Investment and Financing in Incomplete Markets

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Abstract

I propose a model of production with incomplete financial markets, in which a firm can act as a financial innovator by issuing claims against its stock. In this environment, market value maximization may be against the firm's shareholders' interests. I propose instead a new measure of *adjusted* value, which is the sum between the market value and the shareholders' surplus from their trades in the stock market. If a firm maximizes its adjusted value, then its financial policy is relevant (that is, Modigliani-Miller theorem does not hold), equilibrium outcomes are stable to shareholders' renegotiation, and endogenously incomplete markets can arise at the equilibrium. If the firm is competitive, the adjusted value coincides with the objective proposed by Grossman and Hart (1979). In a competitive market with *no production-specific uninsurable risk* (that is, spanning property holds), the adjusted value coincides with the market value.

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JEL classification: D21, D52, G20, L21

1 Introduction

When the financial markets are complete, all shareholders of a perfectly competitive firm attach the same value to any given investment plan and unanimously agree that market value maximizing production plans are the optimal ones. For a firm which has market power, the price effect of a production decision may

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generate a conflict between the interests of shareholders as consumers and as receivers of the firm's dividends. That is because higher profits may come at the expense of higher prices for some goods a shareholder consumes (or lower prices for the goods he/she owns and sells). This may render value maximization undesired –from shareholders' point of view– and thus an unjustified objective. Alternative objectives for the imperfectly competitive firm were defined, for example, in Bejan (2008) and Dierker and Grodal (1999).

Market incompleteness adds a new dimension to the problem because in such an environment, aside from modifying the total supply of consumption goods in the economy, a firm may also change the asset span. The firm's equity contract as well as other securities that the firm may issue to finance its production are risk-hedging instruments which may not be replicable by a portfolio of the other traded securities. In that case, the firm-issued securities cannot be priced in the existing markets and, as a result, different shareholders may attach different values to the same investment/production plan. When evaluating a particular production plan, every shareholder assesses not only its impact on the firm's market value and the change in relative prices that it induces, but also the risk-hedging opportunities that the firm-issued securities offer. It may be that the value maximizing production plan is a riskier alternative than some other plan, which generates a lower value. Depending on their wealth, preferences and attitudes toward risk, different shareholders may favor lower risk over higher market value, or the other way round. These three effects (which I call the income, price and risk effects, respectively) may influence a shareholder's wealth differently – this is the source of shareholder's conflict of interests – and have different impact on different shareholders – which generates the disagreement among them. Note that the first two effects can arise in complete *and* incomplete markets, while the third effect arises *only* in incomplete markets.

A special case of production under incomplete markets occurs when the firm's production set is contained in the asset span (a condition known in the literature as the "spanning property"). In this case, the firm's decisions do not affect the asset market structure and firm's problem is then essentially the same as in complete markets (see, for example Ekern and Wilson (1974), Leland (1974), Radner (1974) and Magill and Quinzii (1996, chapter 6)). Therefore, a firm's production decision can have only an income and/or a price effect on its shareholders' wealth, but no risk effect.

Shareholders unanimously approve profit maximization whenever the firm's choices have only an income effect on their wealth. Disagreement among shareholders concerning the firm's objective and incompatibility of profit maximization with the shareholders' interests may arise if two or more effects are present. The above discussion points out to the fact that, as soon as one leaves the idealistic environment of complete and perfectly competitive markets, profit/value maximization by a privately owned firm may be incompatible with the preferences of that firm's shareholders and thus it is an unjustified and inadequate objective for firms that have market power or act in incomplete markets environment (or both).

Radner (1974) was the first to draw attention to the need to formulate an appropriate goal for such firm. The first attempts to solve the problem (in a more general framework than multiplicative uncertainty or spanning) were made by Dréze (1974) and later Grossman and Hart (1979). For two-period economies with uncertainty and incomplete financial markets, Dréze (1974) proposed a Pareto-Nash criterion to discard "unreasonable" (from the shareholders' point of view) choices of a perfectly competitive firm. The criterion requires that the firm's decision respect the unanimity among shareholders, provided that they can make side payments in consumption good at date 0 to achieve unanimity. Shareholders are not allowed, however, to make any changes in their portfolio holdings, *including* their share holdings. Dréze (1974) interpreted that as respecting the firm's *final* shareholders' interests and proved that this requirement is equivalent to maximizing firm's value, taking as given a system of state prices that is a weighted average of the shareholders' marginal rates of substitution, with weights equal to their final shares.

The equilibrium in which firms follow Dréze's (1974) objective satisfies the first order conditions for constrained Pareto optimality. However, because of the non-convexity of the constrained feasible set, the first order conditions may not be sufficient for optimality. In fact, Zierhut (2017) shows that this happens generically. As a consequence, at a Drèze equilibrium, firm's final shareholders may have incentives to re-trade their shares. In other words, there is a conflict between the interests of a consumer as a final shareholder of the firm and his interests as an initial shareholder.[1] This conflict creates problems for the extension of the model to environments with multiple trading periods, where one period's final shareholders become the next period's initial shareholders.[2]

As a first step toward an extension of Dréze's (1974) model to an environment with more than one trading period, Grossman and Hart (1979) pointed out the necessity of investigating the problem from the perspective of the *initial* shareholders. Since initial shareholders may eventually trade their shares, they need to have some expectations, or perceptions about shares' prices, as being related to the choice of the production plans. Hence, the authors introduced the so-called "competitive price perceptions" assumption.[3] The assumption requires that every consumer use his own present-value vector (normalized gradient of his utility) as state prices to evaluate payoffs that are not in the asset

[1]Dierker, Dierker, and Grodal (2005) proved that when following Dréze's (1974) objective, the firm may in fact select a production plan at which the initial shareholders' surplus is *minimized* (see also Dierker, Dierker, and Grodal (2002)).

[2]Bonnisseau and Lachiri (2004) extend Dréze's (1974) criterion to a multiperiod environment. The authors propose an objective for the firm that is derived from the first order conditions of the non-convex constrained Pareto optimal problem (as Drèze's objective is). However, the authors do not relate that objective to the preferences of any group of shareholders.

[3]As explained later, calling such price perceptions "competitive" may be misleading. I will use the term GH-price perceptions in the sequel.

span. As pointed out by the authors themselves, such price perceptions make shareholders expect that the benefits they accrue from any dividend stream are exactly compensated by its price. The shareholders are therefore neglecting the spanning, or risk-hedging opportunities that a certain dividend stream offers. A shareholder with GH-price perceptions wants his/her firm to maximize its value, computed at his/her vector of state prices, and is indifferent among the policies that finance that plan. Value maximization under the state prices that are the weighted average of shareholders' marginal rates of substitution (with weights equal to their *initial* shares) is proved to achieve efficiency, from the initial shareholders' point of view, *given their price perceptions*.

Dréze's (1974) and Grossman and Hart's (1979) formulations of a firm's objective eliminate a firm's incentives to financially innovate, even in incomplete markets. In both models, a firm's financial policy is irrelevant. In Dréze (1974), final shareholders make the decisions about the firm's policy *after* the financial markets close and no further trade in securities takes place. In Grossman and Hart's (1979) model, the absence of the incentives to innovate comes from the particular form of shareholders' price perceptions.

The goal of this paper is to analyze the decision problem of a firm which acts in an incomplete markets environment, and needs to choose a production plan together with a way to finance it in accordance to the preferences of a specific group of investors, called the firm's *the control group*. I propose, as the objective of such a firm, the maximization of its C*-adjusted value* (where C stands for the set of firm's control group members). The C-adjusted value measures the value of the firm as perceived by its control group members, and it is equal to the sum of the firm's net market value and the surplus gained by the members of the control group from their transactions in the financial markets. This surplus can come from two different sources. One is the difference in the prices of traded securities that the firm's choices might generate. The second is the firm's equity contract, which may not be replicable by a portfolio of the other traded securities and thus represents a new risk-hedging instrument. The C-adjusted value does not, in general, coincide with the standard market value. By taking into account the surplus to the control group members, the C-adjusted value accounts not only for the income effect (captured by the firm's market value), but also the price and risk effects that the firm's choices can have on its stakeholders' wealth. The C-adjusted value generalizes Grossman and Hart's (1979) objective in the sense that it coincides with it if the firm does not have market power. In this sense, the model described here provides a strategic foundation for Grossman and Hart's (1979) objective. If, in addition, markets are complete or the spanning property holds, then the C-adjusted value coincides with profit maximization.

I show that equilibria in which the firm maximizes its C-adjusted value generate production-financial plans which are Pareto undominated from the point of view of the members of the control group. That is, control group members cannot all achieve higher utilities by switching to a different production plan and making after-trade, date-0 transfers among themselves. Although this efficiency concept is based on the same idea as the criteria used by Dréze (1974) and Grossman and Hart (1979), it differs from those in several respects. It differs from Dréze's (1974) notion of shareholder-efficiency in that shareholders are allowed to change their portfolio holdings when pondering two production-financial policies. It also differs from Grossman and Hart's (1979) criterion in the way shareholder's preferences over production plans are defined. In Grossman and Hart's (1979) model, shareholders use GH-price perceptions to derive their preferences over firm's production plans. GH-price perceptions are (partly) rational in the sense that they are fulfilled *at* the equilibrium point; however, they may be false everywhere else.

In this paper, I am imposing a stronger version of rationality: as the governing body of the firm, the control group understands (or anticipates) the effect of the firm's production-financial decisions on securities prices. By doing that, they are able to account for the benefits of the new financial instruments that their firm creates and thus, their "price perceptions" are assumed to be fulfilled at *every* point in the firm's feasible set, not only at the equilibrium one. The underlying idea is that *any* firm, whether it has market power or not, makes an effort to understand the effect of its actions on market prices. The magnitude of that effect (and whether one exists at all) depends on the market structure and the firm's characteristics. For instance, a "production-infinitesimal" firm which cannot create any financial innovation will have no effect on market prices, and its control group will anticipate that fact correctly. Alternatively, a monopolistic firm will also understand the full effect of its actions on market prices. Of course, this level of rationality is extreme. The model is, nevertheless, useful because it offers a flexible benchmark to analyze (and contrast) the implications of various behavioral assumptions (about the firm) on the equilibrium outcomes of a production economy with incomplete markets.

In contrast to Dréze's (1974) and Grossman and Hart's (1979) objectives, maximization of the C-adjusted value makes the financial policy of the firm relevant. Therefore, the firm can act as a financial innovator by issuing claims against its stock. It should be emphasized here that the model accommodates a large variety of types of firms. In particular, it is not restricted to production units in the usual sense. The firm can be, for example, a financial firm which does not supply any consumption goods to the market, but only has a "technology" for creating securities. Financial intermediaries, as described for example in Bisin (1998) and Carvajal, Rostek, and Weretka (2012) are particular firms, whose "production" set consists of the null vector, and which are owned by a single individual who only cares about, and is endowed with some amount of the date 0 consumption good. More generally, a financial intermediary can be modeled as a firm with a singleton (but non-zero) production set, $Y = \{(y_0, y_1, ...y_S)\}$ where $(y_1, ...y_S)$ is the payoff of a security the intermediary owns and y_0 is the cost it incurs from selling that (or the claims against it) in the market. It should also be pointed out that the model abstracts from more sophisticated transaction cost schemes, bid-ask spreads and taxing issues that are generally associated with issuing securities and focuses instead on the investors' need for better risk-hedging opportunities and general equilibrium price effects as sources of the incentives to innovate.

Although members of the firm's control group are assumed to be risk-averse, and the maximization of the C-adjusted value is consistent with their interests, markets can remain incomplete, and the final allocations Pareto suboptimal (even *constrained* suboptimal) at an equilibrium. The reason for this somewhat surprising result is that the price of consumption can (and, typically, does) change with the availability of more assets. In particular, an agent's equilibrium consumption allocation obtained under an incomplete markets structure can become unaffordable at the equilibrium prices arising under a complete markets structure. In addition, the firm's market value can be lower under complete markets than under incomplete markets (see Carvajal, Rostek, and Weretka (2012) for conditions under which this could happen). Therefore, both the income and the price effects could go against (and dominate) the control group's need for risk-hedging, thus leading to an incomplete market structure at the equilibrium.

The paper is organized as follows. Several examples that illustrate the inadequacy of value maximization as the objective of a monopolistic firm in incomplete markets is presented in section 2. Section 3 introduces the notion of C-adjusted value for a simplified model, in which firm's financial policy is restricted to issuing equity only. The full model, in which the firm has access to different financial policies to finance its production is presented in section 4. Section 5 identifies sufficient conditions for the Modigliani-Miller theorem to hold for C-adjusted values, and describes when shareholders' unanimity can be obtained. Section 6 tackles the problem of the existence and optimality of an equilibrium when the firm maximizes its C-adjusted value, and Section 7 concludes.

2 Shareholders' Interest and Firm's Objective: Some Examples

The following examples illustrate the complexities associated with the production and financing decisions of a privately owned firm which acts in an incomplete markets environment. The first two are slight modifications of an example presented in Duffie (1988)[page 121].

Example 1: Consider an economy that lasts for two periods, $t = 0, 1$, with two possible states at date 1: $s = 1, 2$. There is only one consumption good per date and state. The economy is populated by two consumers/investors and one firm with the following characteristics:

$$u^1(c) = (c_0)^3 c_1 (c_2)^2, \; u^2(c) = (c_0)^3 (c_1)^2 c_2,$$
$$\omega^1 = \omega^2 = (1,0,0), \; \delta^1 = \delta^2 = \frac{1}{2},$$
$$Y = \left\{ y \in \mathbb{R}_- \times \mathbb{R}^2_+ \mid (y_1)^2 + (y_2)^2 \leq -y_0 \right\},$$

where Y is the production set and u^i, ω^i and δ^i denote investor i's utility, endowment of goods and, respectively, initial shares of the firm's stock. The only way to transfer consumption among dates/states is through trade in firm's shares. Thus, markets are incomplete. Investors choose share holdings and consumption for each date/state. It is assumed that they behave competitively and thus maximize their utilities within their budget constraints, taking the price of the firm's shares as given.

If the firm chooses production plan $y = (y_0, y_1, y_2) \in Y$, a market clearing price exists if and only if $y_0 > -2$ and in that case it is equal to $2 + y_0$. At this price, investors do not trade firm's shares, and their indirect utilities are: $U^1_{im}(y) = \frac{1}{64}(2+y_0)^3 y_1 (y_2)^2$, $U^2_{im}(y) = \frac{1}{64}(2+y_0)^3 (y_1)^2 y_2$. The firm's equilibrium net market value is $V(y) = y_0 + (2 + y_0) = 2(1 + y_0)$ for $y_0 < 0$, and 0 for $y_0 = 0$. Figure 1 depicts the pairs of exchange equilibrium indirect utilities for every possible value of $y \in Y \cap \left((-2, 0] \times \mathbb{R}^2_+\right)$.

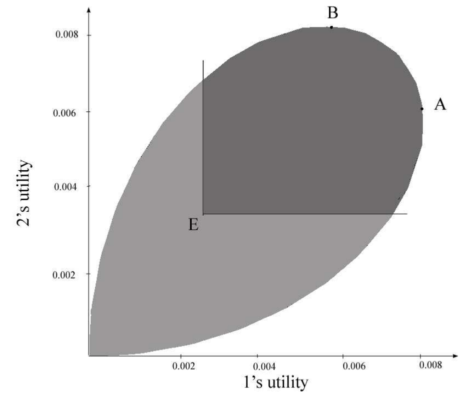

Figure 1: Exchange equilibrium utilities

As the picture shows, the two owners of the firm disagree on the most preferred production plan. If investor 1 were the only one deciding on the firm's production plan, then he would choose $\overline{y}_A = \left(-\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{2}{3}\right) = \arg\max U^1(y)$, which corresponds to point A on the graph. If, instead, investor 2 controlled the firm, she would choose $\overline{y}_B = \left(-\frac{2}{3}, \frac{2}{3}, \frac{\sqrt{2}}{3}\right) = \arg\max U^2(y)$, that is, point B. If they *both* control the firm, reconciling their disagreement becomes the main challenge for defining an appropriate objective for the firm.

Note that the firm's market value is discontinuous, at 0, as a function of the firm's initial investment, $|y_0|$, but increases with lower investments as long as $y_0 \neq 0$. The reason for this discontinuity is the well-known "drop-of-rank" problem pointed out by Hart (1974): as y_0 changes from a negative value to 0, the dimension of the market asset span drops from 1 to 0. Notice also that, as the firm's market value increases (that is, as y_0 increases towards 0) the utilities of the firm's owners converge to 0, indicating that the objective of value maximization is not in the best interest of *any* owner.

The next example illustrates that shareholders may also have strict preferences over the way in which their firm's date-0 investment is financed. The example shows that, even when the different financing choices have no effect on the firm's value (that is, the classical Modigliani-Miller theorem holds), shareholders may (unanimously) strictly prefer one type of financing over another.

Example 2: The economy is the same as in Example 1, except that the firm can now use *both* debt and equity to finance its production plan. Therefore, aside from choosing a production plan, $(y_0, y_1, y_2) \in Y$, the firm can also choose a value of debt, $d \geq 0$, and thus supply 1 unit of the following two securities (at date 0): a risky bond, with payoff $B_s := \min\{d, y_s\}$, and its equity, with payoff $E_s := y_s - B_s$, $s = 1, 2$. If r denotes the price of the bond and v denotes the price of equity, then the bond-equity combination finances the firm's production if $-y_0 \leq r + v$. If $-y_0 < r + v$, then the difference, $D_0 := r + v + y_0$, is paid to the initial shareholders in the form of date-0 dividends.[4] For given choices of the production plan, $(y_0, y_1, y_2) \in Y$, and debt level, $d \geq 0$, and for given bond and equity prices, (r, v), shareholder i's state-by-state budget constraints become

$$\begin{aligned} c_0^i + rb^i + v\theta^i &= 1 + \frac{1}{2}(r + v + y_0), \\ c_s^i &= b^i \cdot B_s + \theta^i \cdot E_s, s = 1, 2, \end{aligned}$$

with b^i and θ^i denoting, respectively, i's bond and equity holdings.

If the firm chooses y and d such that the payoffs of the bond and equity are linearly-independent (and thus markets are complete), shareholders choose the following consumption bundles: $\left(1 + \frac{y_0}{2}, \frac{y_1}{3}, \frac{2y_2}{3}\right)$ and $\left(1 + \frac{y_0}{2}, \frac{2y_1}{3}, \frac{y_2}{3}\right)$. Their indirect utilities in this case are: $U^1_{cm}(y) = \frac{1}{54}\left(2 + y_0\right)^3 y_1 \left(y_2\right)^2$ and $U^2_{cm}(y) = \frac{1}{54}\left(2 + y_0\right)^3 \left(y_1\right)^2 y_2$.

A comparison of these results to those of Example 1 reveals that, for every production plan y, each shareholder is better off under a complete market structure than an incomplete one: $U^1_{cm}(y) > U^1_{im}(y)$, and $U^2_{cm}(y) > U^2_{im}(y)$. Therefore, when given the choice, *both* shareholders would prefer it if their firm financed its date-0 investment by a combination of debt and equity, rather than equity alone. Note that the firm's market value under the complete markets structure remains $V(y) = 2\left(1 + y_0\right)$, as in Example 1, and thus shareholders' strict preference for the debt-equity financing is not due to a higher market value for their firm.

[4] Note that Example 1 corresponds to the firm being forced to choose $d = 0$.

The reason for the conflict between market value maximization and shareholders' interest illustrated by the two examples above is the following: although a higher market value increases shareholders' date-0 wealth (*income effect*), the higher value is also associated with lower date-1 production, which makes that date's consumption more expensive (*price effect*). Since both shareholders want to transfer some of their date-0 wealth into the future, the higher firm market value conflicts with their need for higher consumption at date 1. This is captured by Example 1, which documents an instance in which the price effect dominates the income effect. Example 2, on the other hand, illustrates the importance of the *risk-hedging* effect: even when a firm's choice between debt and equity financing has no effect on its market value, the shareholders may nevertheless be unanimous in their strict preference of a particular financial plan because of the risk-hedging opportunities it offers.

Example 2 may suggest that, as long as the firm's shareholders are risk-averse, they would prefer more risk-hedging opportunities to less, and thus a complete market structure. This intuition is false, as illustrated by Example 3 below: a firm's shareholders may strictly prefer imperfect risk-sharing (that is, incomplete markets) because of the possible negative price and/or income effects associated with completing the markets.

Example 3: Consider now the following modification of the firm's and consumers' characteristics. The production set consists of only one plan:[5] $Y = \{(-1,1,1)\}$. The firm can be interpreted as being a financial intermediary who owns an asset with payoff $(1,1)$ and incurs a cost of 1 unit of date-0 consumption for trading it in the market. Investors' characteristics are:

$$u^1(c) = u^2(c) = c_0 + \ln c_1 + \ln c_2,$$
$$\omega^1 = (3,0,1), \omega^2 = (2,1,0),$$
$$\delta^1 = 1, \delta^2 = 0.$$

There are no other exogenously given assets that can be traded in the market.

Suppose first that the firm chooses to finance its plan by issuing only equity. Given the symmetry of the model, it is immediate to see that, at the equilibrium, the first investor sells half of the firm to the second investor and the price of equity settles at $\frac{8}{3}$. The equilibrium consumption bundles of the two agents are $\left(\frac{10}{3}, \frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{2}{3}, \frac{3}{2}, \frac{1}{2}\right)$, and the utility of the firm's owner is $U^1_{im} = \frac{10}{3} + \ln\frac{3}{4}$.

On the other hand, if investor 1 splits the asset into the Arrow securities $(0,1)$ and $(1,0)$ and sells those, the markets are complete. In this case, the investors fully insure each other, and each ends up consuming the bundle $(1,1)$ at date 1. In this case, the market value of the asset is 2 and investor 1 obtains utility $U^1_{cm} = 3$ at the equilibrium, which is lower than what this investor gets under incomplete markets.

[5]Although this production set violates the condition $\mathbf{0}_{S+1} \in Y$ imposed later for the general model (and used to establish the existence of an equilibrium), the conclusion of Theorem 6.1 remains valid for this example because the production set is, trivially, compact.

Hence, the owner of the firm is better off under an incomplete market structure and does not want to issue enough securities to complete the markets. The reason for this somewhat surprising result is that the price of consumption and the value of the firm change with the availability of more assets. When markets are complete, the unique equilibrium vector of state prices is $(1, 1)$. When markets are incomplete, there are continuously many equilibrium state price vectors, but $(1, 1)$ is not one of them. Because of this price change, agent 1's incomplete markets equilibrium consumption bundle is no longer affordable at the complete markets equilibrium state prices: $\frac{10}{3} + \frac{1}{2} + \frac{3}{2} > 3 + 1 + 1$. Part of the reason is that the firm's value is also lower under complete markets, $1 < \frac{5}{3}$, and thus agent 1's initial wealth is lower under complete markets. In Section 6 I comment more on the robustness of this examples and the forces that drive the result.

When a unanimously approved plan does not exist (as it happens in the first two examples presented above), a weaker but desirable property of an equilibrium outcome is that of being, at least, Pareto non-dominated from the shareholders' point of view. On these grounds, in the context of Example 1, one must eliminate, as possible equilibrium outcomes, all points that do not lie along the frontier between the points A and B. If, for instance, point E were the equilibrium outcome then shareholders would have an incentive to renegotiate to a better alternative in the shaded area of figure 1, and thus the firm would be vulnerable to a take-over. For this reason, point E is viewed as "unstable" and thus undesirable as an equilibrium outcome.

This paper proposes a new objective for the firm, which selects production plans that are efficient from the shareholders' point of view. Clearly, the maximization of a weighted average of the shareholders' indirect utilities would do that as well. However, different utility representations of shareholders' preferences would generate different objectives for the firm and different optimal choices: a highly undesirable feature. The goal is to define an objective for the firm which is *independent* of the utility representations of shareholders' preferences, while still delivering equilibrium outcomes that are stable to shareholders' renegotiation.[6] The paper proposes a way of constructing such an objective.

3 A Benchmark Model

Consider an economy that lasts for two periods, $t = 0, 1$. There are S possible states of nature at date 1, and only one non-storable consumption good per date and state. For any vector $x = (x_0, ..., x_S) \in \mathbb{R}^{S+1}$, I will use the notation $x^{\mathbf{1}} := (x_1, ..., x_S)$ and $x = (x_0, x^{\mathbf{1}})$.

[6] Profit (or value) maximization depends only on the market prices and thus is utility-independent. However, it cannot be a candidate for such an objective because, as proved by Example 1, it is not consistent with the shareholders' preferences.

The economy is populated by I consumers/investors[7] and one firm. Firm's production possibilities are described by the convex and closed subset $Y \subseteq \mathbb{R}_{-} \times \mathbb{R}^{S}$. A typical element of Y is of the form $y = (y_0, y_1, ...y_S)$, where $-y_0$ represents the investment made at date 0, and $\mathbf{y^1} := (y_1, ...y_S)$ is the netput vector of state-contingent production at date 1. It is assumed that $\mathbf{0}_{S+1} \in Y$ and $Y \cap \mathbb{R}_{+}^{S+1} = \{\mathbf{0}_{S+1}\}$, where $\mathbf{0}_{S+1}$ is the zero vector in $\mathbb{R}^{S+1}$.

Investors are characterized by their preferences over state-contingent consumption plans and their endowments of goods and shares in the firm's profits. Investor i is endowed with the vector $\omega^i = (\omega_0^i, \omega_1^i, ..., \omega_S^i) \in \mathbb{R}_{++}^{S+1}$ of state-contingent consumption goods and $\delta^i \in [0, 1]$ shares in firm's profits. His/Her preferences over consumption streams $c = (c_0, c_1, ...c_S) \in \mathbb{R}_{+}^{S+1}$ are represented by the continuously differentiable, increasing in every argument and strictly quasi-concave utility function $u^i : \mathbb{R}_{+}^{S+1} \to \mathbb{R}$.

Let $C \subseteq I$ be the set of investors who control firm's decisions. The set C will be referred to as the *control group.* One could think of C as being, for example, the Board of Directors, which could include shareholders as well as non-shareholders of the firm. Alternatively, one could also think of C as being the group of all stakeholders of the firm, as in Magill, Quinzii, and Rochet (2015). Let $\delta^C := \sum_{i \in C} \delta^i$ be the aggregate initial share holdings by members of the control group (clearly, $\delta^C = 1$ if C stands for the group of all stakeholders). The set C is exogenously given and fixed throughout the paper. While the actual equilibrium outcome will depend on the constituency of the firm's control group (as illustrated by Example 1 in Section 2), the qualitative results of the paper hold for arbitrary control group structures (unless otherwise specified).

For now, it is assumed that the firm's sole decision is to select a feasible production plan $y \in Y$. This assumption is relaxed in Section 4 where it is assumed that the firm takes not only production, but also more complex financial decisions. It is also assumed that the firm's shares are publicly traded, and thus they represent the only financial instrument which allows agents to transfer consumption between dates/states. Firm's shares are available for trade at date 0 and pay dividends at date 1. Short-selling is allowed for up to L shares, where $L \in [0, +\infty]$ is exogenously given, and $L = +\infty$ means that unlimited short sales are allowed.

Investors choose their share holdings and the amount of goods they want to consume in every date/state to maximize their utilities subject to their budget constraints being satisfied at every date and state. It is assumed that the number of investors is large enough, so that their behavior can be approximated by the familiar price taking hypothesis. This means that they act on the belief that their *portfolio and consumption decisions* do not affect market prices. However, those investors who are members of the control group do understand the effect of *firm*'s production choices on its market value. It is therefore assumed that,

[7]By the usual abuse of notation, the same symbol will be used to denote a finite set and the number of its elements. Therefore, $I = \{1, 2, ..., I\}$ also denotes the set of consumers/investors.

for every production plan $y \in Y$, they anticipate the corresponding share price to be given by $\Pi(y) \in \mathbb{R}$ as described below.

Given some production plan, $y \in Y$, and price of shares, v, agent i's optimization problem as an investor is,

$$\left\{\begin{array}{l} \max_{\theta,c} u^i(c) \\ \text{s.t.} \quad c_0 + \theta v = \omega_0 + \delta^i (v + y_0) \\ \qquad c_s = \omega_s + \theta y_s \quad s = 1, .., S \\ \qquad \theta \geq -L, c_0, c_1, ...c_S \geq 0. \end{array}\right. \tag{1}$$

Let $(c^i(y,v), \theta^i(y,v))$ be its solution. An *equilibrium price for the stock-exchange economy corresponding to* y is any real number v that solves the equation: $\sum_{i=1}^I \theta^i(y,v) = 1$. If $\widehat{Y} \subseteq Y$ is a subset of production plans for which a stock-exchange equilibrium exists,[8] the share price anticipated by the control group is assumed to be *rational* in the sense that $\sum_{i=1}^I \theta^i(y, \Pi(y)) = 1$ for every $y \in \widehat{Y}$. Therefore, the control group's rational price anticipation is a particular (measurable) selection, Π, from the set of possible stock-exchange equilibrium prices. This is a simplification made for this Section only. The next Section analyzes a more general case in which the control group holds non-degenerate beliefs over possible equilibrium prices.

Let $c^i(y) := c^i(y, \Pi(y))$ and $\theta^i(y) := \theta^i(y, \Pi(y))$ be investor i's anticipated consumption and, respectively, shareholdings as functions of the firm's production choice, y. Given these anticipations, the preferences of the control group's members over the firm's production plans can be represented by the indirect utilities $V_\Pi^i : \widehat{Y} \to \mathbb{R}$, where $V_\Pi^i(y) := u^i(c^i(y))$, for every $i \in C$. As illustrated by Examples 1 and 2 above, these utilities may not have a common maximizer.

The next definition introduces a notion of efficiency from the point of view of the members of the control group. The basic idea is the following. Suppose that the economy is at a status-quo at which the firm's production plan is $\widetilde{y}$. If there exists an alternative plan, y, and a system of date-0, after-trade side payments which improve every member's utility, then the control group has an incentive to move away from the status-quo plan by adopting y and implementing the transfers. The goal is to identify production plans $\bar{y}$ that are stable to such arrangements. Those production plans are called C-efficient.

Definition 3.1 *A production plan $\widetilde{y} \in \widehat{Y}$ is C-efficient (given the price anticipation Π) if there does not exist a vector $\left(y, (\tau^i)_{i \in C}\right)$ consisting of a production plan $y \in \hat{Y}$ and date-0 transfers $(\tau^i)_{i \in C} \in \mathbb{R}^C$ satisfying:*

1. $\sum \tau^i \leq 0$,
2. $u^i(c^i(y) + \tau^i e_0) \geq V_\Pi^i(\widetilde{y})$ *for every* $i \in C$,
3. $u^j(c^j(y) + \tau^j e_0) > V_\Pi^j(\widetilde{y})$ *for some* $j \in C$,

[8] A more detailed description of such set will be provided in Section 4.

where $e_0 = (1, 0, ..., 0) \in \mathbb{R}^{S+1}$.

Note that this concept of C-efficiency is different from Dréze's (1974) original concept in that investors are allowed to adjust their shares in response to a proposed change in the production plan of the firm. It also differs from Grossman and Hart's (1979) concept in the type of price perceptions used for defining members' preferences over production plans.

As the previous examples suggested, maximization of the firm's market value does not lead, in general, to C-efficient production plans. As shown next, the missing piece is some measure of the risk and price exposure to which members of the control group are subjected under various production plans. The objective proposed below captures the effects of these exposures.

Definition 3.2 *Fix a status-quo production plan* $\widetilde{y} \in \widehat{Y}$ *and the control group's price anticipation* Π*. For every* $y \in \widehat{Y}$*, define the firm's* C*-adjusted value as*

$$\mathcal{V}^C_{\widetilde{y},\Pi}(y) := \delta^C \left(\Pi(y) + y_0\right) + \sum_{i \in C} \theta^i\left(y\right)\left(MRS^i\left(\widetilde{y}\right) y^{\mathbf{1}} - \Pi(y)\right),$$

where $MRS^i(y) := \left(\frac{\partial u^i(c^i(y))}{\partial c_0}\right)^{-1} \cdot \left(\frac{\partial u^i(c^i(y))}{\partial c_s}\right)_{s=1,..,S}$ *denotes the vector of marginal rates of substitution for investor* i *at his anticipated equilibrium consumption corresponding to* y*.*

The C-adjusted value is the firm's value *as perceived* by the members of the control group. It is the sum between the anticipated *market* value of the control group's initial share of the firm, $\delta^C \left(\Pi(y) + y_0\right)$, and the control group's *personalized* value

$$\sum_{i \in C} \theta^i\left(y\right)\left(MRS^i\left(\widetilde{y}\right) y^{\mathbf{1}} - \Pi(y)\right). \tag{2}$$

This last term captures the surplus that members of C derive from firm's production plan *beyond* the market value of their shares. It aggregates the value that members of the control group attach to trading in the firm's equity contract as a *risk-hedging instrument*, as well as the consumption *price effect* that the firm's choice generates. For every i, $MRS^i\left(\widetilde{y}\right) y^{\mathbf{1}}$ is a local estimate of the investor i's valuation of the security with payoff $y^{\mathbf{1}}$, and thus $MRS^i\left(\widetilde{y}\right) y^{\mathbf{1}} - \Pi(y)$ is a measure of the surplus, to investor i, from purchasing one unit of the security $y^{\mathbf{1}}$. Therefore, expression (2) represents, locally, the control group's anticipated aggregate surplus from trading firm's equity.

As shown next, accounting for this surplus in the firm's objective is essential for obtaining C-efficient production plans at the equilibrium. The price perceptions used in Grossman and Hart (1979) imply that investors behave *as if* trading in firm's common stock generates no surplus (or loss) for them. Therefore, the second term of the C-adjusted value (that is, expression (2)) is neglected from the computation of the firm's objective. As shown in Section 5, this is a justified omission if the firm has no market power. However, if the firm has any kind of market power (as a financial innovator or as a producer), the surplus given by formula (2) is different than zero and should be taken into account.

If $C = I$, the C-adjusted value becomes

$$\mathcal{V}^{I}_{\widetilde{y},\Pi}(y) = y_0 + \left[\sum_{i\in I} \theta^i\left(y\right) MRS^i\left(\widetilde{y}\right)\right] y^{\mathbf{1}}.$$

This is the firm's market value computed under a system of state prices equal to the average of the shareholders' marginal rates of substitutions weighted by their final share holdings. This is similar to Dréze's (1974) objective, except that the weights vary with y. Thus, in contrast to both Dréze's (1974) and Grossman and Hart's (1979) objectives[9] , the C-adjusted value is not a linear function of y.

Definition 3.3 *An equilibrium for the production economy (given the control group's price anticipation* Π*) consists of an allocation* $\left(\overline{y}, \left(\overline{c}^i, \overline{\theta}^i\right)_{i\in I}\right) \in \widehat{Y} \times \left(\mathbb{R}^{S+1}_{+} \times \mathbb{R}\right)^I$ *and a price* $\bar{v}$ *for the firm's shares such that the the following conditions are satisfied:*

1. $\left(\overline{c}^i, \overline{\theta}^i\right)_{i\in I}$ *solves* (1) *given the share price* $\bar{v}$*,*
2. $\overline{y} \in \arg\max_{y\in\widehat{Y}} \mathcal{V}^{C}_{\overline{y},\Pi}(y)$*,*
3. $\bar{v} = \Pi(\bar{y})$*,*
4. $\sum_{i\in I} \overline{c}^i = \overline{y} + \sum_{i\in I} \omega^i$.

The equilibrium definition is standard, except for conditions 2 and 3. Condition 2 requires that the firm maximizes its C-adjusted value, rather than the market value, and condition 3 insures that the control group's price anticipation is fulfilled at the equilibrium. As shown by the next theorem, if a firm maximizes its C-adjusted value, the production plan it selects must be efficient from the point of view of its control group's members.

Theorem 3.4 *Every equilibrium production plan is* C*-efficient.*

The proof is given in Section 4 in the context of a more general model and it is therefore omitted here.

For $C = I$ theorem 3.4 delivers a weak first welfare theorem. It says that the equilibria of the production economy are *minimally* constrained efficient[10]

[9] With the notation of this paper, Dréze's (1974) objective is $\mathcal{D}_{\widetilde{y}}(y) = y_0 + \left[\sum_{i\in I} \theta^i\left(\widetilde{y}\right) MRS^i\left(\widetilde{y}\right)\right] y^{\mathbf{1}}$, while Grossman and Hart's (1979) is $\mathcal{GH}_{\widetilde{y}}(y) = y_0 + \left[\sum_{i\in I} \delta^i MRS^i\left(\widetilde{y}\right)\right] y^{\mathbf{1}}$.

[10] The notion of minimal constrained efficiency was first introduced (to the best of my knowledge) by Dierker, Dierker, and Grodal (2005)

in the sense that the equilibrium allocations cannot be improved upon by a social planner who can choose the production plan and redistribute consumption at date 0, *after* using the markets to purchase date-1 consumption for every investor. The formal definition of this efficiency concept follows.

Definition 3.5 *An allocation* $\left(\overline{y}, \left(\overline{c}_0^i, \overline{c}^{i1}\left(\overline{y}\right)\right)_{i\in I}\right)$ *is minimally constrained efficient if and only if:*

1. $\sum_{i\in I} \overline{c}_0^i = \sum_{i\in I} \omega_0^i + \overline{y}_0,$
2. *there does not exist a Pareto superior allocation of the form* $\left(y, \left(c_0^i, c^{i1}\left(y\right)\right)_{i\in I}\right)$ *such that* $\sum_{i\in I} c_0^i = \sum_{i\in I} \omega_0^i + y_0.$

The next Corollary is an immediate consequence of Theorem 3.4.

Corollary 3.6 *If* $C = I$*, every equilibrium is minimally constrained efficient.*

Note that minimal constrained efficiency is weaker than constrained Pareto efficiency. A planner could, potentially, improve upon a minimally constrained efficient allocation by redistributing initial shares (or, equivalently, using *before*-trade date-0 transfers). Therefore, even when $C = I$, equilibria in which the firm maximizes its C-adjusted value are not, in general, constrained Pareto efficient.

Example 1 (continued): To illustrate the results of this section, consider again Example 1 of Section 2. If $C = \{1\}$ (or $C = \{2\}$), the unique C-efficient outcome is point A (respectively B). This corresponds to the production plan $\overline{y}_A = \left(-\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{2}{3}\right)$ (respectively $\overline{y}_B = \left(-\frac{2}{3}, \frac{2}{3}, \frac{\sqrt{2}}{3}\right)$). The firm's C-adjusted value (at $\overline{y}_A$) is $\mathcal{V}_{\overline{y}_A}^{\{1\}}(y) = 2\left(1 + y_0\right) + \frac{1}{3}\left(\sqrt{2}y_1 + 2y_2\right)$, which is maximized at $\left(-\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{2}{3}\right)$. Thus $\overline{y}_A$ is the unique equilibrium production plan if $C = \{1\}$.

Suppose now that $C = \{1, 2\}$. An allocation $\left(\overline{y}, \left(\overline{c}_0^1, \frac{1}{2}\overline{y}_1, \frac{1}{2}\overline{y}_2\right), \left(\overline{c}_0^2, \frac{1}{2}\overline{y}_1, \frac{1}{2}\overline{y}_2\right)\right)$ is minimally constrained efficient if and only if $(\overline{y}, \overline{c}_0^1, \overline{c}_0^2)$ solves

$$\max\left\{ \lambda u^1\left(c_0^1, \frac{1}{2}y_1, \frac{1}{2}y_2\right) + \left(1-\lambda\right) u^2\left(c_0^2, \frac{1}{2}y_1, \frac{1}{2}y_2\right) \;\middle|\; \begin{array}{c} c_0^1 + c_0^2 = 2 + y_0 \\ c_0^1, c_0^2 \geq 0 \end{array} \right\}$$

for some $\lambda \in [0, 1]$. The parametric equation of the minimally constrained efficient frontier (in utility coordinates) is given by

$$\left(\left(\frac{8}{3} - 6\left(y_1\right)^2\right)^3 \frac{y_1}{2}\left(\frac{1}{6} - \frac{\left(y_1\right)^2}{4}\right), \left(6\left(y_1\right)^2 - \frac{4}{3}\right)^3 \frac{\left(y_1\right)^2}{4} \frac{\sqrt{\frac{2}{3} - \left(y_1\right)^2}}{2}\right).$$

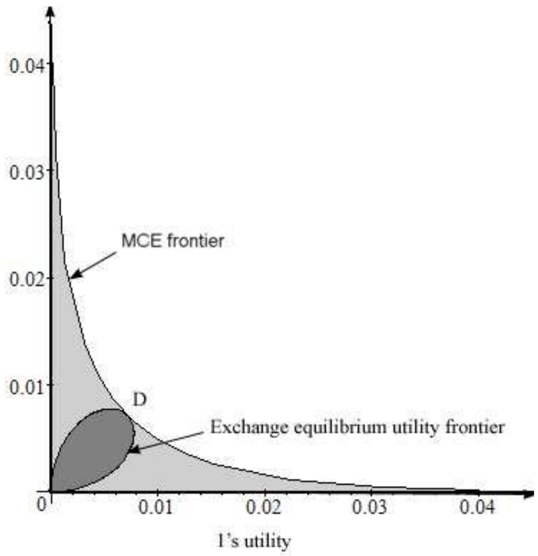

Figure 2: Minimally constrained efficient frontier

As figure 2 illustrates, the minimally constrained efficient frontier intersects the set of exchange equilibrium indirect utilities (computed in Example 1) at point D. Thus D is the only C-efficient outcome, and therefore the only candidate for an equilibrium production plan. Straightforward computations show that $\overline{y}_D = \left(-\frac{2}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ maximizes the adjusted value $\mathcal{V}_{\overline{y}_D}^{\{1,2\}}$, and thus it is an equilibrium production vector.

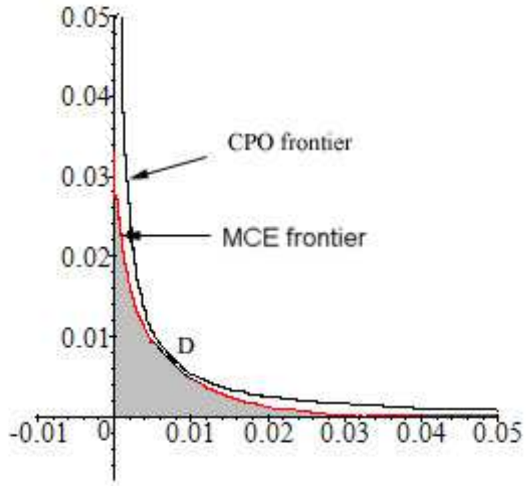

Figure 3: Constrained Pareto optimality vs. minimal constrained efficiency

The constrained Pareto optimal and the minimal constrained efficient frontiers are different, as illustrated by Figure 3. However, since the two frontiers intersect precisely at point D, the equilibrium corresponding to $C = \{1, 2\}$ is *constrained* Pareto optimal. This is an artifact of the particular economy considered here, rather than a general result. In this example, the firm does not influence the markets by making its equity contract available for trade because in every exchange equilibrium consumers choose *not* to trade firm's shares. Therefore, firm lacks any power in the financial markets and it is thus competitive in those markets, according to the definition given in Section 5.

4 The General Model

The model of this section extends the previous one along two directions. First, it allows for more complex financial policies. The firm can finance its production plan not only by selling equity, but also by borrowing in the market and issuing new securities. Second, it generalizes the notion of a price anticipation: the control group may be unsure about the exact equilibrium asset prices and may assign positive probability to one or several possible prices.[11]

4.1 The Economy

As in the previous Section, the economy lasts over 2 periods and it is populated by a large number, I, of consumers/investors and one firm. There are S possible states of the world at date 1. Market participants have the same characteristics as described in Section 3. The market structure is different in that there are $0 \leq J < S$ exogenously given securities available for trade at date 0. Each of them is assumed to be in zero net supply. When $J > 0$, the payoff of security $j \in J$ is described by the (column)[12] vector $A^j = (a_{sj})_{s=1,..,S} \in \mathbb{R}^S$, where a_{sj} represents the units of the state-contingent consumption good paid by security j if state s occurs. The $(S \times J)$-matrix $A := (a_{sj})_{\substack{s=1..S \\ j=1..J}}$ is called the *payoff matrix*. It is assumed throughout that A has full rank.

The firm can finance its production plan by borrowing in the existing security markets, issuing new securities,[13] and/or trade its equity in the market. It is assumed that the firm is allowed to design exactly $N \geq 1$ new securities.[14] The payoffs of the securities that the firm may issue to finance a production plan y are constrained to belong to some exogenously given set

[11] On the one hand, allowing for degenerate as well as non-degenerate beliefs over possible equilibrium prices increases the generality of the model. On the other hand, it makes the equilibrium notion weaker. The discussion following the main Theorem in Section 6 gives more insights into this assumption.

[12] It is assumed that all portfolio payoffs and holdings are column vectors, while prices are row vectors.

[13] Examples of such firm-issued securities are convertibles, warrants, floating-rate debt, zero-coupons, primes and scores.

[14] This assumption is made to simplify the technicalities of the model. However, since the firm is allowed to issue securities with zero payoff in all states, the constraint merely imposes an upper bound on the number of new securities the firm may issue.

$\mathcal{K}(y) \subseteq \mathbb{R}^S$. For example, the firm may be constrained to issue only securities with positive payoff in every state, in which case $\mathcal{K}(y) \subseteq \mathbb{R}^S_+$. Alternatively, $\mathcal{K}(y) \subseteq \{(f(y_1), ..., f(y_S)) \mid f : \mathbb{R} \to \mathbb{R}\}$ means that the firm can issue only derivatives on its equity (such as options on equity). It is assumed that $\mathbf{0}_S \in \mathcal{K}(y)$, so that the firm can always choose to issue no new securities.

The firm has to decide on: (i) the production plan $y \in Y$, (ii) the stream of dividends $D \in \mathbb{R}^S$ to be paid to its shareholders at date 1, (iii) the $(S \times N)$-matrix of payoffs, X, for the N securities it issues, (iv) the portfolio $b^f \in \mathbb{R}^J$ of asset holdings, and (v) whether to trade its shares publicly ($\theta^f = 1$) or remain privately held ($\theta^f = 0$). The vector $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right)$ is called a *production-financial plan.*

To capture, within the same framework, both the publicly traded and the privately held firms, it will be assumed that a security with payoff $\theta^f \cdot D$ is always traded in the market. For every $j \in \{1, ..., J + N + 1\}$, investors are allowed to short-sell up to $L_j \in [0, +\infty]$ units of the security j, with the understanding that, when $L_j = +\infty$, unlimited short-selling of security j is allowed. The same constraints are faced by the firm for its trades in the exogenously given securities. Let $L := (L_1, ..., L_{N+J+1})$ denote the vector of portfolio bounds.

Definition 4.1 *Let $\mathcal{K} : Y \rightrightarrows \mathbb{R}^S$ and $N \geq 1$ be given. A production-financial plan $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right)$ is called feasible if the following conditions are met:*

1. *$y \in Y$*
2. *$\theta^f \cdot D, X^n \in \mathcal{K}(y)$ for every $n \in \{1, ..., N\}$, where X^n denotes the n-th column of X,*
3. *$\theta^f \in \{0, 1\}$ and $b^f_j \geq -L_j$ for every $j \in \{1, ..., J\}$,*
4. *$y^{\mathbf{1}} = D + A \cdot b^f + X \cdot \mathbf{1}_N$, where $\mathbf{1}_N = (1, ..., 1)^t \in \mathbb{R}^N$.*

The set of all feasible plans is denoted by $\mathcal{F}$.

If the firm chooses production-financial plan $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right)$ and $(q, p) \in \mathbb{R}^J \times \mathbb{R}^N$ are the market prices of the already existing securities and, respectively, the firm-issued securities, then the firm's profit at date 0 is

$$D_0 := y_0 + qb^f + p\mathbf{1}_N. \tag{3}$$

This is distributed to firm's shareholders according to their initial shares.

Let $v \in \mathbb{R}$ denote the price of security $\theta^f \cdot D$. If the firm is publicly traded (that is, $\theta^f = 1$), then v is the price of the firm's equity. If the firm is privately held ($\theta^f = 0$) then, at the equilibrium, $v = 0$. Given prices $(q, p, v) \in \mathbb{R}^J \times \mathbb{R}^N \times \mathbb{R}$, consumer i's budget constraint is described by the following equalities:

$$\begin{aligned} c^i_0 + qb^i + pr^i + v\theta^i &= \omega^i_0 + \delta^i (D_0 + v) \\ c^{i\mathbf{1}} = \omega^{i\mathbf{1}} + D \cdot \delta^i(1 - \theta^f) &+ A \cdot b^i + X \cdot r^i + \theta^f \cdot D \cdot \theta^i \\ c^i \geq 0, &\left(b^i, r^i, \theta^i\right) \geq -L, \end{aligned} \tag{4}$$

which, using (3), can be written, equivalently, as:

$$\begin{aligned} c_0^i + qb^i + pr^i + v\theta^i &= \omega_0^i + \delta^i y_0 + \delta^i \left(qb^f + p\mathbf{1}_N + v\right) \quad (5) \\ c^{i\mathbf{1}} &= \omega^{i\mathbf{1}} + D\cdot\delta^i(1-\theta^f) + A\cdot b^i + X\cdot r^i + \theta^f\cdot D\cdot\theta^i \\ c^i &\geq 0, \left(b^i, r^i, \theta^i\right) \geq -L. \end{aligned}$$

Therefore, if the firm chooses policy $\mathcal{P} = (y, D, X, b^f, \theta^f)$, having an initial endowment of δ^i shares is equivalent, in terms of wealth, to receiving $(\delta^i y_0, D\cdot\delta^i(1-\theta^f)) \in \mathbb{R}^{S+1}$ additional units of state-contingent consumption and being endowed with a portfolio $(\delta^i b^f, \delta^i\mathbf{1}_N, \delta^i\theta^f)$ of the $J+N+1$ traded securities. For every feasible production-financial plan, $\mathcal{P}$, let $\mathcal{E}_\mathcal{P}$ denote the (artificial) stock-*exchange* economy in which investors trade in securities with payoffs given by $(A, X, \theta^f\cdot D)$ and their endowments of goods and assets are as described above.

Definition 4.2 *An equilibrium of the stock-exchange economy $\mathcal{E}_\mathcal{P}$ consists of prices $(\overline{q}, \overline{p}, \overline{v})$, consumption allocations $\left(\overline{c}^i\right)_{i\in I}$, and securities holdings, $\left(\overline{b}^i, \overline{r}^i, \overline{\theta}^i\right)_{i\in I}$, such that the following conditions are satisfied:*

1. $\left(\overline{c}^i, \overline{b}^i, \overline{r}^i, \overline{\theta}^i\right)_{i\in I}$ *maximizes agent i's utility within the budget constraint defined by* (5)*, given prices $(\overline{q}, \overline{p}, \overline{v})$,*

2. *all markets clear, that is,*

$$\begin{aligned} \sum_{i=1}^{I} \overline{b}^i &= b^f, \ \sum_{i=1}^{I} \overline{r}^i = 1, \ \sum_{i=1}^{I} \overline{\theta}^i = \theta^f, \\ \sum_{i=1}^{I} \overline{c}^i &= \sum_{i=1}^{I} \omega^i + y. \end{aligned}$$

This economy will be used as a tool to define the firm's objective and the equilibrium of the original economy.

4.2 Firm's Objective and the Equilibrium Concept

The firm's objective is to maximize its *expected* C-adjusted value with respect to the beliefs of its control group over the set of possible equilibrium prices. The control group's beliefs are positive probability measures over the space of asset prices. The beliefs are assumed to be rational in the sense that they assign zero probability to all prices which are not among the equilibrium ones. Their precise definition is given below.

For every $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right) \in \mathcal{F}$, let $\widetilde{\prod}(\mathcal{P}) = \left\{\Pi(\mathcal{P}) = \left(q(\mathcal{P}), p(\mathcal{P}), v(\mathcal{P})\right)\right\} \subseteq \mathbb{R}^{J+N+1}$ be the set of equilibrium asset prices of $\mathcal{E}_\mathcal{P}$. Define

$$\widehat{Y} := \left\{ y \in Y \mid y_s \geq -\min_{i\in I} \frac{\omega_s^i}{\delta^i} + \varepsilon, \forall s = 0, 1, ..., S \right\}, \quad (6)$$

and $\widehat{\mathcal{F}} := \left\{\mathcal{P} = \left(y, D, X, b^f, \theta^f\right) \in \mathcal{F} \mid y \in \widehat{Y}\right\}$, where $\varepsilon > 0$ is arbitrarily small. Then $\widetilde{\prod}(\mathcal{P}) \neq \emptyset$ for every $\mathcal{P} \in \widehat{\mathcal{F}}$ (see the Appendix for a proof). The firm's choices are restricted, in the sequel, to the feasible production-financial plans[15] in $\widehat{\mathcal{F}}$. Let $\mathcal{M}$ be the space of all measurable selections from $\widetilde{\prod}$, endowed with the product topology and the associated Borel σ-algebra, $\mathcal{B}(\mathcal{M})$. Any probability measure with finite support, μ, over $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ is called a *rational (price) belief* for the control group.

As in the model of Section 3, the C-adjusted value is defined as the sum of the fraction of firm's market value received by the members of the control group as initial shareholders and a measure of the members' surplus from their transactions in the stock markets. Some notation is introduced first.

If μ is the control group's price belief, $\Pi = (q, p, v) \in \text{supp}(\mu)$ is an arbitrary price functional in the support of μ, and $\mathcal{P} \in \widehat{\mathcal{F}}$ is an arbitrary production-financial plan, then:

1. $\Pi(\mathcal{P}) = (q(\mathcal{P}), p(\mathcal{P}), v(\mathcal{P})) \in \mathbb{R}^J \times \mathbb{R}^N \times \mathbb{R}$ denotes the vector of security prices for the $J + N + 1$ securities, with $\Pi_l(\mathcal{P})$ denoting the component corresponding to the price of security $l \in \left\{(A^j)_j, (X^n)_n, \theta^f \cdot D\right\}$.

2. $Z^i(\mathcal{P}) \in \mathbb{R}^J \times \mathbb{R}^N \times \mathbb{R}$ is consumer i's vector of optimal security holdings at prices $\Pi(\mathcal{P})$, with $Z^i_l(\mathcal{P})$ denoting i's holdings of security $l \in \left\{(A^j)_j, (X^n)_n, \theta^f \cdot D\right\}$;

3. $Z^C(\mathcal{P}) := \sum_{i \in C} Z^i(\mathcal{P})$;

4. $c^i(\mathcal{P})$ is investor i's optimal consumption at prices[16] $\Pi(\mathcal{P})$;

5. span$(\mathcal{P})$ is the linear subspace (in $\mathbb{R}^S$) generated by the asset payoffs $\left\{(A^j)_j, (X^n)_n, \theta^f \cdot D\right\}$ and it is called the *asset span.*

Definition 4.3 *Fix $\overline{\mathcal{P}} \in \widehat{\mathcal{F}}$ and a pricing functional $\Pi \in$supp(μ). For every $\mathcal{P} \in \widehat{\mathcal{F}}$, the C-adjusted value corresponding to $\mathcal{P}$ is defined as:*

$$\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) := \sum_{i \in C} \delta^i \left(D_0 + \theta^f v(\mathcal{P}) + (1-\theta^f) MRS^i(c^i(\overline{\mathcal{P}})) \cdot D\right) + \mathcal{W}_{\overline{\mathcal{P}}}(\mathcal{P}), \quad (7)$$

where

$$\begin{aligned} D_0 &= y_0 + q(\mathcal{P}) \cdot b^f + p(\mathcal{P}) \cdot \mathbf{1}_N \quad \textit{and} \\ \mathcal{W}_{\overline{\mathcal{P}}}(\mathcal{P}) &= \sum_{i \in C} Z^i(\mathcal{P}) \cdot MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right) \cdot (A, X, \theta^f \cdot D) - Z^C(\mathcal{P}) \cdot \Pi(\mathcal{P}). \end{aligned}$$

[15]This restriction can be relaxed considerably by enlarging the set of "permissible" production plans to a superset of $\widehat{Y}$. However, that level of generality is beyond the scope of this paper.

[16]Clearly, optimal portfolio holdings and consumption allocations depend on both $\mathcal{P}$ and the price selection Π and thus a more precise notation would be $Z^i(\Pi(\mathcal{P}), \mathcal{P})$, $c^i(\Pi(\mathcal{P}), \mathcal{P})$. However, the notational simplification used above should create no confusion.

As in Section 3, the expression

$$Z^i(\mathcal{P})\left(MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\cdot(A,X,\theta^f\cdot D)-\Pi(\mathcal{P})\right)$$

measures i's surplus from trading in the markets, and $\mathcal{W}_{\overline{\mathcal{P}}}(\mathcal{P})$ is the control group's surplus from all its transactions in the stock markets. The first term,

$$\sum_{i\in C}\delta^i\big(D_0+\theta^f v(\mathcal{P})+(1-\theta^f)MRS^i(c^i(\overline{\mathcal{P}}))\cdot D\big)$$

represents the fraction of the firm's value received by the initial shareholders. If the firm is publicly traded (that is, $\theta^f=1$), then that is the share of the firm's *market* value owned originally by the control group. If the firm is privately held ($\theta^f=0$), then the term measures the *personalized* value members of the control group attach to their shares.

According to definition 4.3, the C-adjusted value depends on the control group's set of optimal portfolios. If there are redundant securities in the market, the optimal portfolios may not be unique. Investors who trade in at least one redundant security are, in fact, indifferent among a continuum of portfolios. To be well-defined, the C-adjusted value has to be invariant to the choice of optimal portfolios. The following proposition shows that this is indeed the case.

Proposition 4.4 *The C-adjusted value is well-defined and depends on $\overline{\mathcal{P}}$ and $\mathcal{P}$ only through their corresponding production plans and asset spans.*

Proof. Let $\Pi\in\operatorname{supp}(\mu)$, $(\mathcal{P},\overline{\mathcal{P}})\in\widehat{\mathcal{F}}\times\widehat{\mathcal{F}}$ and $(Z^i(\mathcal{P}))_{i\in C}$ a vector of optimal portfolios for the members of the control group. Using every investor i's budget constraint (4) for all $i\in C$ and rearranging terms in (7) we obtain:

$$\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})=\sum_{i\in C}\left(c^i_0(\mathcal{P})-\omega^i_0\right)+\sum_{i\in C}MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\left(c^{i\mathbf{1}}(\mathcal{P})-\omega^{i\mathbf{1}}\right). \tag{8}$$

Since u^i is strictly quasi-concave, the optimal consumption stream $c^i(\mathcal{P})$ is unique and depends only on the asset span and the production plan chosen under plan $\mathcal{P}$. Thus, the value of $\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})$ is independent of the choice of the members' optimal portfolios and depends on $\mathcal{P}$ and $\overline{\mathcal{P}}$ only through their corresponding production plans and asset spans. ∎

The firm's objective is to maximize its *expected* C-adjusted value, given its control group's beliefs about the market prices. This means, $\max_{\mathcal{P}} E^\mu(\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}))=\int_{\mathcal{M}}\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\,d\mu(\Pi)$. Note that the integral $\int_{\mathcal{M}}\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\,d\mu(\Pi)$ is well-defined because, given the product topology on $\mathcal{M}$, the mapping $\Pi\longmapsto\Pi(\mathcal{P})$ from $\mathcal{M}$ to $\mathbb{R}^{J+N+1}$ is continuous (and thus integrable) for every $\mathcal{P}$. Moreover, consumers' demands for goods are continuous functions of prices.

Definition 4.5 *An equilibrium for the production economy consists of rational beliefs for the control group,* μ*, a production-financial plan for the firm,* $\overline{\mathcal{P}} \in \widehat{\mathcal{F}}$*, consumption-portfolio allocations for the investors,* $\left(\overline{c}^i, \overline{b}^i, \overline{r}^i, \overline{\theta}^i\right)_{i \in I}$*, and a vector of prices,* $(\overline{q}, \overline{p}, \overline{v})$*, such that:*

1. *consumption-portfolio allocations are optimal within each investor's budget constraint, given* $(\overline{q}, \overline{p}, \overline{v})$*, (that is,* $\left(\overline{c}^i, \overline{b}^i, \overline{r}^i, \overline{\theta}^i\right)_{i \in I}$ *solves* (4)*),*

2. *beliefs are consistent with the equilibrium prices (that is, there exists* $\overline{\Pi} \in$ *supp*(μ) *such that* $(\overline{q}, \overline{p}, \overline{v}) = \overline{\Pi}(\overline{\mathcal{P}})$*),*

3. *the firm maximizes its expected* C*-adjusted value at* $\overline{\mathcal{P}}$*, given the beliefs* μ*, i.e.,* $\overline{\mathcal{P}} \in \arg\max_{\mathcal{P}} \int_{\mathcal{M}} \mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\, d\mu(\Pi)$*,*

4. *all markets clear.*

The equilibrium defined above is a generalization of the concept introduced in Section 3, which corresponds to $\mathcal{K}(y) = \{y^1, \mathbf{0}_S\}$, $J = 0, N = 1$ and μ being a degenerate probability distribution with singleton support. In this case, firm's decision can be reduced to a choice between the production-financial plans $\mathcal{P} = (y, y^{\mathbf{1}}, \mathbf{0}, 1)$ and $\mathcal{P} = (y, y^{\mathbf{1}}, \mathbf{0}, 0)$, with $y \in \widehat{Y}$.

4.3 C-efficient Policies

A production-financial plan is dominated from the point of view of the firm's control group if there exists another production-financial plan and a system of date-0 transfers that lead to an improvement in every member's utility for *every* price in the support of the group's beliefs. Those production-financial plans that are not dominated in this sense are called C-efficient. The formal definition follows.

Definition 4.6 *Let* μ *be the control group's beliefs over* $\mathcal{M}$*. A production-financial plan* $\overline{\mathcal{P}} \in \widehat{\mathcal{F}}$ *is called* C*-efficient (given* μ*) if there does not exist a production-financial plan* $\mathcal{P} \in \widehat{\mathcal{F}}$ *together with a system of date-*0 *transfers* $(\tau^i)_{i \in C}$ *such that the following are true for every* $\Pi \in$ *supp*(μ)*:*

1. $\sum \tau^i \leq 0$,

2. $u^i\left(c^i(\mathcal{P}) + \tau^i e_0\right) \geq u^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)$ *for every* $i \in C$*,*

3. $u^j\left(c^i(\mathcal{P}) + \tau^i e_0\right) > u^j\left(c^j\left(\overline{\mathcal{P}}\right)\right)$ *for some* $j \in C$*.*

Theorem 4.7 *Every equilibrium production-financial plan is* C*-efficient.*

Proof. Let $\overline{\mathcal{P}}$ be an equilibrium production-financial plan corresponding to belief μ. If it is not C-efficient then there exists an alternative plan, $\mathcal{P}$, and a system of transfers at time 0, $(\tau^i)_{i\in C}$, such that $\sum_{i\in C}\tau^i \leq 0$ and, for every $\Pi \in \mathrm{supp}(\mu)$, $u^i\left(c^i(\mathcal{P})+\tau^i e_0\right) \geq u^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)$ $\forall i \in C$, with at least one strict inequality for some $j \in C$.

Strict quasi-concavity of u^i implies that:

$$c_0^i(\mathcal{P})+\tau^i-c_0^i\left(\overline{\mathcal{P}}\right)+MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\left[c^{i\mathbf{1}}(\mathcal{P})-c^{i\mathbf{1}}\left(\overline{\mathcal{P}}\right)\right] \geq 0,$$

for every $i \in C$, with strict inequality for some $j \in C$.

Adding up over $i \in C$ and using $\sum_{i\in C}\tau^i \leq 0$ yields

$$\sum_{i\in C}\left(c_0^i(\mathcal{P})-c_0^i\left(\overline{\mathcal{P}}\right)\right)+\sum_{i\in C}MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\left[c^{i\mathbf{1}}(\mathcal{P})-c^{i\mathbf{1}}\left(\overline{\mathcal{P}}\right)\right] > 0,$$

or, equivalently,

$$\sum_{i\in C}\left(c_0^i(\mathcal{P})-\omega_0^i\right)+\sum_{i\in C}MRS^i\left(c^i\left(\overline{\mathcal{P}}\right)\right)\left[c^{i1}(\mathcal{P})-c^{i1}\left(\overline{\mathcal{P}}\right)\right] > \sum_{i\in C}\left(c_0^i\left(\overline{\mathcal{P}}\right)-\omega_0^i\right).$$

Using (8), the above inequality becomes $\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) > \mathcal{V}^C_{\overline{\mathcal{P}}}\left(\overline{\mathcal{P}}\right)$. Since the support of μ is finite, integrating this last inequality over $\Pi \in \mathrm{supp}(\mu)$ leads to $E^\mu\left(\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P})\right) > E^\mu\left(\mathcal{V}^C_{\overline{\mathcal{P}}}\left(\overline{\mathcal{P}}\right)\right)$, which contradicts the expected C-adjusted value maximization condition. ■

5 Shareholders' Unanimity and Perfect Competition

At its most basic level, perfect competition is understood as that strategic situation in which no individual player has the ability to influence prices. Traditionally, that has been identified with the existence of a large number of players (a non-atomic space of agents, more precisely), an intuition that was formalized, in the context of pure-exchange economies, by Aumann (1964). In a complete markets environment with finitely many commodities, the existence of a non-atomic space of firms (whose production sets and ownership structures satisfy certain conditions) is enough to guarantee that no firm has any effect on market prices (see Hildenbrand (1974) and Bejan and Bidian (2012)). Thus, the appropriate behavioral assumption that captures the idea of perfect competition under complete markets (even when the model is finite) is that firms take prices as given and maximize their value. Market value maximization is, in this case, unanimously supported by all shareholders of the firm.

The goal of this section is to derive a similar assumption to capture perfectly competitive behavior in incomplete markets. In such a framework, having a non-atomic space of firms is no longer enough to guarantee that no firm has an effect on prices. That is because when markets are incomplete, a firm may affect the market prices through *two* channels: its production choices, and its financial policy. When short selling is allowed, an infinitesimal firm can still have a significant impact on market prices by introducing a new security which enlarges the asset span (see Essay II in Kreps (1987) and Allen and Gale (1991)). The failure of "large numbers" to achieve perfect competition in this environment is related to the notion of market *thickness* pointed out by Gretsky and Ostroy (1985) (see also Ostroy and Zame (1994) and Rustichini and Yannelis (1991)) in the context of economies with infinitely many commodities. The idea put forward by that literature is that, for perfect competition to prevail, markets need to be both *physically* thick (i.e., large number of potential buyers and sellers) and *economically* thick (i.e., physical differences are superseded by conditions of economic substitutability among commodities). When securities are traded in an incomplete markets environment, two securities whose payoffs may be "close" in terms of the Euclidian distance on $\mathbb{R}^S$ can, typically, be very "far apart" as substitutes in terms of fulfilling investors' risk-hedging needs. This violates the second thickness requirement and it is at the root of the reason why a continuum of firms is not enough to achieve perfect competition in incomplete markets (when short-selling is allowed).

Rather than trying to construct the appropriate market structure that would lead to perfect competition in this environment, I am following a line of reasoning similar to that of Makowski (1983) and, still focusing on the decision problem of one firm, I give specific conditions for *that* firm to be negligible as a production unit, or as a financial innovator, or both. As in Makowski (1983), perfect competition (seen as equivalent to the absence of market power) is therefore described here via exact definitions, rather than derived from primitive assumptions. Unlike in Makowski's (1983) paper, unlimited short-selling is allowed in this Section (that is, $L_j = +\infty$ for every $j = 1, ..., J + N + 1$), and a distinction will be made between the market power that comes from production (price effects due to changes in the production plan which trigger changes in the aggregate consumption of the state-contingent goods) and the market power that comes from altering the asset span.

Let μ be some rational belief for the control group, which will remain fixed throughout this Section.

Definition 5.1 *A firm is said to be production-negligible if, for every feasible plans $\mathcal{P}$ and $\mathcal{P}'$ for which $span(\mathcal{P}) = span(\mathcal{P}')$, it is true that $\Pi(\mathcal{P}) = \Pi(\mathcal{P}')$, for every $\Pi \in supp(\mu)$.*

This condition is analogous to Makowski's (1983) condition (2a) and captures the idea that the firm's control group members perceive their firm's production capacity to be too small relative to the size of the economy to have any significant effect on market prices. Unless additional conditions are imposed, being production-negligible does not imply, in general, that the C-adjusted value coincides with the market value, or that the firm behaves as a pricetaker. A production-negligible firm may still have a significant effect on the markets through its financial innovation. Conditions that prevent such market power are derived below.

For every $y \in Y$ let $\mathcal{P}_y^0 = (y, y^{\mathbf{1}}, \mathbf{0}_{SN}, \mathbf{0}_J, 0)$ and $\mathcal{E}_{\mathcal{P}_y^0}$ be the associated stock-exchange economy in which only the J exogenously-given assets are traded, the firm has committeed to producing y, but its stock is not traded in the market (if $J = 0$, $\mathcal{E}_{\mathcal{P}_y^0}$ is the trivial autarky economy in which each investor i consumes $\omega^i + \delta^i y$). Let $\widehat{Y}$ be defined as in (6). By Lemma 8.1, $\mathcal{E}_{\mathcal{P}_y^0}$ has a stock-exchange equilibrium for every $y \in \widehat{Y}$.

Definition 5.2 *A firm is said to be competitive in the financial markets if, for every $y \in \widehat{Y}$, $\mathcal{P} = (y, D, X, b^f, \theta^f) \in \widehat{\mathcal{F}}$ and $\Pi \in supp(\mu)$, $\Pi_l(\mathcal{P}) = \Pi_l(\mathcal{P}_y^0)$ for every $l \in (A^j)_{j \in J}$, and $c^i(\mathcal{P}) = c^i(\mathcal{P}_y^0)$ for every $i \in I$.*

According to the definition, if the firm is competitive in the financial markets, any security it may issue has no effect on the equilibrium prices of the exogenously-given securities or the equilibrium consumption allocations. Therefore, investors are indifferent between having the markets for the firm-specific securities open or closed. The firm's financial policy is thus irrelevant, in the sense that it does not affect the consumers' welfare or the firm's C-adjusted value. This is made precise by the following proposition, which can be seen as an analogue of the Modigliani-Miller theorem to this environment.

Proposition 5.3 *Fix some $\Pi \in supp(\mu)$ and $\overline{\mathcal{P}} \in \widehat{\mathcal{F}}$. If the firm is competitive in the financial markets, then $c^i(\mathcal{P}) = c^i(\mathcal{P}')$ and $\mathcal{V}_{\overline{\mathcal{P}}}(\mathcal{P}) = \mathcal{V}_{\overline{\mathcal{P}}}(\mathcal{P}')$, for every $i \in I$ and every $\mathcal{P}, \mathcal{P}' \in \widehat{\mathcal{F}}$ with $\mathcal{P} = (y, D, X, b^f, \theta^f)$, $\mathcal{P}' = (y, D', X', b^{f'}, \theta^{f'})$.*

Proof. Since the firm is competitive in the financial markets, $q(\mathcal{P}) = q(\mathcal{P}') = q(\mathcal{P}_y^0)$. Moreover, since $(c^i(\mathcal{P}))_{i \in I}$ and $(c^i(\mathcal{P}'))_{i \in I}$ can be supported as equilibrium consumption allocations for the economy $\mathcal{E}_{\mathcal{P}_y^0}$ at prices $q(\mathcal{P}_y^0)$, it must be that $c^i(\mathcal{P}) = c^i(\mathcal{P}')$ for every $i \in I$, and there exist portfolio holdings $\tilde{b}^i \in \mathbb{R}^J$ such that, for every $i \in I$,

$$\begin{aligned} c_0^i(\mathcal{P}') - \omega_0^i = c_0^i(\mathcal{P}) - \omega_0^i &= \delta^i y_0 - q(\mathcal{P}_y^0) \cdot \tilde{b}^i, \\ c^{i\mathbf{1}}(\mathcal{P}') - \omega^{i\mathbf{1}} = c^{i\mathbf{1}}(\mathcal{P}) - \omega^{i\mathbf{1}} &= \delta^i y^{\mathbf{1}} + A \cdot \tilde{b}^i. \end{aligned}$$

Using equation (8) and the equilibrium condition $MRS^i(c^i(\overline{\mathcal{P}})) \cdot A = q(\overline{\mathcal{P}}) = q(\mathcal{P}_{\bar{y}}^0)$, one can write

$$\mathcal{V}_{\overline{\mathcal{P}}}^C(\mathcal{P}') = \mathcal{V}_{\overline{\mathcal{P}}}^C(\mathcal{P}) = \sum_{i \in C} \delta^i \big(y_0 + MRS^i\left(c^i(\overline{\mathcal{P}})\right) \cdot y^{\mathbf{1}}\big) + \tilde{b}^C\big(q(\mathcal{P}_{\bar{y}}^0) - q(\mathcal{P}_y^0)\big), \quad (9)$$

which completes the proof. ■

If the firm is competitive in the financial markets then, for every feasible production-financial plan $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right)$, there exists an equilibrium of $\mathcal{E}_{\mathcal{P}}$ in which investor i's holdings of every firm-issued security (including its equity, when traded) is δ^i. Since holding a share δ^i of the firm is equivalent to holding that fraction of the firm's portfolio, this implies that, if the firm is competitive in the financial markets, investors' net trades in the securities issued by the firm are zero. Therefore, financial market competitiveness, as described here, is similar to condition (2b) in Makowski (1983).

Note that being competitive in the financial markets does *not* mean that the firm's decisions have no influence on the markets. Changes in the production plan can still affect prices if the firm is not production-negligible.

Definition 5.4 *A firm is said to be competitive in all markets (or simply, competitive) if it is competitive in the financial markets and production-negligible.*

If the firm is competitive then the last term of equation (9) is equal to zero and thus the C-adjusted value becomes

$$\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) = \sum_{i \in C} \delta^i \big(y_0 + MRS^i \left(c^i(\overline{\mathcal{P}})\right) \cdot y^{\mathbf{1}}\big), \tag{10}$$

which coincides with the firm's objective proposed by Grossman and Hart (1979). This positions the C-adjusted value as a generalization of Grossman and Hart's (1979) objective: the two coincide when the firm is competitive and the control group includes all initial shareholders of the firm. This also provides an argument of why Grossman and Hart's (1979) objective is an appropriate firm objective to use when modeling perfect competition in an incomplete markets framework. It should be emphasized, that firm's competitiveness does *not* imply that GH-price perceptions are correct everywhere. They only generate some preferences for the control group's members whose peaks coincide with their actual most preferred production plans. In other words, although there is no clear justification for what Grossman and Hart (1979) call "competitive price perceptions" (see more details on this critique in Magill and Quinzii (1996, chapter 6)), the firm objective that Grossman and Hart (1979) derive using those price perceptions lies on a more solid foundation.

Whenever the firm is not competitive, following Grossman and Hart's (1979) objective can generate less social welfare for the members of the control group than by maximizing the C-adjusted value. This is illustrated by Example 2 of Section 2. The firm of that example is not competitive in the financial markets: the equilibrium consumption allocation $\left(\left(1 + \frac{y_0}{2}, \frac{y_1}{3}, \frac{2y_2}{3}\right), \left(1 + \frac{y_0}{2}, \frac{2y_1}{3}, \frac{y_2}{3}\right)\right)$ corresponding to the production-financial plan $\mathcal{P} = (y, E, B, 1)$ cannot be supported by the autarky economy generated by $\mathcal{P}^0_y$. As argued in Section 2, both shareholders *strictly* prefer a combination of debt *and* equity (and thus complete markets) to finance *every* production plan. A comparison of the shareholders' indirect utilities between Examples 1 and 2 (derived in Section 2) reveal that those are scaled versions of each other. Therefore, the production plan that maximizes the firm's C-adjusted value (for $C = \{1, 2\}$), must be the same between the two examples. As argued in Section 3, the firm of Example 1 chooses production plan $\left(-\frac{2}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$. Therefore, the firm of Example 2 chooses the same plan, but it finances that by issuing *both* debt and equity. Since the Grossman and Hart's (1979) objective does not differentiate between the various financial policies that support a particular production plan, it leads to multiple equilibria, which are Pareto ranked.

It should be noted that competitiveness, by itself, is not sufficient to guarantee that the C-adjusted value and the market value coincide, nor is it sufficient for obtaining the unanimity of shareholders with regards to the choice of a production-financial plan. The following proposition shows that the absence of production-specific risk (that is $\widehat{Y}^1 := \{y^1 \in \mathbb{R}^S \mid y = (y_0, y^1) \in \widehat{Y}\} \subseteq \text{span}(A)$, a condition known in the literature as *spanning*) is sufficient to guarantee that the objective of market value maximization is unanimously approved by the shareholders of a *competitive* firm.

Proposition 5.5 *If the firm is competitive and* $\widehat{Y}^{\mathbf{1}} := \{y^{\mathbf{1}} \in \mathbb{R}^S \mid y = (y_0, y^{\mathbf{1}}) \in \widehat{Y}\} \subseteq span(A)$ *then, for every* $\overline{\mathcal{P}}, \mathcal{P} \in \widehat{\mathcal{F}}$,

$$\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) = \delta^C\big(y_0 + \Pi_{y^{\mathbf{1}}}(\mathcal{P})\big).$$

Moreover, for every i *with* $\delta^i > 0$ *and every* $\mathcal{P}, \mathcal{P}' \in \widehat{\mathcal{F}}$, $u^i(c^i(\mathcal{P})) > u^i(c^i(\mathcal{P}'))$ *holds if and only if* $y_0 + \Pi_{y^{\mathbf{1}}}(\mathcal{P}) > {y_0}' + \Pi_{y^{\mathbf{1}\prime}}(\mathcal{P}')$.

Proof. If the firm is competitive then its C-adjusted value is given by (10). Since $y^{\mathbf{1}} \in span(A)$, $MRS^i\left(c^i(\overline{\mathcal{P}})\right) \cdot y^{\mathbf{1}} = MRS^j\left(c^j(\overline{\mathcal{P}})\right) \cdot y^{\mathbf{1}} = \Pi_{y^{\mathbf{1}}}(\overline{\mathcal{P}})$ for every $i, j \in I$ and all $y \in \widehat{Y}$. On the other hand, the assumption of competitiveness also implies that

$$\Pi_{y^{\mathbf{1}}}(\overline{\mathcal{P}}) = \Pi_{y^{\mathbf{1}}}(\mathcal{P}^0_{\bar{y}}) = \Pi_{y^{\mathbf{1}}}(\mathcal{P}^0_{y}) = \Pi_{y^{\mathbf{1}}}(\mathcal{P}),$$

and thus

$$\mathcal{V}^C_{\overline{\mathcal{P}}}(\mathcal{P}) = \delta^C\big(y_0 + \Pi_{y^{\mathbf{1}}}(\mathcal{P})\big).$$

For every $\mathcal{P} \in \widehat{\mathcal{F}}$, let $MV(\mathcal{P}) := y_0 + \Pi_{y^{\mathbf{1}}}(\mathcal{P})$ be the firm's market value when the production-financial plan $\mathcal{P}$ is chosen. Let now $i \in I$ be such that $\delta^i > 0$, and let $\mathcal{P}, \mathcal{P}' \in \widehat{\mathcal{F}}$ be such that $u^i(c^i(\mathcal{P})) > u^i(c^i(\mathcal{P}'))$. By Proposition 5.3, it can be assumed, without loss of generality, that $\mathcal{P} = (y, y^{\mathbf{1}}, \mathbf{0}_{SN}, \mathbf{0}_J, 0)$ and $\mathcal{P}' = (y', y^{\mathbf{1}\prime}, \mathbf{0}_{SN}, \mathbf{0}_J, 0)$. Then, for $\mathcal{X} \in \{\mathcal{P}, \mathcal{P}'\}$,

$$\begin{aligned} c^i_0(\mathcal{X}) &= \omega^i_0 + \delta^i y_0 - q(\mathcal{X}) b^i(\mathcal{X}), \\ c^{i\mathbf{1}}(\mathcal{X}) &= \omega^{i\mathbf{1}} + \delta^i y^{\mathbf{1}} + A \cdot b^i(\mathcal{X}), \end{aligned} \tag{11}$$

for some $b^i(\mathcal{X}) \in \mathbb{R}^J$.

Since u^i is quasi-concave, $u^i(c^i(\mathcal{P})) > u^i(c^i(\mathcal{P}'))$ implies that

$$c^i_0(\mathcal{P}) + MRS^i(c^i(\mathcal{P}')) \cdot c^{i\mathbf{1}}(\mathcal{P}) > c^i_0(\mathcal{P}') + MRS^i(c^i(\mathcal{P}')) \cdot c^{i\mathbf{1}}(\mathcal{P}')$$

which, using (11) and the assumption of competitiveness, delivers $y_0+\Pi_{y^{\mathbf{1}}}(\mathcal{P}) > {y_0}' + \Pi_{y^{\mathbf{1}\prime}}(\mathcal{P}')$.

Assume now that $MV(\mathcal{P}) > MV(\mathcal{P}')$. As before, it can be assumed, without loss of generality, that $\mathcal{P} = (y, y^{\mathbf{1}}, \mathbf{0}_{SN}, \mathbf{0}_J, 1)$ and $\mathcal{P}' = (y', y^{\mathbf{1}'}, \mathbf{0}_{SN}, \mathbf{0}_J, 1)$. For $\mathcal{X} \in \{\mathcal{P}, \mathcal{P}'\}$, define

$$\mathcal{BC}^i(\mathcal{X}) := \left\{c \in \mathbb{R}^{S+1}_+ \mid c^{i\mathbf{1}} - \omega^{i\mathbf{1}} \in span(A), \Pi_{c^{i\mathbf{1}}-\omega^{i\mathbf{1}}}(\mathcal{X}) \le \omega^i_0 + \delta^i MV(\mathcal{X})\right\}.$$

Then

$$u^i(c^i(\mathcal{X})) = \max\{u^i(c^i) \mid c^i \in \mathcal{BC}^i(\mathcal{X})\}. \tag{12}$$

Since $MV(\mathcal{P}) > MV(\mathcal{P}')$ and $\Pi(\mathcal{P}) = \Pi(\mathcal{P}')$, $\mathcal{BC}^i(\mathcal{P}) \supsetneq \mathcal{BC}^i(\mathcal{P}')$ and thus, using (12), $u^i(c^i(\mathcal{P})) > u^i(c^i(\mathcal{P}'))$, which completes the proof. ■

6 Existence and Optimality of Equilibria

The next theorem identifies conditions under which an equilibrium for the incomplete markets production economy exists. The existence result in Kelsey and Milne (1996) cannot be applied here because the firm's preference relation, as described by its C-adjusted value, fails to satisfy the convexity property required by their theorem.

Theorem 6.1 *Assume that the production set and consumers' preferences satisfy the assumptions of Section 3. If $L \ll +\infty$ and the mapping $y \mapsto \mathcal{K}(y)$ is upper hemi-continuous with compact-values, then an equilibrium in which the firm maximizes the expected C-adjusted value exists.*

The full details of the proof are relegated to the Appendix. The main idea is the following. For any rational belief of the control group, μ, one can construct a two-player, normal-form, imitation game, Γ_μ, such that its Nash equilibria are in a one-to-one and onto relationship with the equilibria of the production economy which are consistent with μ. Each player in Γ_μ chooses a production-financial plan from the firm's feasible set. The first player's goal is to choose his plan to maximize the firm's C-adjusted value computed at the status-quo plan chosen by the second player, while the second player's goal is to "guess" the choice of the first player. Using then the main theorem in Simon and Zame (1990), it is shown that there must exist a rational belief μ such that Γ_μ has a Nash equilibrium.

Theorem 6.1 guarantees that an equilibrium with C-adjusted value maximizing firms exists for *some* belief of the control group regarding market clearing prices. That belief might be degenerate (assigning full probability to a single equilibrium pricing functional) or non-degenerate. The theorem cannot be strengthened further to guarantee the existence of an equilibrium for some *degenerate* (or for an arbitrarily *fixed*) belief. The main reason for the difficulty of obtaining such results is that, typically, the equilibrium price correspondence is not convex-valued, and thus it may not admit a continuous selection. Allowing for the full set of rational beliefs (degenerate and non-degenerate) introduces some convexity in the model, on which the existence result relies. Besides helping with the technicalities of the model, allowing for non-degenerate beliefs seems also natural in an environment in which the equilibrium prices are not unique.

The importance and limitations imposed by (some of) the hypotheses of Theorem 6.1 deserve further comment. The assumptions on the production set and consumers' preferences are standard in the general equilibrium literature and do not require further discussion. The assumption of limited short-selling is also standard. It is needed here to prevent agents' portfolios from becoming unbounded when the firm's policy causes a drop in the rank of the asset span (see Hart (1979) for a similar problem in a different context). The role of the requirement that the correspondence $y \mapsto \mathcal{K}(y)$ be upper hemi-continuous with compact values is less transparent and deserves further comment. From a technical point of view, the assumption is needed to guarantee that the firm's choice set is "well-behaved." Although the requirement does impose some limitations on the financial policies the firm can choose (for example, it does not allow $\mathcal{K}(y) = \mathbb{R}^S_+$), it is still a mild assumption in the sense that it does *not* exclude the "typical" financial instruments most firms use to finance their investments. In particular, the ubiquitous debt-equity financing strategy satisfies the requirement, as argued below.

Indeed, a firm's choice of financing its production by issuing debt and selling its equity in the market can be modeled by taking

$$\mathcal{K}(y) := \left\{x \in \mathbb{R}^S \mid \exists d \in \mathbb{R}_+ \text{ s.t. } x_s = \min\{d, y_s\}, \forall s = 1, ..., S\right\}.$$

Clearly, the set $\mathcal{K}(y)$ is compact for every $y \in Y$. Since $\widehat{Y}$ is compact, $\bigcup_{y \in \widehat{Y}} \mathcal{K}(y)$ can be included in a compact set as well and thus, to prove that $\mathcal{K}(\cdot)$ is upper hemi-continuous, it is enough to show that it has closed graph. Let therefore $y^n \to \hat{y}$ in $\widehat{Y}$ and $x^n \in \mathcal{K}(y^n)$, such that $x^n \to \hat{x}$. Let also $(d^n)_n \subseteq \mathbb{R}_+$ be such that $x^n_s = \min\{d^n, y^n_s\}$, for every $s = 0, ..., S$ and every $n \in \mathbb{N}$. Since $y^n \to \hat{y}$ and $x^n \to \hat{x}$, it can be assumed, without loss of generality, that $(d^n)_n$ contains a convergent subsequence. To simplify notation, assume that $d^n \to \hat{d} \geq 0$. Then, continuity of the function $(a, b) \mapsto \min\{a, b\}$ implies that $\hat{x}_s = \min\{\hat{d}, \hat{y}_s\}$ for every $s = 0, ..., S$, and thus $\hat{x} \in \mathcal{K}(\hat{y})$.

An important characteristic of the equilibrium in which the firm maximizes its C-adjusted value is that the policy used by the firm to finance its production becomes relevant: two policies that finance the same production plan but generate different asset spans may give different utilities to the members of the control group and thus may be ranked differently. This implies that an analogue of the Modigliani-Miller theorem for C-adjusted values (as opposed to market values) does not necessarily hold unless, as proved by Proposition 5.3, the firm has no market power in the financial markets. One implication of this result is that, as illustrated by Example 3 of Section 2, an endogenously incomplete financial market structure might arise at the equilibrium. Given that markets can be incomplete at an equilibrium, it should not be surprising that, except in special circumstances, equilibrium allocations of incomplete markets production economies in which the firm maximizes its C-adjusted value are Pareto suboptimal. One may wonder how robust such result is. While sufficient conditions for endogenous market incompleteness are difficult to obtain for the general model (and beyond the scope of this paper), the following specific framework provides some insight into the robustness of the result and identifies a specific characteristic of investors' preferences, their *prudence*, that appears to be responsible for endogenous market incompleteness.

Consider an economy with two states of the world at date 1, one firm and two consumers. Consumers have identical utilities given by

$$U^1(c) = U^2(c) = c_0 + u(c_1) + u(c_2), \tag{13}$$

where $u : \mathbb{R} \to \mathbb{R}$ is an arbitrary function that is increasing and concave. Consumer 1 owns the firm, whose production set consists of the single vector $(-1, 1, 1)$. Date-1 endowments for the two consumers are $(0, x)$ and, respectively $(x, 0)$ with $x > 0$ arbitrary. Endowments at date 0 are assumed to be large enough so that date-0 positivity constraints on consumption are not binding for either consumer. Since there is no income effect on agents' date-1 consumption, their date-0 endowments are reflected only as a shift in their equilibrium indirect utilities and thus their magnitudes can be omitted. Example 3 of Section 2 is a particular case of this economy in which $u(c) = \ln c$ and $x = 1$.

If the firm issues another security (besides its equity) which completes the market, then agents' date-1 consumption is $\left(\frac{x+1}{2}, \frac{x+1}{2}\right)$, the vector of state prices is $u'\left(\frac{x+1}{2}\right) \cdot (1, 1)$, and agent 1's indirect utility at equilibrium is given by:

$$U^1_{cm}(x) = u'\left(\frac{x+1}{2}\right) + 2u\left(\frac{x+1}{2}\right) - 1. \tag{14}$$

If the firm issues only equity, then that is divided equally between the consumers, which renders an equilibrium indirect utility of

$$U^1_{im}(x) = \frac{u'\left(\frac{1}{2}\right) + u'\left(x + \frac{1}{2}\right)}{2} + u\left(\frac{1}{2}\right) + u\left(x + \frac{1}{2}\right) - 1. \tag{15}$$

for the firm's owner.

Note that, since u is concave, if u' is also concave (that is, $u''' < 0$), then complete markets are preferred by the firm's owner to incomplete markets. Therefore $u''' > 0$ is a *necessary* condition for endogenous market incompleteness. The following proposition provides a sufficient condition.

Proposition 6.2 *Assume that u is twice continuously differentiable, increasing and concave. If $-\frac{u'''}{u''} > 2$ then $U^1_{cm} < U^1_{im}$.*

The proof is immediate. The inequality $-\frac{u'''}{u''} > 2$ is equivalent to $\frac{1}{2}u''' + u'' > 0$ which implies that $\frac{1}{2}u' + u$ is convex. Jensen's inequality applied for the convex combination of $\frac{1}{2}$ and $x + \frac{1}{2}$ delivers then the result.

These results suggest that the third derivative of the utility function plays an important role in shaping the equilibrium market structure. The economic significance of the third derivative of the utility function has been recognized since several decades ago (Leland (1968), Sandmo (1970), Kimball (1990); see also Gollier (2004)) and it has been related to a consumer's propensity for precautionary savings (increased savings as a precaution against higher future income risk). The results above indicate that *prudence* (that is, $u''' > 0$) is a necessary condition for market incompleteness. While a positive third derivative indicates a propensity for precautionary savings, the *absolute prudence* coefficient, $-\frac{u'''}{u''}$, measures its intensity (see Kimball (1990) for a formalization of this idea). The condition in Proposition 6.2 requires an absolute prudence greater than two. Therefore, while prudence is necessary for incomplete markets to prevail at equilibrium, *high "enough"* prudence is what guarantees the markets remain incomplete at equilibrium.

In a similar model, Carvajal, Rostek, and Weretka (2012) found that prudence ($u''' > 0$) is a necessary and sufficient condition for the value of the firm to be higher under incomplete markets. These results show that an incomplete market structure cannot arise as an equilibrium outcome unless the firm's market value is higher under that structure than complete markets. But a higher market value is not sufficient. *"Enough"* prudence is required, on top of that, to obtain a sufficient condition.

Constant absolute risk aversion utilities of the form $u(c) = -e^{-\alpha c}$ with $\alpha > 2$ satisfy the sufficient condition described in Proposition 6.2 and thus such utilities always lead to endogenously incomplete markets. Constant relative risk aversion utilities do not satisfy the condition on their entire domain. Numerical computations reveal that in the case of both, $u(c) = \ln c$ and $u(c) = c^{1-\theta}$, endogenously incomplete markets prevail only for small enough values of the initial endowment, x. For instance, if date-1 endowments of Example 3 are modified to $(0, 2)$ and $(2, 0)$, the conclusion of the example is reversed: the owner of the firm prefers a complete market structure.

These results show that endogenous market incompleteness and equilibrium suboptimality are robust features of the model. They appear under specific (but meaningful) conditions on the endowments and preferences, which are not pathological in any way.

7 Final Remarks

Market incompleteness opens up the possibility for financial innovation. As argued in Section 5, when short selling is allowed, any firm that creates a new security in such market can have a significant impact on prices (and allocations) even if the firm is otherwise "small" in terms of its production possibilities.

Market power, rather than perfect competition, seems thus a more natural starting point for the analysis of production under incomplete markets,[17] yet much of the literature has focused on the latter.[18] This paper fills the gap by modeling the decision problem of a firm which has market power and acts in the best interest of its control group. Except under special circumstances (for instance when the members of the control group do not care about their future consumption) market value maximization does not perform well, and it needs to be adjusted to match the interests of the control group's members. This is precisely what the new objective proposed here, the C-adjusted value, does: it reconciles the three possibly conflicting effects (income, price, risk-hedging) of the firm's action on its control group welfare.

Since the focus of the paper is on the decision problem of an individual firm, it is assumed throughout that only one firm is active in the market. However, the model can easily capture important aspects of a more general framework. In the setting of Section 4, (some of) the exogenously given assets can be interpreted as securities issued by other firms. Such interpretation can be reconciled with the assumption of zero net supply imposed on those assets as long as portfolio holdings of the existing securities represent net trades rather than final asset holdings. As such, the model of Section 4 describes the reaction of one firm to other firms' decisions, when the latter are taken as given. The equilibrium concept defined in Section 4.2 describes then a *partial* equilibrium, in which it is assumed that the choices of the other firms remain fixed. Modeling the fully-fledged Cournot competition among firms that maximize their C-adjusted value will be subject of future research.

8 Appendix

Lemma 8.1 *The economy $\mathcal{E}_{\mathcal{P}}$ has an equilibrium for every $\mathcal{P} \in \widehat{\mathcal{F}}$.*

Proof. Note first that, using part (iv) of Definition 4.1 and formula (3), the budget constraints (4) can be written, equivalently, as:

$$\begin{aligned} c_0^i + q(b^i - \delta^i b^f) + p(r^i - \delta^i \mathbf{1}_N) + v(\theta^i - \delta^i(1+\theta^f)) &= \omega_0^i + \delta^i y_0 \qquad (16) \\ c^{i\mathbf{1}} = \omega^{i\mathbf{1}} + \delta^i y^{\mathbf{1}} + A(b^i - \delta^i b^f) + X(r^i - \delta^i \mathbf{1}_N) + \theta^f \cdot D(\theta^i - \delta^i) & \\ c^i \geq 0, \left(b^i, r^i, \theta^i\right) &\geq -L. \end{aligned}$$

Therefore, if the firm chooses policy $\mathcal{P} = \left(y, D, X, b^f, \theta^f\right)$, having an initial endowment of δ shares is equivalent, in terms of generating the same consumption streams, to receiving an additional endowment of goods δy and trading in

[17]The same point is made by Kreps (1987) in Essay II. Allen and Gale (1991) also point out that for incentives to innovate to exist in the financial markets, competition must be imperfect.

[18]For a more recent attempt, see Bisin, Gottardi, and Ruta (2016).

an economy in which all the $J+N+1$ securities are in zero net supply. Therefore, the economy $\mathcal{E}_{\mathcal{P}}$ is equivalent to a standard stock-exchange economy, $\mathcal{E}^0_{\mathcal{P}}$, in which

1. consumers' endowments of goods are $(\omega^i + \delta^i y)_{i \in I}$,
2. asset structure is given by $\left(A, X, \theta^f \cdot D\right)$,
3. there is no initial endowment of assets,
4. consumers face "personalized" short-sale bounds, $L^{i,\mathcal{P}} \in \mathbb{R}^{J+N+1}$, given by:
   (a) $L_j + \delta^i b^f$ for every exogenously given security A^j,
   (b) $L_n + \delta^i$ for every firm-issued security X^n,
   (c) $L_D + \delta^i$ for the security $\theta^f \cdot D$.

Since $y \in \widehat{Y}$, every consumer's endowment of goods in $\mathcal{E}^0_{\mathcal{P}}$ is strictly positive.

The proof of the existence of an equilibrium for $\mathcal{E}^0_{\mathcal{P}}$ is similar to standard general equilibrium existence proofs. The reader is referred to Debreu (2000 [1962]) and, especially, Geanakoplos and Polemarchakis (1986) for the details.[19] An important step in the proof is the construction of an appropriate convex and compact price space. For this specification of the model, the set can be defined as follows.

Let λ_0 be the price of date-0 consumption, expressed in some arbitrary unit of account, and let π be the price vector for the $J+N+1$ assets (expressed in the same units). Let

$$Q := \left\{ (\lambda_0, \pi) \in \mathbb{R}_+ \times \mathbb{R}^{J+N+1} \mid \exists \lambda \in \mathbb{R}^S_+ \text{ s.t. } \pi = \lambda \left(A, X, D\right) \right\}.$$

If (λ_0, π) are equilibrium prices for $\mathcal{E}^0_{\mathcal{P}}$, then $(\lambda_0, \pi) \in Q$. Clearly, Q is a convex and closed cone. If Q does not contain a full line then there exists a hyperplane $H \subseteq \mathbb{R}^{J+N+2}$ (of dimension $J+N+1$) such that $0 \neq (\lambda_0, \pi) \in Q$ if and only if $\alpha\left(\lambda_0, \pi\right) \in Q \cap H$ for some $\alpha > 0$, and $Q^0 := Q \cap H$ is compact. If Q contains a full line, take H to be half the unit sphere in $\mathbb{R}^{J+N+2}$, centered at origin and let $Q^0 := Q \cap H$ be the price space. Then Q^0 is a convex and compact set (or an acyclic absolute neighborhood retract if H is the half sphere) and the technique used by Geanakoplos and Polemarchakis (1986) can be applied here. ■

Let $\widetilde{\prod}^0(\mathcal{P})$ be the set of normalized (to lie in Q^0) date-0 consumption and asset equilibrium prices of $\mathcal{E}^0_{\mathcal{P}}$, so that $\widetilde{\prod}^0 : \widehat{\mathcal{F}} \rightrightarrows Q^0$. Then $\widetilde{\prod}(\mathcal{P}) = \left\{ \frac{\pi}{\lambda_0} \mid (\lambda_0, \pi) \in \widetilde{\prod}^0(\mathcal{P}) \right\}$.

[19] Geanakoplos and Polemarchakis's (1986) proof is given for economies with unlimited short-sales. Portfolio constraints only simplify the problem, as it is enough to prove existence of an equilibrium for the truncated economy.

Lemma 8.2 *The equilibrium price correspondence $\widetilde{\prod} : \widehat{\mathcal{F}} \rightrightarrows \mathbb{R}^{J+N+1}$ is upper hemi-continuous, with compact values.*

Proof. It is enough to show that $\widetilde{\prod}^0 : \widehat{\mathcal{F}} \rightrightarrows Q^0$ has closed graph. Since Q^0 is compact, this implies that $\widetilde{\prod}^0$ is upper hemi-continuous with compact values. On the other hand, since the utility functions of the agents are assumed to be strictly increasing in date-0 consumption, for every $\mathcal{P} \in \widehat{\mathcal{F}}$ and every $(\lambda_0, \pi) \in \widetilde{\prod}^0(\mathcal{P})$, $\lambda_0 > 0$. Thus, if $\widetilde{\prod}^0$ is upper hemi-continuous and compact-valued, $\mathcal{P} \mapsto \widetilde{\prod}(\mathcal{P})$ must be upper hemi-continuous and compact-valued as well.

To prove that $\widetilde{\prod}^0 : \widehat{\mathcal{F}} \rightrightarrows Q^0$ has closed graph, notice first that the equilibrium portfolios are bounded, due to the short sale constraints. Let K be a cube in $\mathbb{R}^{J+N+1}$, large enough so that it contains all the portfolio bounds. Consider the truncated portfolio demands $Z_K^i : \widehat{\mathcal{F}} \times Q^0 \rightrightarrows K$. Then Z_K^i has non-empty, convex and compact values, and it is upper hemi-continuous at every $(\mathcal{P}, \pi) \in \widehat{\mathcal{F}} \times Q^0$ with $\lambda_0 (\omega_0 + \delta^i y_0) + \pi L^{i,\mathcal{P}} \neq 0$.

To overcome the possible discontinuity of the demand at points $(\mathcal{P}, \pi) \in \widehat{\mathcal{F}} \times Q^0$ for which $\lambda_0 (\omega_0 + \delta^i y_0) + \pi L^{i,\mathcal{P}} = 0$, it is enough to construct a smoothed demand correspondence, $\widehat{Z_K^i}$, and a quasi-equilibrium as in Debreu (2000 [1962]). It can be shown that every quasi-equilibrium of $\mathcal{E}_{\mathcal{P}}^0$ is an equilibrium, and that the smoothed demand correspondence is upper hemi-continuous everywhere.

The closed graph property of $\widetilde{\prod}^0$ follows now immediately from the upper-hemi-continuity of the smoothed aggregate demand. ■

Proof of Theorem 6.1. For a given $\mu \in \mathcal{M}$, let Γ_μ be the normal-form, two-player game defined as follows:

- The strategy set of each player is

$$\widehat{\mathcal{F}}' = \left\{ \mathcal{P} = \left(y, D, X, b^f, \theta^f\right) \in \widehat{\mathcal{F}} \mid \left(b^f, \mathbf{0}_N, 0\right) \in K \right\},$$

  where K is a cube in $\mathbb{R}^{J+N+1}$ which is large enough to contain all the portfolio bounds.

- The first player's payoff function is

$$\Phi_\mu^1 (\mathcal{P}_1, \mathcal{P}_2) := \int_{\mathcal{M}} \mathcal{V}_{\mathcal{P}_2}^C (\mathcal{P}_1) \, d\mu (\Pi) .$$

- The second player's payoff function is

$$\Phi_\mu^2 (\mathcal{P}_1, \mathcal{P}_2) := - \|\mathcal{P}_1 - \mathcal{P}_2\| ,$$

  where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{2S+SN+J+2}$ ($\mathcal{P}_1$ and $\mathcal{P}_2$ are viewed as $(2S + SN + J + 2)$-dimensional vectors here).

It is easy to see that $\mathcal{P}$ is an equilibrium production-financial plan consistent with the belief μ if and only if $(\mathcal{P}, \mathcal{P})$ is a Nash equilibrium of the game Γ_μ. Therefore, it is enough to prove that there exists $\mu \in \mathcal{M}$ such that Γ_μ has a Nash equilibrium. The proof will proceed in two steps.

Step 1: The strategy space $\widehat{\mathcal{F}}'$ is compact.

I prove first that $\widehat{Y}$ is compact. Since $\widehat{Y}$ is a closed subset of $\mathbb{R}^{S+1}$, it is enough to prove that it is bounded. Suppose it is not. Then, there exists a sequence $(y^n)_n \subseteq \widehat{Y}$ such that $\|y^n\| > n$, $\forall n \geq 1$. Convexity of $\widehat{Y}$, together with $0 \in \widehat{Y}$, implies that:

$$\frac{1}{\|y^n\|} y^n + \left(1 - \frac{1}{\|y^n\|}\right) 0 \in \widehat{Y}, \quad \forall n \geq 1.$$

Since $\left\|\frac{1}{\|y^n\|} y^n\right\| = 1$, it can assumed, without loss of generality, that $\frac{1}{\|y^n\|} y^n \to \hat{y} \in \mathbb{R}^{S+1}$, with $\|\hat{y}\| = 1$. $\widehat{Y}$ closed implies then that $\hat{y} \in \widehat{Y}$.

On the other hand, $y^n \in \widehat{Y} \Longrightarrow y_s^n \geq -\min_i \frac{\omega_s^i}{\delta^i} + \varepsilon$ for every $s = 0, 1, ..., S$ and therefore

$$\lim_{n \to \infty} \frac{1}{\|y^n\|} y^n \geq -\lim_{n \to \infty} \frac{\left(-\min_i \frac{\omega_s^i}{\delta^i} + \varepsilon\right)_{s=0,...,S}}{\|y^n\|} = \mathbf{0}_{S+1}.$$

Since $Y \cap \mathbb{R}_+^{S+1} = \{\mathbf{0}_{S+1}\}$, the above inequality implies that $\hat{y} = \mathbf{0}_{S+1}$, which contradicts $\|\hat{y}\| = 1$. Compactness of $\widehat{\mathcal{F}}'$ follows now immediately from compactness of $\widehat{Y}$ and K, and the assumption that $y \longmapsto \mathcal{K}(y)$ is upper hemi-continuous with compact values.

Step 2: The game Γ_μ has a Nash equilibrium, for some $\mu \in \mathcal{M}$.

I will show that the family of games $(\Gamma_\mu)_{\mu \in \mathcal{M}}$ induces a "game with endogenous sharing rules" which satisfies all the hypotheses of the main theorem in Simon and Zame (1990).

Define the payoff correspondences $Q^1, Q^2 : \widehat{\mathcal{F}}' \times \widehat{\mathcal{F}}' \rightrightarrows \mathbb{R}$ as follows:

$$\begin{aligned} Q^1(\mathcal{P}_1, \mathcal{P}_2) &:= \left\{ \int_{\mathcal{M}} \mathcal{V}_{\mathcal{P}_2}^C(\mathcal{P}_1)\, d\mu(\Pi) \mid \mu = \text{rational belief} \right\}, \\ Q^2(\mathcal{P}_1, \mathcal{P}_2) &:= -\|\mathcal{P}_1 - \mathcal{P}_2\|. \end{aligned}$$

The game satisfies the hypotheses of the main theorem in Simon and Zame (1990) if: (a) the strategy sets are compact metric spaces, and (b) correspondences Q^1 and Q^2 are upper hemi-continuous with compact and convex values. The above conditions are indeed satisfied for the following reasons:

a. The strategy space $\widehat{\mathcal{F}}'$ can be organized as a metric space with the distance induced by the Euclidian metric of $\mathbb{R}^{2S+SN+J+2}$. According to step 1, $\widehat{\mathcal{F}}'$ is also compact.

b. Q^2 is a continuous function and thus upper hemi-continuous as a correspondence. Clearly, it has compact and convex values. Upper hemi-continuity of Q^1 (as well as compactness of its values) follows from the upper hemi-continuity and compactness of the values of $\widetilde{\prod}$, together with the continuity of the optimal consumption as a function of prices and endowments. Convexity of values follows from the linearity of the integral with respect to μ.

Therefore, there exists μ such that the game Γ_μ has a Nash equilibrium. ■

References

Allen, F., and D. Gale (1991): "Arbitrage, Short Sales, and Financial Innovation," *Econometrica*, 59(4), 1041–68.

Aumann, R. J. (1964): "Markets with a Continuum of Traders," *Econometrica*, 32(1/2), 39–50.

Bejan, C. (2008): "The Objective of a Privately Owned Firm under Imperfect Competition," *Economic Theory*, 37(1), 99–118.

Bejan, C., and F. Bidian (2012): "Ownership Structure and Efficiency in Large Economies," *Economic Theory*, 50(3).

Bisin, A. (1998): "General Equilibrium with Endogenously Incomplete Markets," *Journal of Economic Theory*, 82, 19–45.

Bisin, A., P. Gottardi, and G. Ruta (2016): "Equilibrium Corporate Finance," SSRN: https://ssrn.com/abstract=2815066 or http://dx.doi.org/10.2139/ssrn.2815066.

Bonnisseau, J., and O. Lachiri (2004): "On the objective of firms under uncertainty with stock markets," *Journal on Mathematical Economics*, 40, 493–513.

Carvajal, A., M. Rostek, and M. Weretka (2012): "Competition in Financial Innovation," *Econometrica*, 80(5), 1895–1936.

Debreu, G. (2000 [1962]): *New Concepts and Techniques for Equilibrium Analysis*pp. 297–313, Equilibrium. Volume 1. Introduction to equilibrium in economics. Elgar; distributed by American International Distribution Corporation, Williston, Vt., Elgar Reference Collection. Critical Ideas in Economics, vol. 2. Cheltenham, U.K. and Northampton, Mass.

Dierker, E., H. Dierker, and B. Grodal (2002): "Nonexistence of Constrained Efficient Equilibria When Markets Are Incomplete," *Econometrica*, 70(3), 1245–1251.

——— (2005): "Are Incomplete Markets Able to Achieve Minimal Efficiency?," *Economic Theory*, 25(1), 75–87.

Dierker, E., and B. Grodal (1999): "The Price Normalization Problem in Imperfect Competition and the Objective of the Firm," *Economic Theory*, 14(2), 257–284.

Dréze, J. (1974): "Investment under Private Ownership: Optimality, Equilibrium and Stability," in *Allocation under Uncertainty: Equilibrium and Optimality*, ed. by J. Dréze. Macmillan, New York, NY.

Duffie, D. (1988): *Security Markets: Stochastic Models*. Academic Press, Inc., Boston, MA.

Ekern, S., and R. Wilson (1974): "On the Theory of the Firm in an Economy with Incomplete Markets," *Bell Journal of Economics*, 5, 171–180.

Geanakoplos, J., and H. Polemarchakis (1986): "Regularity and Constrained Suboptimality of Competitive Allocations when Markets are Incomplete," in *Uncertainty, Information and Communication, Essays in Honor of Kenneth Arrow*, ed. by W. Heller, R. Ross, and D. Starret, vol. 3, pp. 65–95. Cambridge University Press, Cambridge, MA.

Gollier, C. (2004): *The Economics of Risk and Time*. The MIT Press.

Gretsky, N., and J. Ostroy (1985): *Thick and Thin Market Non-Atomic Exchange Economies*Advances in Equilibrium Theory.

Grossman, S. J., and O. D. Hart (1979): "A Theory of Competitive Equilibrium in Stock Market Economies," *Econometrica*, 47(2), 293–329.

Hart, O. D. (1974): "On the existence of equilibrium in a securities model," *Journal of Economic Theory*, 9, 293–311.

——— (1979): "On Shareholder Unanimity in Large Stock Market Economies," *Econometrica*, 47(5), 1057–1083.

Hildenbrand, W. (1974): *Core and Equilibria of a Large Economy*. Princeton University Press, Princeton, NJ.

Kelsey, D., and F. Milne (1996): "The Existence of Equilibrium in Incomplete Markets and the Objective Function of the Firm," *Journal of Mathematical Economics*, 25, 229–245.

Kimball, M. (1990): "Precautionary Savings in the Small and in the Large," *Econometrica*, 58, 53–73.

Kreps, D. (1987): "Three Essays on Capital Markets," *La Revista Espanola de Economia*, 4, 111–145.

LELAND, H. (1968): "Saving and Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics*, 82, 465–473.

——— (1974): "Production Theory and the Stock Market," *The Bell Journal of Economics and Management Sciences*, 5, 125–144.

MAGILL, M., AND M. QUINZII (1996): *The Theory of Incomplete Markets.* The MIT Press, Cambridge, MA.

MAGILL, M., M. QUINZII, AND J.-C. ROCHET (2015): "A Theory of the Stakeholder Corporation," *Econometrica*, 85(5), 1685–1725.

MAKOWSKI, L. (1983): "Competitive Stock Markets," *Review of Economic Studies*, 50, 305–330.

OSTROY, J. M., AND W. R. ZAME (1994): "Nonatomic Economies and the Boundaries of Perfect Competition," *Econometrica*, 62(3), 593–633.

RADNER, R. (1974): "A Note on Unanimity of Stockholders' Preferences among Alternative Production Plans: A Reformulation of the Ekern-Wilson Model," *Bell Journal of Economics*, 5(1), 181–184.

RUSTICHINI, A., AND N. YANNELIS (1991): "What is Perfect Competition?," in *Equilibrium Theory in Infinite Dimensional Spaces. Studies in Economic Theory*, ed. by M. Khan, and N. Yannelis, vol. 1 of *Equilibrium Theory in Infinite Dimensional Spaces. Studies in Economic Theory.* Springer, Berlin, Heidelberg.

SANDMO, A. (1970): "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies*, 37, 353–360.

SIMON, L., AND R. ZAME (1990): "Discontinuous Games with Endogenous Sharing Rules," *Econometrica*, 58, 861–872.

ZIERHUT, M. (2017): "Constrained efficiency versus unanimity in incomplete markets," *Economic Theory*, 64(1), 23–45.