbitrarily large horizon, and the visits do not need to happen within a strict or fixed time frame. Rather, the service provider must design a schedule of tours that optimally balances frequency and cost. We will refer to the problem as the *Infinite Period Traveling Salesman Problem* and will abbreviate it as ∞-TSP.

Examples of this sort are abundant in a variety of industries, such as healthcare, utilities, agriculture, or consumer and subscription services. For example, electric utilities must periodically check infrastructure but can often flexibly route crews; delivery trucks, buses, or rental cars need regular servicing, but providers can batch visits flexibly; agronomists need to check crop health throughout the season, but timing can be somewhat flexible; home healthcare providers want to plan efficient routes across their patients with chronic conditions, but the exact timing can shift within a window; many commercial or residential appliance systems need inspections and cleanings every so often, but not at rigidly fixed dates. In all these cases, providers must repeatedly visit a set of customers/assets, but with some flexibility in the exact timing between visits. The provider wants to optimize travel routes to these customers across an ongoing, rolling horizon.

We argue first that this problem is not equivalent to the classical TSP. This is illustrated in the following example.

Example 1.1 *A service provider must visit nine different cities indefinitely many times. The cities are connected by roads according to the Petersen graph (see Tamir (1989) and Petersen (1898)) as shown in Figure 1. The trips must start and end at the home node, labeled as* 0. *The cost of using any road is \$1.*

We start by identifying the solution of the classical TSP for this graph. That corresponds to the case where each city requires only one visit. However, since the graph is not complete, the provider will need to pass through some cities more than once during the same tour. In this case, the minimum

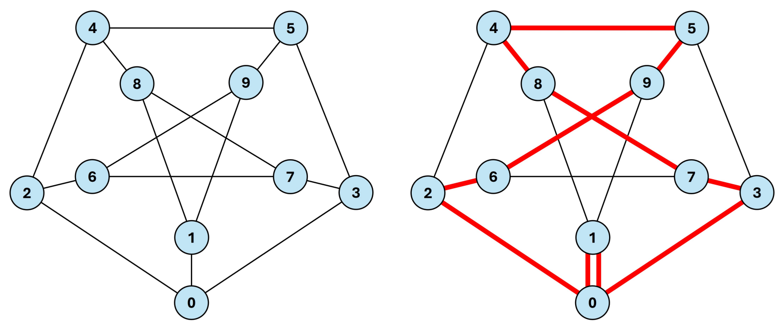
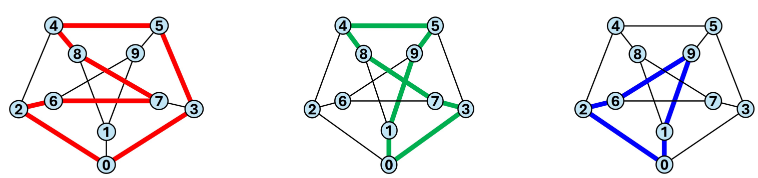


Figure 1: The Petersen graph

cost to visit all cities at least once is \$11. A sample route that achieves this cost is shown above.

One may think that the ∞-TSP is equivalent to the classical TSP because a possible approach to the infinite period scenario would be to repeatedly use an optimal route for the classical (one-shot) TSP. However, this solution may be suboptimal. Indeed, for this example, it is possible to build a schedule of tours that visits each city twice for a total cost of less than $2 \times 11 = \$22$. The schedule consists of the three (Hamiltonian) tours depicted in Figure 2:

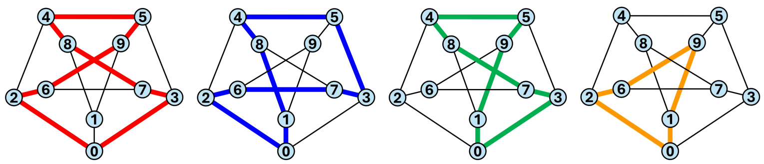


	node 1	node 2	node 3	node 4	node 5	node 6	node 7	node 8	node 9	Cost
Tour 1		1	1	1	1	1	1	1		\$ 8
Tour 2	1		1	1	1		1	1	1	\$ 8
Tour 3	1	1				1			1	\$ 5
Total	2	2	2	2	2	2	2	2	2	\$ 21

Figure 2: Optimal schedule with 2 visits

Iterating this schedule of tours reduces the cost per visit to $(8+8+5)/2 = \$10.50 < \$11$.

With three visits to every city, the cost savings increase even more. Indeed, the schedule of four tours depicted in Figure 3 brings the cost per visit down to $(9+9+8+5)/3 = 10\frac{1}{3}$.



	node 1	node 2	node 3	node 4	node 5	node 6	node 7	node 8	node 9		Cost
Tour 1		1	1	1	1	1	1	1	1		$ 9
Tour 2	1	1	1	1	1	1	1	1			$ 9
Tour 3	1		1	1	1		1	1	1		$ 8
Tour 4	1	1				1			1		$ 5
Total	3	3	3	3	3	3	3	3	3		$ 31

Figure 3: Optimal schedule with 3 visits

The previous example prompts questions about the asymptotic behavior of the average cost as the number of visits increases indefinitely. For the visiting frequencies discussed earlier, the optimal average cost per visit declined as the number of visits grew. Although such monotonicity in the number of visits is not a general result, we will show that the average cost eventually converges as the number of visits increases. To capture the idea of an infinite time horizon, we let the number of visits tend to infinity and study the asymptotic behavior of the minimum average cost per visit. We then ask whether this limiting cost can be achieved via an iterative schedule of tours, and whether it can be distributed to customers in a "fair" way.

We show that the ∞-TSP problem can be solved by using a novel link between the problem described above and a solution concept in cooperative game theory, called the *aspiration core.* This concept addresses not only cost-allocation vectors but also coalition formation which, in the context of the problem analyzed here, translates into subtours of subsets of the existing customer base. When the core is non-empty, the aspiration core recommends forming the grand coalition (i.e., visiting all customers in one tour) and aligns its proposed cost allocations with those of the core. Conversely, if the core is empty, it suggests a family of proper coalitions (i.e., subgroups of customers that should be visited in one tour) along with likely cost-allocation vectors that are stable to coalitional deviations. In this way, the aspiration core generalizes the core to cover the entire space of cooperative games and informs the construction of a solution for the ∞-TSP.

Our Theorem 5.2 together with Proposition 3.2 establishes that the cost per visit can be lowered if and only if the corresponding (one-shot) Traveling

Salesman Game (TSG) has an empty core. Given that the cost per visit might sometimes increase with the number of visits, we also ask about the asymptotic behavior of the cost savings when the number of visits grows large. Proposition 5.3 finds the limit towards which such savings converge.

Our results highlight the close relation between the aspiration core solution concept and the ∞-TSP. In particular, we show that:

1. The asymptotic value of the minimum cost per visit is determined by the aggregate cost allocation of the aspiration core vectors.

2. The optimal schedule of tours that achieves the asymptotic minimum average cost per visit can be built using the coalitions suggested by the aspiration core.

3. The cost of the optimal schedule can be allocated to customers in a way that it cannot be improved upon by any coalition.

Since the aspiration core is typically not single-valued, multiple optimal visiting strategies may exist. The challenge, then, is how to choose among them. We propose a specific single-valued selection from the aspiration core, known in the literature as the aspiration nucleolus Bennett (1983). The aspiration nucleolus –which is related but different from the original nucleolus defined by Schmeidler (1969)– has a number of desirable properties, including non-emptiness, uniqueness, and anonymity. Each of these properties is of practical relevance for the problem studied here.

The aspiration nucleolus can be described as identifying the coalitionally rational payoff distribution considered the most "stable" among all players because it iteratively maximizes the "savings" of all coalitions, starting with the least advantaged one. The "savings" in this context refers to the difference between the cost incurred by serving a coalition and its allocated cost charge. Because it is a solution concept that does not depend on the names of the players (and thus, it is *anonymous*), it preserves all the symmetries of the game.

The paper is organized as follows. Section 3 defines the problem and shows examples in which the core of the associated (one-shot) TSG is empty. Section 4 introduces the aspiration core and a number of its properties. Section 5 contains our main results. It applies the aspiration core to the associated TSGs and analyzes the alternative solutions proposed by this cooperative solution concept. Section 6 describes the aspiration nucleolus as a single-valued selection of the aspiration core, and Section 7 concludes.

2 Literature

The ∞-TSP we present here shares some similarities with, but it is different from the TSP with multiple visits, mTSP (see Bektas (2006) and Cheikhrouhou & Khoufi (2021) for surveys of that literature). Like in the mTSP, every node has to be visited multiple times but, unlike in the mTSP, there is no fixed bound on the number of visits and, more importantly, there is only one traveling service provider who has to visit all nodes with equal frequency, indefinitely many times.

The papers that are closest to ours are Sun & Karwan (2015) and Sun et al. (2018). Sun & Karwan (2015) studied the problem of repeated visits to each node when the required frequency is the same for all nodes, and attempted to identify a set of tours and number of visits to each node that minimize the *average* cost of visiting all nodes. Sun et al. (2018) extend that analysis to a situation where different nodes require different visit frequency. These papers also make the observation that the average cost per visit may be lowered below the cost of visiting each node once. However, they do not investigate the asymptotic behavior of the average cost and do not make the connection between the optimal schedules and the aspiration core. Moreover, both papers restrict attention to complete weighted graphs that satisfy the triangle inequality. We focus on the case of equal frequencies, but our results apply to *arbitrary* connected graphs.

The problem of finding the minimum traveling cost of serving a finite set of customers (i.e., the solution of the classical TSP) is intrinsically related to that of identifying "fair" divisions of that cost among the serviced customers. The game theoretical concept of a *core* has been used for that purpose. A traveling salesman game (TSG) assigns to each subset of customers $S \subseteq N$ the minimal cost $c(S)$ required to serve them. A *core* cost allocation $x \in \mathbb{R}^N$ satisfies:

1. $\sum_{i \in N} x_i = c(N)$.
2. $\sum_{i \in S} x_i \leq c(S)$.

It is known that many TSG-s have empty cores. Tamir (1989) gives examples of TSP-s on undirected graphs with $|N| = 6, 7, 8$ for which the associated games have empty cores. On the other hand, if the cost matrix is asymmetric, even games defined by graphs with four nodes can have an empty core, as shown in Potters et al. (1992).

The literature on cooperative games has provided two fundamentally different approaches to deal with empty-core games. The reason for the core emptiness is that proper coalitions are too powerful with respect to the grand coalition, therefore being able to improve upon any proposed allocation. A first option, suitable when the formation of the grand coalition is a required modeling feature, is to restrict the blocking ability of proper coalitions. This gives rise to solution concepts such as the ϵ-core (Shapley & Shubik (1966)[1] and the least core (Maschler et al. (1979)).

Alternatively, one can focus the analysis on the formation of proper coalitions. Such a mindset led researchers to the Aspiration Core solution concept (Cross (1967), Bennett (1983), Bejan & Gómez (2012)). Together with a stable cost allocation, this concept proposes the formation of a number of (possibly overlapping) proper coalitions. If the core is not empty, the grand coalition can form as, in that case, it is at least as powerful as other combinations of proper coalitions. The aspiration core coincides then with the core. However, when the core is empty, there must exist a family of proper coalitions which, together, are more powerful than the grand coalition. The aspiration core solution concept indicates, in that case, that those coalitions would form.

We show that the Aspiration Core and the optimal solution to the ∞-TSP are equivalent. Both propose the same smaller tours and lower the cost per visit in the same amount. Moreover, the aspiration core cost allocations prevent proper coalitions from leaving.

3 The formal ∞-TSP problem

The TSP can be represented mathematically via a weighted, connected, not necessarily complete, directed graph $G = (V, A, l)$ where V, A, and l are defined as follows. $V := \{0\} \cup N$ is a finite set of *nodes*, with 0 representing the location of the hub (or the home city) and each $i \in N$ being the location of a customer. $A \subseteq \{(i, j) \subset V \times V \mid i \neq j\}$ is the set of *arcs* between the nodes in V. A graph is called *complete* if the above inclusion holds with equality. The function $l : A \to \mathbb{R}_+$ associates a length (or weight, or cost) $l(a) \in \mathbb{R}_+$ to each arc $a \in A$. The arcs (i, j) and (j, i) may have different costs in a general directed weighted graph. An undirected weighted graph can be seen as a particular case of a directed weighted graph for which $(i, j) \in A$ if and only if $(j, i) \in A$ and $l(i, j) = l(j, i)$ for every $(i, j) \in A$.

[1] Faigle et al. (1988) also propose a concept of ϵ-core, but that is different from the earlier one introduced by Shapley & Shubik (1966) in the economics literature. Both concepts capture the idea of "taxing" proper coalitions to prevent blocking, but Shapley & Shubik (1966) consider a lump-sum tax that is equal across coalitions, while Faigle et al. (1988) consider a tax that is proportional to the cost of serving each coalition.

In light of this observation, we focus on the more general case of directed graphs, with the understanding that the analysis also applies to the subclass of undirected weighted graphs.

Let $G = (\{0\} \cup N, A, l)$ be a connected, directed, weighted graph. There are no restrictions imposed on the length function l. In particular, l need not satisfy the triangle inequality, and it may be that $l(i,j) \neq l(j,i)$ for some $(i,j) \in A$. For every $k \in \mathbb{N}$, a *k-path* (or simply a *path*) is defined as a vector of nodes $p := (v_1, v_2, \ldots, v_k, v_{k+1})$ such that, for every $i \in \{1, \ldots k\}$, $a_i := (v_i, v_{i+1}) \in A$. The *length* of the k-path p is then $l(p) := \sum_{i=1}^{k} l(a_i)$.

For every non-empty $S \subseteq N$, a *tour* of S in G is a path that starts and ends at node 0 and goes at least once through each node of S. Thus, $(v_1, v_2, \ldots, v_k, v_{k+1})$ is a tour of S if $v_1 = v_{k+1} = 0$, $(v_i, v_{i+1}) \in A$ for every $i = 1, ..., k$ and $S \subseteq \{v_2, \ldots, v_k\}$. Note that a tour may use the same node or the same arc more than once and may visit nodes that are not in S. For every non-empty $S \subseteq N$ we let $\mathbb{T}(S)$ denote the set of all tours of subset S.

For every non-empty $S \subseteq N$, we define the cost of visiting the nodes in S as

$$c(S) := \min \{l(t) \mid t \in \mathbb{T}(S)\}, \tag{1}$$

and extend the functional c to 2^N by letting $c(\emptyset) = 0$. The pair (N, c) is called a *traveling salesman game* (TSG). In the terminology of cooperative game theory, N is referred to as the *set of players* and $c : 2^N \longrightarrow \mathbb{R}_+$ is the *characteristic function*.

The *classical* Traveling Salesman Problem (TSP) associated with $G = (\{0\} \cup N, A, l)$ amounts to finding a tour $t^* \in \mathbb{T}(N)$ such that $l(t^*) = c(N)$. Once the minimal-cost tour $t^* \in \mathbb{T}(N)$ is found, the question is how to allocate the cost $c(N)$ among the customers in N. The *core* of the TSG (N, c), denoted by $Core(c)$, is the set of all cost-allocation vectors $x \in \mathbb{R}_+^N$ that satisfy

$$x(N) = c(N) \tag{2}$$

$$x(S) \leq c(S) \qquad \text{for every } S \subseteq N. \tag{3}$$

Any vector $x \in \mathbb{R}_+^N$ that belongs to the core is a natural candidate to solve the TSP cost allocation question. First, as $x(N) = c(N)$, the total travel cost is covered. Second, no subset of players could obtain a lower cost by acting on their own (i.e., by contracting separately with a different service provider) because $x(S) \leq c(S)$ for every $S \subseteq N$. As we will see from the examples presented below, some TSG-s may have an empty core.

We introduce next some notation that will be used to formulate the infinite variant of the TSP. A *finite schedule* is a vector $(T_k)_{k=1,\ldots,K}$ such that K is a positive integer, $T_k \subseteq N$ for every k, and $\bigcup_{k=1}^{K} T_k = N$. Note that repetitions of subsets are allowed in a schedule. The *frequency* of node $i \in N$ in the finite schedule $\sigma = (T_1, \ldots, T_k, \ldots, T_K)$ is defined as $f_i(\sigma) = \sum_{k=1}^{K} I_{T_k}(i)$, where I_{T_k} denotes the indicator function of the set T_k. Clearly, $f_i(\sigma) \geq 1$, as $(T_1, \ldots, T_k, \ldots, T_K)$ is a cover of N. The *frequency* of the finite schedule σ is then defined as $f(\sigma) = \min_{i \in N} f_i(\sigma)$. The set of all finite schedules with frequency F is denoted by Σ_F, and $\Sigma = \bigcup_{F=1}^{\infty} \Sigma_F$ is the set of all finite schedules.

The *cost* of the finite schedule $\sigma = (T_1, \ldots, T_k, \ldots, T_K) \in \Sigma$ is $c(\sigma) = \sum_{k=1}^{K} c(T_k)$ and we define its *average cost per visit* as $ACV(\sigma) := \frac{c(\sigma)}{f(\sigma)}$. A finite schedule σ offers potential cost savings with respect to the original TSP if $c(\sigma) < f(\sigma)c(N)$ or, equivalently, if its average cost per visit $ACV(\sigma)$ is strictly lower than $c(N)$.

As illustrated by Example 1.1, the lowest average cost per visit, $\inf_{\sigma \in \Sigma_F} \frac{c(\sigma)}{f(\sigma)}$ may vary with the number of visits, F. We are interested in characterizing the asymptotic behavior of the average cost when the number of visits grows arbitrarily large. That is,

$$\lim_{F \to \infty} \inf_{\sigma \in \Sigma_F} \frac{c(\sigma)}{f(\sigma)}. \tag{4}$$

We will show that the limit in (4) exists, and denote by $c^*(N)$ its value. We will refer to $c^*(N)$ as the *asymptotically optimal cost per visit* of serving the customers in N. Clearly, $c^*(N) \leq c(N)$.

We then ask whether $c^*(N)$ can be achieved through some infinite schedule. An infinite schedule is an infinite-dimensional vector of tours $(T_k)_{k=1}^{\infty}$ with $\cup_{k=1}^{\infty} T_k = N$. We let Σ^{∞} denote the set of infinite schedules. For every schedule $\sigma = (T_k)_{k=1}^{\infty} \in \Sigma^{\infty}$ and $K \in \mathbb{N}$ large enough (i.e., $K \geq K_0$ where K_0 is such that $\cup_{k=1}^{K_0} T_k = N$), we define its K-th truncation as the finite schedule $\sigma_K := (T_1, T_2, ..., T_K)$, and its average cost per visit as

$$ACV(\sigma) := \lim_{K \to +\infty} ACV(\sigma_K),$$

whenever the limit exists. We say that an infinite schedule $\sigma^* \in \Sigma^{\infty}$ solves the ∞-TSP if $ACV(\sigma^*) = c^*(N)$.

Note, however, that if some infinite schedule $\sigma^* \in \Sigma^{\infty}$ achieves the asymptotically optimal cost, then so does any infinite schedule obtained from σ^* by adding to it finitely many arbitrary tours. Therefore, the class of infinite schedules that achieve the asymptotically optimal average cost, if non-empty, is infinite itself. Moreover, some of the schedules in that set are clearly undesirable from a practical point of view. However, we will show that there always exists a finite schedule that achieves the asymptotically optimal average cost, when repeated indefinitely. For a finite schedule $\sigma = (T_1, ..., T_K) \in \Sigma$, we let $\sigma^\infty \in \Sigma^\infty$ denote the infinite schedule obtained through the indefinite repetition of the tours in σ. That is, $\sigma^\infty := (T_1, ..., T_K, T_1, ..., T_K, ...)$. We will show that there exists $\sigma^* \in \Sigma$ such that $ACV(\sigma^{*\infty}) = c^*(N)$.

Finally, we ask if there exists a cost allocation vector $x^* \in \mathbb{R}_+^N$ of per-visit charges that covers the asymptotically optimal average cost (that is, $x^*(N) = c^*(N)$) and that is stable to coalitional deviations. A subgroup of customers, $S \subset N$, who might explore entering into a long-run contract with a different service provider takes into account the (possibly lower) average cost per visit of such contract, $c^*(S)$, rather than $c(S)$. Therefore, coalitional stability in this context requires that $x^*(S) \leq c^*(S)$ for every non-empty $S \subseteq N$.

The next Proposition demonstrates that the asymptotic average cost of an infinitely repeated finite schedule is equal to the average cost of the finite schedule.

Proposition 3.1 *Let $\sigma \in \Sigma$ be an arbitrary finite schedule and let $\sigma^\infty \in \Sigma^\infty$ be the infinite schedule obtained by repeating σ indefinitely. Then $ACV(\sigma^\infty)$ is well defined and $ACV(\sigma) = ACV(\sigma^\infty)$.*

Proof. Let $\sigma = (T_1, \ldots, T_k, \ldots, T_K) \in \Sigma$ for some $K \in \mathbb{N}$, and let $M \geq K$, arbitrary. Then there exist $\alpha, \beta \in \mathbb{N}$ with $0 \leq \beta < K$ such that $M = \alpha K + \beta$. The following inequalities hold

$$\alpha c(\sigma) \leq c(\sigma_M^\infty) \leq (\alpha + 1)c(\sigma), \tag{5}$$

$$\alpha f(\sigma) \leq f(\sigma_M^\infty) \leq (\alpha + 1)f(\sigma), \tag{6}$$

which imply that

$$ACV(\sigma) - \frac{ACV(\sigma)}{1+\alpha} \leq ACV(\sigma_M^\infty) \leq ACV(\sigma) + \frac{ACV(\sigma)}{\alpha}. \tag{7}$$

Letting $M \to \infty$ (and thus $\alpha \to \infty$) in (7), we obtain that $\lim_{M\to\infty} ACV(\sigma_M^\infty)$ exists and

$$ACV(\sigma^\infty) = \lim_{M\to\infty} ACV(\sigma_M^\infty) = ACV(\sigma),$$

as desired. ■

Proposition 3.1 implies that, to find a solution for the ∞-TSP, it is enough to find a finite schedule $\sigma^* \in \Sigma$ and a cost allocation vector $x^* \in \mathbb{R}_+^N$

such that $ACV(\sigma^*) = c^*(N) = x^*(N)$ and $x^*(S) \leq c^*(S)$ for every $S \subseteq N$. Proposition 3.1 also indicates that a service provider entering into a long-run contract with a set of customers can offer to sell those customers packages of $f(\sigma^*)$ visits at a cost of x_i^* per visit for each customer $i \in N$.

The next proposition identifies a solution of the ∞-TSP when the core of the original TS-game is non-empty.

Proposition 3.2 *Let the weighted graph $G = (\{0\} \cup N, A, l)$ have the associated cost function c. If $Core(c) \neq \emptyset$, then $c^*(N)$ is well defined, $c^*(N) = c(N)$, and the schedule $\{N\}^\infty$ solves the ∞-TSP.*

Proof. Let $\bar{\sigma} = (T_1, \ldots, T_k, \ldots, T_K) \in \Sigma$ and let $x \in Core(c)$. Then,

$$ACV(\bar{\sigma}) = \frac{1}{f(\bar{\sigma})} \sum_{k=1}^{K} c(T_k) \geq \frac{1}{f(\bar{\sigma})} \sum_{k=1}^{K} x(T_k) = \sum_{i \in N} \frac{f_i(\bar{\sigma})}{f(\bar{\sigma})} x_i \geq x(N). \quad (8)$$

Thus, the average cost per visit of any finite schedule is bounded below by $c(N)$ and therefore $\inf_{\sigma \in \Sigma_F} ACV(\sigma) \geq c(N)$ for every F. On the other hand, $\inf_{\sigma \in \Sigma_F} ACV(\sigma) \leq ACV(\{N\}) = c(N)$, which implies that $\inf_{\sigma \in \Sigma_F} ACV(\sigma) = c(N)$. Thus, $c^*(N)$ is well defined and equal to $c(N)$.

As $ACV(\{N\}) = c(N)$, Proposition 3.1 implies that $ACV(\{N\}^\infty) = c(N)$ and therefore the schedule $\{N\}^\infty$ solves the ∞-TSP. ■

However, as argued in the Introduction, many TSP-s generate empty-core games. Next, we propose a methodology to solve the ∞-TSP in those cases.

4 Aspirations and the Aspiration Core

We present here a couple of solution concepts from the cooperative game literature, which are useful for the study of empty-core games. These apply to cooperative games (N, c) where N is a finite set and $c : 2^N \to \mathbb{R}_+$ is a characteristic function with $c(\emptyset) = 0$.

A payoff vector $x \in \mathbb{R}_+^N$ is called *coalitionally rational* if

$$x(S) \leq c(S) \qquad \text{for every } S \subseteq N. \quad (9)$$

In the context of a cost game, such vectors capture the idea of a "fair" cost allocation in the sense that no coalition is asked to pay more than the cost it generates.

An allocation $x \in \mathbb{R}_+^N$ is called *aspirationally feasible* if, for every $i \in N$ there exists $S \subseteq N$ with $S \ni i$ such that $x(S) = c(S)$. The *aspiration* *set*, denoted $Asp(c)$, is the set of all allocations $x \in \mathbb{R}^N_+$ that are coalitionally rational and aspirationally feasible. The elements of $Asp(c)$ are called *aspirations*.

Aspirational feasibility relaxes the traditional feasibility requirement by allowing players to achieve their payoffs within *some* coalition rather than necessarily within the grand coalition. This approach provides an alternative way of thinking about stability and coalition formation as it allows for the possibility that players may form smaller coalitions. Clearly, every payoff that satisfies the standard feasibility condition (i.e., $x(N) = c(N)$) is also aspirationally feasible.

For any aspirationally feasible cost allocation $x \in \mathbb{R}^N_+$ we define its *generating collection* as

$$\mathcal{GC}(x) = \{S \subseteq N \mid x(S) = c(S)\}. \tag{10}$$

These are the coalitions that are asked to cover their entire cost at the allocation vector x. Clearly, $\mathcal{GC}(x)$ is a cover (but not necessarily a partition) of N for every x that is aspirationally feasible. As we will see, the coalitions in the generating collection are the building blocks to create the cost-saving schedules proposed in this work.

The *aspiration core*[2] of a game (N, c), denoted $AC(c)$, is the subset of those aspirations that cover the highest total cost. Thus,

$$AC(c) = \arg\max\{x(N) \mid x \in Asp(c)\}. \tag{11}$$

Let $\bar{c}(N)$ denote the highest total cost[3] that can be covered with an aspiration allocation. Thus, $x(N) = \bar{c}(N)$ for every $x \in AC(c)$.

Let $\mathcal{N} := \{S \subseteq N \mid S \neq \emptyset\}$ be the set of all non-empty *coalitions* of N. A collection of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is called *balanced* (respectively *weakly balanced*) if there exist positive (respectively non-negative) numbers (called *balancing weights*) $(\lambda_S)_{S \in \mathcal{B}}$ such that, for every $i \in N$,

$$\sum_{S \in \mathcal{B}} I_S(i)\lambda_S = 1, \tag{12}$$

[2]The aspiration core is also known in the game theory literature as the *balanced aspiration set*. See Bennett (1983) and Bejan & Gómez (2012)).

[3]The ϵ-core, ϵ-$C(c)$, as defined in Faigle et al. (1988) and Sun & Karwan (2015) is closely related to the aspiration core. It can be easily verified that for $\epsilon = \frac{c(N)}{\bar{c}(N)} - 1$, $x \in \epsilon$-$C(c)$ if and only if $\frac{x}{1+\epsilon} \in AC(c)$. It should be noted that the ϵ-core as defined by the above authors is different from the homonym concept used in economics (see Shapley & Shubik (1966) and Maschler et al. (1979)). The minimal α-core concept described by Toriello & Uhan (2013) also coincides with the aspiration core when $\alpha = \frac{c(N)}{\bar{c}(N)}$.

where I_S denotes the indicator function of subset S.

The following results are proved in Bennett (1983).

Proposition 4.1 *Every cooperative game $c : 2^N \longrightarrow \mathbb{R}_+$ satisfies the following properties:*

1. $\emptyset \neq AC(c)$.
2. *If $Core(c) \neq \emptyset$, then $AC(c) = Core(c)$.*
3. *If $x \in \mathbb{R}_+^N$ is a coalitionally rational payoff, then $x \in AC(c)$ if and only if $\mathcal{GC}(x)$ is balanced.*
4. *There exists a balanced family of coalitions $\mathcal{B}$ with balancing weights $(\lambda_S)_{S\in\mathcal{B}}$ such that $\bar{c}(N) = \sum_{S\in\mathcal{B}} \lambda_S c(S)$.*

A balanced collection of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is called *minimally balanced* if no proper subset of $\mathcal{B}$ is balanced. The following Lemma provides a useful characterization of minimally balanced collections.

Lemma 4.2 *If a balanced collection $\mathcal{B} \subset \mathcal{N}$ is minimal, then $|\mathcal{B}| \leq |N|$ and the balancing weights $(\lambda_S)_{S\in\mathcal{B}}$ are unique, strictly positive, and rational.*

For proofs of these results, we refer the reader to Shapley (1967) and Kannai (1992).

5 Applying the Aspiration Core to the TSG

To fix ideas, consider the following example.

Example 5.1 *A service provider must visit* 6 *different customers indefinitely many times. The customers' locations are connected by roads as shown in Figure 4.*[4] *Trips must start and end at the service provider's hub, labeled as node* 0*. The cost of using any road is* \$1.

The minimum cost to visit all cities is $c(N) = 8$, as illustrated by the red tour depicted in Figure 4. One can verify that $x_0 = (1.6, 1.6, 1.6, 0.9, 0.9, 0.9)$ belongs to the aspiration core of the corresponding TS-game. As x_0 is coalitionally rational, we see that $x_0 \in AC(c)$ because

$$\mathcal{GC}(x_0) = \{\{1, 2, 4, 5\}, \{1, 3, 4, 6\}, \{2, 3, 5, 6\}\}$$

[4] A similar problem on a Euclidian 6-node *complete* graph was presented by Faigle et al. (1988), Sun & Karwan (2015) and Sun et al. (2018).

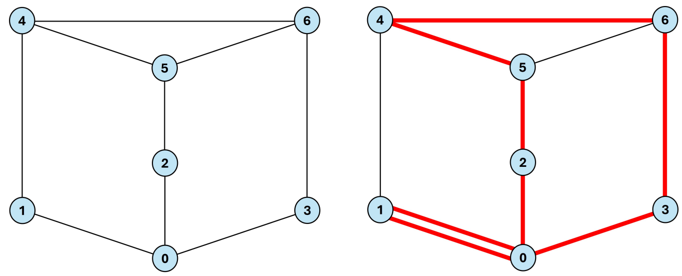
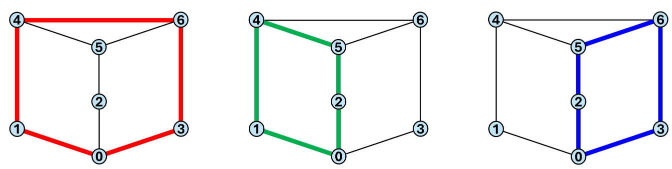


Figure 4: Optimal tour with 1 visit

is a balanced family of coalitions, with a balancing weight of $\frac{1}{2}$ for each of the three sets. Since $x_0(N) > c(N)$, it must be that the core of the game c is empty. Consider the schedule depicted in Figure 5, suggested by the generating collection of x_0. That is, $\sigma_{x_0} := (\{1,2,4,5\}, \{1,3,4,6\}, \{2,3,5,6\})$.



	node 1	node 2	node 3	node 4	node 5	node 6	Cost
Tour 1	1		1	1		1	$ 5
Tour 2	1	1		1	1		$ 5
Tour 3		1	1		1	1	$ 5
Total	2	2	2	2	2	2	$ 15

Figure 5: Optimal schedule

Each customer is visited twice, and the total cost incurred is $x(\{1,2,4,5\}) + x(\{1,3,4,6\}) + x(\{2,3,5,6\}) \leq 5+5+5 = 15 < 16 = 2c(N)$. We will show next that the indefinite iteration of this schedule solves ∞-TSP. More precisely, we show that, if the salesperson's travels have to be repeated indefinitely, the aspiration core vectors can be used to construct solutions for the ∞-TSP, as well as corresponding cost allocations to customers. This is formalized in the following theorems.

Theorem 5.2 *Let the weighted graph $G = (\{0\} \cup N, A, l)$ be associated with the game c and choose any aspiration core vector $x \in AC(c)$. Then, there exists an associated schedule $\sigma_x = (T_1, \ldots, T_k, \ldots, T_K) \in \Sigma$ such that:*

1. All the tours in the schedule σ_x belong to the generating collection $\mathcal{GC}(x)$.

2. $ACV(\sigma_x) = \min_{\sigma \in \Sigma} ACV(\sigma) = \bar{c}(N)$.

Proof. Let $\sigma = (T_1, \ldots, T_k, \ldots, T_K) \in \Sigma$ be an arbitrary schedule and let $x \in AC(c)$. By the definition of the aspiration core we know that $x(N) = \bar{c}(N)$ and $c(T_k) \geq x(T_k)$, for every k. Therefore, the sequence of inequalities shown in (8) shows that $ACV(\sigma) = \frac{c(\sigma)}{f(\sigma)} \geq \bar{c}(N)$.

We construct next a schedule $\sigma_x \in \Sigma$ that satisfies $\frac{c(\sigma_x)}{f(\sigma_x)} = \bar{c}(N)$ and thus it achieves the minimum cost per visit. By item 3 in Proposition 4.1, we know that $\mathcal{GC}(x)$ is balanced. Take a minimally balanced family of coalitions $\mathcal{B} = \{S_m \mid m = 1, \ldots, M\} \subseteq \mathcal{GC}(x)$. For every m, let λ_m be the balancing weight corresponding to coalition S_m. According to Lemma 4.2 these balancing weights are strictly positive and rational so, for every m, we can choose $a_m, b_m \in \mathbb{N}$ such that $\lambda_m = \frac{a_m}{b_m}$ and the greatest common factor g.c.f.(a_m, b_m) is equal to one for every $m = 1, ..., M$. Let $F =$ l.c.m.$(b_1, \ldots, b_M)$ be the least common multiple of the denominators $b_1, \ldots, b_M$, and let $K = \sum_{m=1}^{M} \frac{Fa_m}{b_m}$.

Define the schedule $\sigma_x = (T_1, \ldots, T_k, \ldots, T_K) \in \Sigma(N)$ as follows. Set $T_k = S_1$ if $k \leq \frac{Fa_1}{b_1}$ and set $T_k = S_m$ if

$$\sum_{j=1}^{m-1} \frac{Fa_j}{b_j} < k \leq \sum_{j=1}^{m} \frac{Fa_j}{b_j} \qquad \forall m = 2, \ldots, M,$$

so that every subset $S_m \in \mathcal{B}$ appears $\frac{Fa_m}{b_m}$ times in schedule σ_x.

We show next that all the inequalities in (8) hold with equality when applied to σ_x. First, $c(T_k) = x(T_k)$ because $T_k \in \mathcal{GC}(x)$ for every $k \leq K$. Moreover, $\frac{f_i(\sigma_x)}{f(\sigma_x)} = 1$ because the frequency of every node $i \in N$ in schedule σ_x is F. Indeed,

$$f_i(\sigma) = \sum_{k=1}^{K} I_{T_k}(i) = \sum_{m=1}^{M} I_{S_m}(i) \frac{Fa_m}{b_m} = F \sum_{m=1}^{M} I_{S_m}(i) \lambda_m = F,$$

where the last equality follows from the fact that $\mathcal{B}$ is balanced with the weights $(\lambda_m)_{m=1,\ldots,M}$. Therefore, $\frac{c(\sigma_x)}{f(\sigma_x)} = x(N)$, which proves that σ_x minimizes the average cost per visit. This completes parts (1) and (3) of the theorem. By construction, all the tours in the schedule σ_x belong to $\mathcal{GC}(x)$, which proves part (2). ∎

As an illustration of Theorem 5.2, let us revisit Example 1.1. The corresponding TSG satisfies, for instance, $c(\{1\}) = c(\{2\}) = c(\{3\}) = 2$, $c(\{4\}) = c(\{5\}) = c(\{6\}) = c(\{7\}) = c(\{8\}) = c(\{9\}) = 4$, $c(N \setminus \{i\}) = 9$ for every $i \in N$, and, as argued before, $c(N) = 11$. One can verify that $x = (\frac{9}{8}, \frac{9}{8}, \frac{9}{8}, \frac{9}{8}, \frac{9}{8}, \frac{9}{8}, \frac{9}{8}, \frac{9}{8}, \frac{9}{8})$ is coalitionally rational with corresponding generating collection

$$\mathcal{GC}(x) = \{S \subseteq N \mid |S| = 8\},$$

which is minimally balanced with the weights $\lambda_S = \frac{1}{8}$ for every $S \in \mathcal{GC}(x)$. Therefore, $x \in AC(c)$. This suggests the optimal schedule depicted in Figure 6:

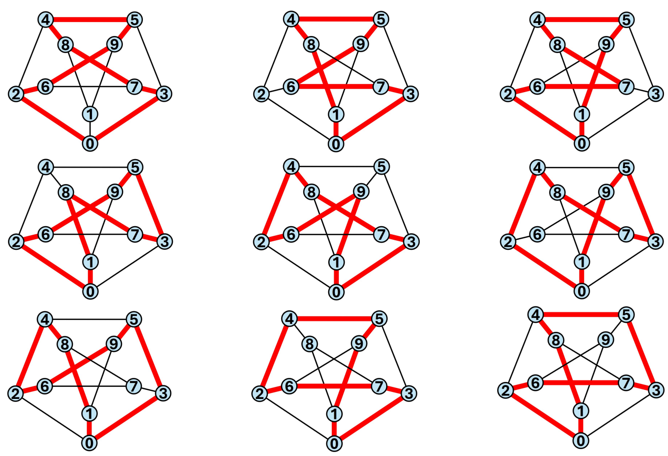


Figure 6: Optimal schedule for the Petersen graph

Each tour in this schedule visits all cities except one. Overall, the service provider visits each city eight times. This suggests that, if the service provider wants to sell packages of bundled visits to its customers, it should sell packages of 8 visits. Theorem 5.2 guarantees that this is the finite schedule that minimizes the average cost per visit to its optimal value of $(9 \times 9)/8 = 10.125$. Furthermore, if the total cost is allocated to customers according to the vector x, then no coalition has an incentive to deviate, as $x \in AC(c)$.

The next Proposition computes the long-run optimal average cost.

Proposition 5.3 *Let* $c^*(N) := \lim_{F\to\infty} \inf_{\sigma\in\Sigma_F} ACV(\sigma)$. *Then* $c^*(N)$ *is well defined and* $c^*(N) = \bar{c}(N)$.

Proof. Let $x \in AC(c)$ and construct the associated finite schedule σ_x as in the proof of Theorem 5.2. Any $F \in \mathbb{N}$ can be written as $F = af(\sigma_x) + r$ where $a, r \in \mathbb{N}$ and $0 \leq r < f(\sigma_x)$. Define then a schedule $\tau \in \Sigma_F$ (with slight abuse of notation) as

$$\tau = (\underbrace{\sigma_x, \ldots, \sigma_x}_{a \text{ times}}, \underbrace{N, \ldots, N}_{r \text{ times}}).$$

Then $f(\tau) = af(\sigma_x) + r = F$ and indeed $\tau \in \Sigma_F$. Using Theorem 5.2, we have the following:

$$\begin{aligned}
\bar{c}(N) &\leq \inf_{\sigma \in \Sigma_F} ACV(\sigma) \leq ACV(\tau) = \frac{1}{F}[ac(\sigma_x) + rc(N)] = \\
&= \frac{1}{F}[af(\sigma_x)\bar{c}(N) + rc(N)] = \frac{1}{F}[F\bar{c}(N) + r(c(N) - \bar{c}(N))] = \\
&= x(N) + \frac{r}{F}(c(N) - \bar{c}(N))
\end{aligned}$$

When F tends to infinity, all terms in this inequality converge to $\bar{c}(N)$. In particular, $\lim_{F\to\infty} \inf_{\sigma\in\Sigma_F} ACV(\sigma)$ exists and it is equal to $\bar{c}(N)$. Thus, $c^*(N) = \bar{c}(N)$ as desired. ■

The next theorem shows how to construct a (per-visit) cost allocation vector that covers the asymptotically optimal average cost and it is coalitionally rational.

Theorem 5.4 *Let the weighted graph $G = (\{0\} \cup N, A, l)$ be associated with the game c. There exists $x^* \in \mathbb{R}^N$ such that $x^*(N) = c^*(N)$ and $x^*(S) \leq c^*(S)$ for every $S \subseteq N$.*

Proof. Let $x \in AC(c)$ be an arbitrary aspiration core vector. Then $x(N) = \bar{c}(N) = c^*(N)$ and $x(S) \leq c(S)$. We prove next that x remains coalitionally rational even when each coalition $S \subseteq N$ expects to pay no more than $c^*(S) \leq c(S)$.

Fix some $S \subseteq N$ and let (S, c_S) be the truncation of game (N, c) to the subset S. That is, $c_S(T) := c(T)$ for every $T \subseteq S$. Applying Proposition 5.3 to the game (S, c_S) for every $S \subseteq N$, we obtain $c^*(S) = \bar{c}(S)$. Moreover, by item (4) of Proposition 4.1 applied to the game (S, c_S), there exists a balanced family of subsets of S, $\mathcal{B}_S$ with the associated weights $(\lambda_T)_{T\in\mathcal{B}_S}$, such that $\sum_{T\ni i} \lambda_T = 1$ for every $i \in S$, and

$$c^*(S) = \bar{c}(S) = \sum_{T\in\mathcal{B}_S} \lambda_T c(T). \tag{13}$$

Since $x \in AC(c)$, $x(T) \leq c(T)$ for every $T \subseteq N$. Using equation (13), we have

$$c^*(S) = \bar{c}(S) \geq \sum_{T \in \mathcal{B}_S} \lambda_T x(T) = \sum_{i \in S} \sum_{T \ni i} \lambda_T x_i = x(S), \tag{14}$$

which completes the proof. ■

6 Selecting a "fair" cost allocation

Since, typically, the aspiration core is not single-valued, we describe next a procedure that selects a single cost allocation vector from the set of aspiration core allocations. Our methodology uses the concept of the *aspiration nucleolus*, which was introduced by Bennett (1981). Among several other desirable properties, the aspiration nucleolus always belongs to the aspiration core.

For every $x \in Asp(c)$, and every coalition $S \subseteq N$, define the *savings* of S with respect to x as $e(x, S) = c(S) - x(S)$ and denote the savings vector by $\mathbf{e}(x) = (e(x, S))_{S \subseteq N} \in \mathbb{R}^{2^N - 1}$. Let $\theta(\mathbf{e}(x))$ be the vector obtained from $\mathbf{e}(x)$ by rearranging its coordinates in non-decreasing order. Thus, $\theta(\mathbf{e}(x))$ implicitly ranks coalitions in terms of their "satisfaction" with the cost saving embedded in the allocation vector x, with the least satisfied coalition listed first. The aspiration nucleolus iteratively maximizes the savings of all coalitions, prioritizing the least satisfied coalitions. It is defined as follows.

For every $x, y \in Asp(c)$, we say that $\theta(\mathbf{e}(x))$ dominates $\theta(\mathbf{e}(y))$ in the lexicographic order, $\theta(\mathbf{e}(x)) \succeq_L \theta(\mathbf{e}(y))$, if and only if

1. $\theta_1(\mathbf{e}(x)) > \theta_1(\mathbf{e}(y))$ or
2. there exists $m \in \mathbb{N}$, $m < 2^N - 1$, such that
   (a) $\theta_j(\mathbf{e}(x)) = \theta_j(\mathbf{e}(y))\ \forall\ j \leq m$, and
   (b) $\theta_{m+1}(\mathbf{e}(x)) > \theta_{m+1}(\mathbf{e}(y))$.

The *Aspiration Nucleolus* is defined as the aspiration associated with the maximal element of the set $\{\theta(\mathbf{e}(x)) \mid x \in Asp(c)\}$ with respect to $\succ_L$. That is,

$$AspNuc(c) = \{x \in Asp(c) \mid \theta(e(x)) \succeq_L \theta(e(y)),\ \forall y \in Asp(c)\}.$$

The aspiration nucleolus is always an element of the aspiration core. Besides maximizing savings for the worse off coalitions, the aspiration nucleolus also satisfies anonymity.[5] This guarantees that the allocations players receive do not depend on their identities. Referring back to Example 4, the aspiration core of the corresponding TS-game is

$$AC(c) = \{(a, b, c, 2.5 - a, 2.5 - b, 2.5 - c) \in \mathbb{R}^6 \mid a, b, c \in [1.5, 2]\}.$$

Given the symmetry of the graph, it seems reasonable that nodes that play the same role —such as nodes 1, 2 and 3— be charged the same price. More precisely, a single-valued solution concept $\phi(c)$ is said to satisfy *anonymity* if for every cooperative game c on N and bijection $\pi : N \longrightarrow N$, $\phi_i(c) = \phi_{\pi(i)}(c^\pi)$, where the characteristic function $c^\pi : 2^N \to \mathbb{R}_+$ is defined as $c^\pi(S) := c(\pi(S))$ for every $S \subseteq N$, and $c^\pi(\emptyset) = 0$.

In the context of Example 4, with $N = \{1, 2, 3, 4, 5, 6\}$, let $\pi : N \longrightarrow N$ be defined by $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 5$, $\pi(5) = 6$, and $\pi(6) = 4$. It can be readily verified that, given the symmetry of the game, for every $S \subseteq N$, $c(S) = c(\pi(S))$ and thus $c = c^\pi$. Therefore, anonymity implies that if $x \in AspNuc(c)$, then $x_1 = x_2 = x_3$ and $x_4 = x_5 = x_6$, thus addressing the concern raised before. In fact,

$$AspNuc(c) = \{(1.75, 1.75, 1.75, 0.75, 0.75, 0.75)\}.$$

Since every aspiration $x \in Asp(c)$ is coalitionally rational, it provides non-negative savings, $c(S) - x(S)$, to every coalition S. This means that coalitions which obtain zero savings are the ones that are worse off when using x to allocate costs. These coalitions form the generating collection $\mathcal{GC}(x)$. By definition, the aspiration nucleolus minimizes the number of coalitions with zero savings. This shortens the list of candidates that can be used to build a schedule of tours that, when indefinitely repeated, solves ∞-TSP.

As shown earlier, cost-saving tour schedules are possible if and only if the core of the original TS-game is empty. Tamir (1989) and Kuipers (1993) proved that any TSP derived from an undirected graph with $|N| \leq 5$ generates a game with a non-empty core. Therefore, Example 4 describes the smallest undirected graph for which the corresponding TSG has an empty core. However, as the following example illustrates, TSP-s derived from directed graphs can generate empty-core games even if $|N| \leq 5$.

[5] For a list of other properties as well as a an axiomatic characterization of the aspiration nucleolus, the reader is referred to Hokari & Kibris (2003).

Example 6.1 *Consider the set of players* $N = \{1, 2, 3, 4\}$ *and the complete graph* $(\{0\} \cup N, A)$ *where costs are defined as in Figure 7.*[6]

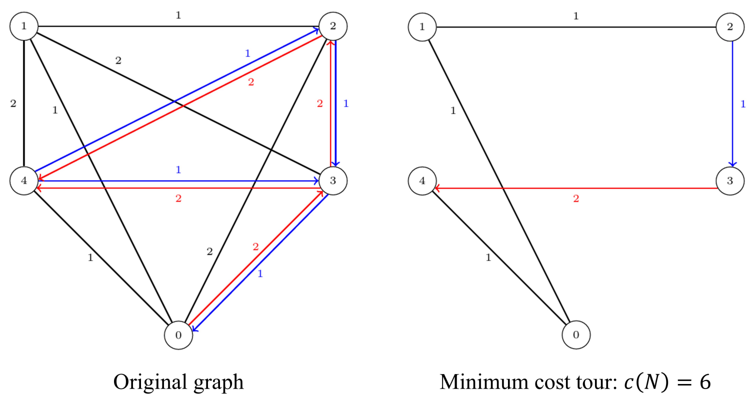


Figure 7: Directed graph with a non-balanced TSG

There is no Hamiltonian circuit that only involves arcs of cost \$1. Therefore, visiting the four cities in a single tour must cost more than \$5. The minimal cost of the one-shot TSP is \$6 and it is achieved, for example, with the tour depicted in Figure 7. The rest of the TSG is defined as follows. $c(\{1\}) = c(\{4\}) = 2$, $c(\{2\}) = 4$, and $c(\{3\}) = 3$. If $|S| = 2$, then $c(S) = 4$ except for $c(\{3, 4\}) = 3$. If $|S| = 3$, then $c(S) = 4$ except for $c(\{1, 3, 4\}) = 5$.

For this TSG, the aspiration core is

$$AC(c) = \{(a, 2.5 - a, 1.5, 1.5) \in \mathbb{R}^4 \mid a \in [1.5, 2]\},$$

which implies that the asymptotic minimal cost is \$5.50. To choose among cost allocations, we can use the aspiration nucleolus

$$AspNuc(c) = \{(1.75, 0.75, 1.5, 1.5)\}$$

with generating collection $\{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}\}$, leading to the optimal tour schedule shown in Figure 8.
The corresponding cost per visit is $(4 + 4 + 3)/2 =$ \$5.50.

[6]This example is taken from Potters et al. (1992).

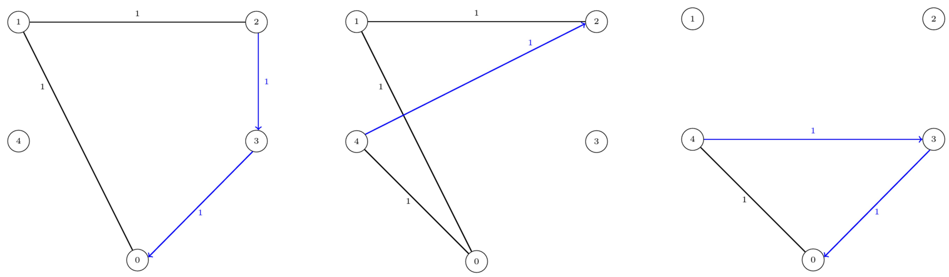


Figure 8: Optimal schedule for the directed graph

7 Concluding remarks

We introduced and analyzed the Infinite Period Traveling Salesman Problem (∞-TSP), which extends the classical TSP by considering an arbitrarily large time horizon with repeated visits to customer locations. Our study focused on minimizing the long-run (or asymptotic) average cost per visit while ensuring a fair cost allocation among customers. By leveraging some cooperative game theory results, we established a new and fundamental link between the ∞-TSP and the aspiration core solution concept.

Our results demonstrate that the feasibility of reducing the average cost per visit over time is directly linked to the core emptiness of the corresponding one-shot TSG.

When the core of the one-shot TSG is empty, multiple stable solutions may exist. To address this ambiguity, we introduced the aspiration nucleolus as a single-valued selection from the aspiration core. This refinement ensures uniqueness, fairness, and anonymity in cost allocation, making it a practically-relevant tool for designing stable long-term service schedules. The aspiration nucleolus identifies the most stable cost distribution by prioritizing the least advantaged coalitions and preserving the symmetries of the problem.

Our findings contribute to both the theoretical and practical understanding of long-term routing problems. The connection between the aspiration core allocations and the optimal long-run cost minimization suggests new avenues for designing efficient and fair service schedules in contexts where repeated visits are required. Future research may explore extensions of this framework to stochastic demand variations, or multi-provider competitive environments.

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