Employment Lotteries, Endogenous Firm Formation and the Aspiration Core

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Abstract

The paper shows that the aspiration core of a TU-game coincides with the set of competitive wages arising in a labor market economy in which time is indivisible, but workers and firms can sign contingent labor contracts and trade in employment lotteries. The set of firms that are active in the market is endogenously determined at equilibrium and it coincides with the generating collection of the corresponding aspiration core allocation.

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1 Introduction

In economies with non-convexities arising from the indivisibility of some commodities, it is often the case that a competitive equilibrium does not exist. When competitive equilibria do exist, the corresponding allocations are in the core of the economy (under some assumptions on preferences and endowments). On the other hand, if equilibria for the economy with indivisible commodities do not exist, a market clearing price may still exist in the economy with a richer market, in which (the same) agents can trade not only the

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indivisible commodities, but also lotteries over bundles of commodities. Such markets have been analyzed, among others by Rogerson (1988), Prescott and Townsend (1984), Shell and Wright (1993), Garratt (1995), (see also Prescott and Shell (2002) for a survey). We consider here a class of private-ownership, finite, production economies with indivisible labor and show that, if workers and firms can trade in employment lotteries, an equilibrium always exists and it belongs to the aspiration core, a non-empty core extension. Moreover, every aspiration core allocation can be supported as a lottery equilibrium of an economy in that class. The analysis furthers our understanding of the relationship between the Walrasian equilibrium, lottery equilibrium and the core.

Most cooperative solution concepts do not address the payoff distribution and coalition formation problems simultaneously. The core, for instance, if non-empty, implicitly assumes the formation of the grand coalition. Zhou (1994) defines a new type of bargaining set which addresses both questions but, as shown by Anderson, Trockel, and Zhou (1997), Zhou's (1994) bargaining set cannot be decentralized using a market economy. Like Zhou's (1994) bargaining set, the aspiration core is defined in such a way that both payoffs and formed coalitions arise endogenously. This paper endows it with the link to competitive equilibrium that Zhou's (1994) bargaining set is lacking and shows that, similar to the core, which it extends, the aspiration core can be identified with the equilibrium outcomes of a specific lottery market. These equilibria involve only degenerate lotteries if and only if the aspiration core coincides with the core (and thus the core is non-empty).

To obtain our results, we exploit the equivalence between coalitional games and economies. We show that there is a natural way to construct a private-ownership, production economy with indivisibe labor from every TU-game, and a TU-game from every quasi-linear economy. Moreover, for super-additive games, these two processes are inverses to each other. The paper is therefore related to the literature on market games initiated by Shapley and Shubik (1969) and Shapley and Shubik (1975), where the equivalence between totally balanced games and convex pure-exchange economies is used to show that there is a bijection between the core payoffs of such games and the Walrasian equilibrium allocations of their corresponding "direct" economy. Later, Garratt and Qin (1997) focused on super-additive, balanced games and showed that core elements can be supported as lottery equilibria of an associated pure-exchange economy with indivisible goods. Their results imply that some indivisible-good economies may have no lottery equilibrium (precisely, those economies that generate super-additive, non-balanced games) and therefore non-convexities generated by indivisibilities are not eliminated by the use of lotteries.

We define here a weaker notion of lottery equilibrium for production economies with indivisible labor and show that such equilibria *always exist*. Moreover, for *every* super-additive game, its aspiration core (Cross 1967, Bennett 1983) is in a bijection with the set of lottery equilibria of the indivisible-labor production economy that represents the game. Our construction allows for the endogenous creation of firms through a process that mimics the coalition formation approach described in the aspiration literature. For games that are not balanced, the grand coalition cannot form because its worth cannot be divided among the individual players in such a way that the demands of all smaller coalitions are satisfied. By contrast, the standard literature on market games (Shapley and Shubik 1969, Shapley and Shubik 1975, Billera 1970, Qin 1993, Garratt and Qin 1997) analyzes only balanced games, where the final allocation of payoffs can always be realized by the grand coalition.

The production economy we associate to any TU-game makes the coalition formation process explicit. Given a TU-game v, members of coalition S may join efforts working in a firm whose productivity depends on $v(S)$. Since many economic examples do not allow an agent to simultaneously be part of two different enterprises, we assume here that labor is indivisible. Due to the inherent non-convexity introduced, such economies do not always have a Walrasian equilibrium, but we show that if agents and firms are allowed to trade lottery contracts specifying a positive probability of unemployment, an equilibrium always exists. Equilibrium wages prevailing in such markets for employment lotteries map, through a bijection, into the aspiration core of the game. Firms that form with positive probability in equilibrium are those whose coalitions of workers belong to the corresponding generating collection (the family of those coalitions that can satisfy the demands of their members). Our results posit the aspiration core payoffs as being the competitive market values of the individual players' participation into various coalitions.

Sun, Trockel, and Yang (2008) analyzed the role of labor indivisibilities too, but their analysis focuses on coalition production economies in which no trade in lotteries is allowed. They showed that competitive equilibria of such economies are in a bijective correspondence with the core vectors of the super-additive completion of the game, whenever the super-additive completion is balanced. We obtain the same result for private-ownership economies. Moreover, since the aspiration core of a game v coincides with the core of v (or the core of the super-additive completion of v) whenever the latter is not empty, our results are a generalization of both Garratt and Qin's (1997) and Sun, Trockel, and Yang's (2008) results to arbitrary TU-games.

2 An Illustrative Example

Consider an economy with three agents (we will call them truck drivers) and four firms. Agents have identical skills and each is endowed with one unit of time which can be supplied as labor. Each firm owns a truck and hires labor to produce the same output good (deliveries). The first three firms have production functions of the form $F_j : \mathbb{R}^3_+ \to \mathbb{R}_+$, with $F_j(l_1, l_2, l_3) = \min\{l_{[j]}, l_{[j+1]}\}$, where $[k] := k(\text{mod } 3) + 1$.[1] The fourth firm's production function is $F_4 : \mathbb{R}^3_+ \to \mathbb{R}_+$, defined as $F_4(l_1, l_2, l_3) = \min\{l_1, l_2, l_3\}$. Thus, every delivery needs the labor input of at least two truck drivers. Firm 4 is equally owned by all agents, while firm j, with $j \in \{1, 2, 3\}$, is equally owned by agents $\{[j], [j + 1]\}$. Truck drivers care only about their wealth and have no disutility of labor.

If labor is indivisible (agents cannot receive part-time contracts) then an equilibrium does not exist. Indeed, if such equilibrium existed, its corresponding allocation would be Pareto optimal, and therefore either all drivers would be employed by firm 4, or two drivers would be employed by one of the first three firms while the others (firms and worker) would remain idle. The first allocation requires a wage of $\frac{1}{3}$ to be supported as an equilibrium, while the second allocation requires a wage of $\frac{1}{2}$. However, at a wage $w = \frac{1}{3}$, all firms would want to hire, and therefore labor market would be in excess demand. At a wage of $w = \frac{1}{2}$, firm 4 shuts down and each of the first three firms is indifferent between shutting down or hiring two drivers. Therefore, $\frac{1}{2}$ cannot be an equilibrium wage either and thus an equilibrium does not exist.

There is a natural way to associate a TU-game to this economy by assuming that a coalition of agents can choose how to operate every firm they fully own (and only those). Then, the TU-game associated to this economy is defined by: $v(\emptyset) = v(1) = v(2) = v(3) = 0, v(1, 2) = v(1, 3) = v(2, 3) = v(1, 2, 3) = 1$. The game has an empty core. As we will show later, this is intrinsically related to the non-existence of a Walrasian equilibrium for the production economy. The reason for the emptiness of the core is that the

[1] We denote by $k(\text{mod } 3)$ the remainder of the Euclidean division of k by 3.

grand coalition is not powerful enough (in terms of the value it can generate) relative to the two-player coalitions. Since players are identical and each can form a two-player coalition with another, every player naturally "aspires" to receive a payoff of (at least) $\frac{1}{2}$. Clearly, such demands cannot be satisfied simultaneously. Yet, their sum is the minimum needed to satisfy all proper coalitions (Zhao (2001) calls it the "minimum no-blocking payoff"). We are going to show that these are precisely the payoffs that arise as equilibrium wages in the original production economy if lottery trading is allowed.

Assume therefore that truck drivers can submit job applications to more than one firm and randomize over which offer to accept. Firms can also offer employment contracts that stipulate a probability of being laid off (or a probability of delivery cancellation). In this case $w = \frac{1}{2}$ is an equilibrium wage. At this wage, each driver chooses to submit exactly two job applications to firms $1, 2$ or 3 and accepts each firm's offer with equal probability. Firm 4 shuts down, and each of the first three firms hires two drivers, offering them employment contracts that carry a 50% chance of delivery cancellation (or, equivalently, job termination). There are three essentially different outcomes arising from these equilibrium wages, each occurring with probability $\frac{1}{3}$. In each of the outcomes, two workers are employed by one firm and the other worker and firms are inactive.

The vector of wages the agents receive at this employment lottery equilibrium coincides with the vector of aspirations described before. Moreover, identifying each firm with the set of workers it employs, the set of potentially active firms coincides with the family of coalitions –of the associated game– that can pay their members their aspiration payoffs.

The vector of aspirations, together with the coalitions that can support it, describes a solution concept for TU-games called the *aspiration core* (which will be defined properly in the next section). The concept, suggested first by Cross (1967) and later formalized by Bennett (1983), is a core extension: it is always non-empty and it coincides with the core when the latter is non-empty. Our results show that this TU-game solution concept is intrinsically related to lottery equilibria of economies with indivisible commodities, in the same way the core is related to Walrasian equilibria. The rest of the paper formalizes and generalizes these results, making them applicable to arbitrary TU-games.

3 TU-games and the Aspiration Core

Let $N = \{1, 2, ..., N\}$[2] be a finite set of players, $\mathcal{N}$ the collection of all non-empty subsets of N, and for every $i \in N$ define $\mathcal{N}_i = \{S \in \mathcal{N} \mid S \ni i\}$. Let Δ_N (respectively $\Delta_{\mathcal{N}}$) be the unit simplex in $\mathbb{R}^N$ (respectively $\mathbb{R}^{\mathcal{N}}$), and $e_i \in \Delta_N$ (respectively $e_S \in \Delta_{\mathcal{N}}$) the vertex corresponding to $i \in N$ (respectively $S \in \mathcal{N}$). For every $S \in \mathcal{N}$, let $\mathbf{1}_S \in \{0,1\}^N$ denote the indicator function of S.

A *TU-game* (or simply a *game*) is a pair (N, v) with $v : \mathcal{N} \to \mathbb{R}_+$. For every $S \in \mathcal{N}$, $v(S)$ is called the *worth of coalition* S. A game (N, v) is called *super-additive* if for every $S, T \in \mathcal{N}$ with $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$. Given a game (N, v), a possible outcome is represented by a *payoff vector* $x \in \mathbb{R}^N$. Given $x \in \mathbb{R}^N$ and $S \in \mathcal{N}$, let $x(S) := \sum_{i \in S} x_i$. A payoff vector $x \in \mathbb{R}^N$ is *feasible* for coalition S if $x(S) \leq v(S)$. It is *individually feasible* if for every $i \in N$, there exists $S \in \mathcal{N}_i$ such that x is feasible for S. We say that coalition S is able to *improve upon* the outcome $x \in \mathbb{R}^N$ if $x(S) < v(S)$. A vector $x \in \mathbb{R}^N$ is *stable* if it cannot be improved upon by any coalition. The *core* of a game (N, v) is the set of stable outcomes that are feasible for N, that is,

$$\mathcal{C}(N, v) := \{x \in \mathbb{R}^N \mid x(S) \geq v(S)\ \forall S \in \mathcal{N},\ x(N) = v(N)\}.$$

A stable payoff vector $x \in \mathbb{R}^N$ that is individually feasible is called an *aspiration*. We denote by $\mathcal{A}sp(N, v)$ the set of aspirations of game (N, v). It is known that for any game (N, v), $\mathcal{A}sp(N, v)$ is a non-empty, compact and connected set (Bennett and Zame 1988). The *generating collection* of an aspiration x is the family of coalitions S that can attain x, that is,

$$\mathcal{GC}(x) := \{S \in \mathcal{N} \mid x(S) = v(S)\}.$$

Given a coalition $S \in \mathcal{N}$, a family of coalitions $\mathcal{B} \subseteq 2^S \setminus \{\emptyset\}$ is called *balanced* if every $T \in \mathcal{B}$ can be associated with a non-negative number λ_T such that, for every $i \in S$, $\sum_{T \in \mathcal{B} \cap \mathcal{N}_i} \lambda_T = 1$. The numbers λ_T are called *balancing weights*. The family $\mathcal{B}_+ := \{T \in \mathcal{B} \mid \lambda_T > 0\}$ is also balanced with respect to the corresponding weights $(\lambda_T)_T$. The *balanced cover* of a game $(N, v) \in \Gamma$ is the game $(N, \bar{v}) \in \Gamma$ defined for every $S \in \mathcal{N}$ as $\bar{v}(S) := \max \sum_{T \in \mathcal{B}} \lambda_T v(T)$, where the maximum is taken over all balanced

[2] By a standard abuse of notation, we use the same symbol for a finite set and the number of its elements.

families of coalitions $\mathcal{B} \subseteq 2^S \setminus \{\emptyset\}$. Given a game $(N, v) \in \Gamma$, a balanced family $\mathcal{B}$ with respect to N and with associated balancing weights $(\lambda_S)_S$ is called *optimally balanced* if $\bar{v}(N) = \sum_{S \in \mathcal{B}} \lambda_S v(S)$. The game (N, v) is called *balanced* if $\bar{v}(N) = v(N)$. It is known that the core of a game is non-empty if and only if the game is balanced (Bondareva 1963, Shapley 1953).

The *aspiration core* (Cross 1967, Bennett 1983) of a game (N, v), denoted $\mathcal{AC}(N, v)$, is the set of those aspirations $x \in \mathcal{A}sp(N, v)$ for which $\mathcal{GC}(x)$ is balanced. It is known that $\mathcal{AC}(N, v) = \mathcal{C}(N, v)$ if and only if v is balanced and $\mathcal{AC}(N, v) = \mathcal{C}(N, \bar{v}) \neq \emptyset$ for every game (N, v). Moreover (see Bennett (1983) and Bejan and Gómez (2012)),

$$\mathcal{AC}(N, v) = \arg\min \left\{ x(N) \mid x \in \mathbb{R}^N, x(S) \geq v(S), \forall S \in \mathcal{N} \right\}.$$

4 Games as Economies with Indivisible Labor

In the spirit of Shapley and Shubik (1969), we are going to establish an equivalence between the family of all super-additive TU-game and a specific class of quasi-linear, private-ownership production economies with indivisible inputs, which will be called *direct production economies*.

We focus here on quasi-linear production economies with one output and several indivisible inputs. A typical economy, $\mathcal{E}$, consists of a finite set of consumers, I, a finite set of firms, J, and $K + 1$ tradable goods. Good 0, the output, is a perfectly divisible composite commodity, denoted by C. Goods $1, ..., K$ are indivisible inputs, which can also serve as consumption goods. For every $j \in J$, $F^j : \mathbb{N}^K \to \mathbb{R}_+$ denotes firm j's production function, which is non-decreasing in every argument and satisfies $F^j(0) = 0$ for every $j \in J$. Each consumer $i \in I$ is characterized by the utility function $U^i : \mathbb{R}_+ \times \mathbb{N}^K \to \mathbb{R}$ with $U^i(c, l) = c + u^i(l)$, endowment of indivisible goods $\omega^i \in \mathbb{N}^K$ and endowment of shares in firm j's profits, $\theta_i^j \in [0, 1]$. It is assumed that $\sum_{i \in I} \theta_i^j = 1$ for every $j \in J$.

A *Walrasian equilibrium* for this economy consists of a vector of relative prices for the indivisible goods, $\bar{w} \in \mathbb{R}_+^K$, an allocation $(\bar{c}^i, \bar{l}^i)_i$ for the consumers, and a vector of labor inputs $\bar{L} = (\bar{L}^j)_j$ such that the following conditions are satisfied:

1. for every $j \in J$, $\bar{\Pi}^j := F^j(\bar{L}^j) - \bar{w} \cdot \bar{L}^j = \max \left\{ F^j(L^j) - \bar{w} \cdot L^j \mid L^j \in \mathbb{N}^K \right\}$,

2. for every $i \in I$, $(\bar{c}^i, \bar{l}^i) \in \arg\max \left\{ U^i(c,l) \mid c + \bar{w} \cdot l = \bar{w} \cdot \omega^i + \sum_{j \in J} \theta_i^j \bar{\Pi}^j \right\}$,

3. $\sum_{i \in I} \bar{l}^i + \sum_{j \in J} \bar{L}^j \leq \sum_{i \in I} \omega^i$, and $\sum_{i \in I} \bar{c}^i \leq \sum_{j \in J} F^j(\bar{L}^j)$.

For every subset of consumers $S \in \mathcal{N}$, let $J_S := \{j \in J \mid \sum_{i \in S} \theta_i^j = 1\}$ be the set of firms that are fully owned by consumers in S. An allocation $(c^i, l^i)_i$ is called *feasible for coalition* S if for every firm $j \in J_S$ there exists $L^j \in \mathbb{N}^K$ such that $\sum_{i \in S} c^i \leq \sum_{j \in J_S} F^j(L^j)$ and $\sum_{i \in S} l^i + \sum_{j \in J_S} L^j \leq \sum_{i \in S} \omega_i$. We denote by $\mathcal{F}(S)$ the set of feasible allocations for coalition S. Given an economy $\mathcal{E}$, we define its associated TU-game, $(I, V_{\mathcal{E}})$ by letting, for every $S \subseteq I$, $S \neq \emptyset$

$$V_{\mathcal{E}}(S) := \max \left\{ \sum_{i \in S} U^i(c^i, l^i) \mid (c^i, l^i)_{i \in S} \in \mathcal{F}(S) \right\}. \tag{1}$$

Proposition 4.1 *If* $\left(\bar{w}, (\bar{c}^i, \bar{l}^i)_{i \in I}, (\bar{L}^j)_{j \in J}\right)$ *is a Walrasian equilibrium for* $\mathcal{E}$*, then* $\bar{U} = (\bar{U}^i)_{i \in I} \in \mathcal{C}(I, V_{\mathcal{E}})$*, where* $\bar{U}^i := \bar{c}^i + u^i(\bar{l}^i)$*,* $\forall i \in I$*.*

Proof. We need to show that $\bar{U}(S) \geq V_{\mathcal{E}}(S)$ for every $S \in \mathcal{N}$ and $\bar{U}(N) = V_{\mathcal{E}}(S)$. The last equality is an immediate consequence of the first welfare theorem and the definition of $V_{\mathcal{E}}$. Assume therefore that for some $S \in \mathcal{N}$, $\bar{U}(S) < V_{\mathcal{E}}(S)$ and let $(c^i, l^i)_{i \in S}$ and $(L^j)_{j \in J_S}$ be such that $\sum_{i \in S} c^i = \sum_{j \in J_S} F^j(L^j)$, $\sum_{i \in S} l^i + \sum_{j \in J_S} L^j = \sum_{i \in S} \omega^i$, and $\bar{U}(S) < \sum_{i \in S} c^i + u^i(l^i)$. One can then construct $(z^i)_{i \in S}$ such that $\sum_{i \in S} z^i = \sum_{i \in S} c^i$ and $\bar{c}^i + u^i(\bar{l}^i) < z^i + u^i(l^i)$ for every $i \in S$. This implies that, for every $i \in S$,

$$\bar{w} \cdot \omega^i + \sum_{j \in J} \theta_i^j \bar{\Pi}^j < z^i + \bar{w} \cdot l^i.$$

Summing up these inequalities over $i \in S$ and using that $\sum_{i \in S} z^i = \sum_{j \in J_S} F^j(L^j)$ and $\sum_{i \in S} l^i + \sum_{j \in J_S} L^j = \sum_{i \in S} \omega^i$, we get:

$$\sum_{j \in J_S} \bar{\Pi}^j + \sum_{j \in J \setminus J_S} \bar{\Pi}^j \sum_{i \in S} \theta_i^j < \sum_{j \in J_S} F^j(L^j) - \bar{w} \cdot L^j,$$

which is a contradiction, because $\bar{\Pi}^j \geq 0$ for every $j \in J$ and $\bar{\Pi}^j \geq F^j(L^j) - \bar{w} \cdot L^j$ for every $j \in J_S$. ■

We show next that a production economy, denoted $\mathcal{E}(v)$, can be constructed from every TU-game (N, v) such that $v = V_{\mathcal{E}(v)}$. The economy is defined as follows. There are N consumers, $2^N - 1$ firms and $N+1$ commodities. Commodity 0, denoted by C, is the output; the other N commodities, denoted $L_1, ..., L_N$, represent agent-specific human capital (or skilled labor). Labor is indivisible, while the output C is perfectly divisible. Therefore, each consumer's consumption set is $\mathbb{R}_+ \times \mathbb{N}^N$. Consumers care only about the amount of composite good C they consume and experience no disutility of labor. Thus $U_i(C, l_1, .., l_N) = C$, $\forall i \in N$. Every consumer i is endowed with one unit of human capital L_i and zero units of the output good. Firms are indexed by $S \in \mathcal{N}$ and each firm S uses human capital (skilled labor) $(L_i)_{i \in S}$ to produce the composite commodity according to the following production function: $F^S(L) := v(S) \cdot \min\{L_i \mid i \in S\}$, for every $L \in \mathbb{N}^N$. Moreover, each consumer i owns an initial share $\theta_i^S = \frac{1}{|S|} \cdot \mathbf{1}_S(i)$ in firm S. [3][4]

Given the description of the economy $\mathcal{E}(v)$, firms $T \subseteq S$ are the only ones fully owned by the set of consumers S and, since the game v is super-additive, this implies that $V_{\mathcal{E}(v)} = v$. Moreover, as we show next, there is a one-to-one and onto correspondence between the competitive equilibria of the economy $\mathcal{E}(v)$ and the core allocations of the game (N, v). Core vectors of the game (N, v) are identified with the equilibrium wages (or utilities) agents receive by selling their time/skills in a competitive market.

Proposition 4.2 *A payoff vector $\bar{w}$ is in the core of the game (N, v) if and only if $\left(\bar{w}, (\bar{w}^i, 0)_{i \in N}, (\bar{l}^S \cdot \mathbf{1}_S)_{S \in \mathcal{N}}\right)$ is a competitive equilibrium for $\mathcal{E}(v)$, where $\bar{l}_S = 0$ for every $S \subsetneq N$ and $\bar{l}_N = 1$.*

Proof. Note that at every equilibrium of the economy $\mathcal{E}(v)$, agents supply all their labor, their consumption of the divisible commodity is equal to the (relative) wage they receive, and firm S's vector of inputs must be of the form $\bar{l}_S \cdot \mathbf{1}_S$, with $\bar{l}_S \in \{0, 1\}$. Therefore, if $v(S) > \bar{w}(S)$ then firm S is active at the vector of relative wages $\bar{w}$. Moreover, its choice of $\bar{l}_S = 1$ is optimal if and only if $v(S) = \bar{w}(S)$. Hence, $\left(\bar{w}, (\bar{w}^i, 0)_{i \in N}, (-\bar{l}^S \cdot \mathbf{1}_S, \bar{l}^S v(S))_{S \in \mathcal{N}}\right)$ with $\bar{l}_S = 0$ for every $S \in \mathcal{N}$, $S \neq N$, and $\bar{l}_N = 1$ is a competitive equilibrium for

[3]Sun, Trockel, and Yang (2008) and Inoue (2010) have proposed an alternative way of relating TU-games to economies by associating a *coalition* production economy to every game.

[4]Since firms' technologies have constant returns to scale, the initial distribution of shares is irrelevant for the competitive equilibrium. Our results remain true for other distributions of shares as long as for every $S \subseteq N$, every consumer in S has ownership in firm S and, together, consumers in S fully own firm S.

$\mathcal{E}(v)$ if and only if $v(N) = \bar{w}(N)$ and $v(S) \leq \bar{w}(S)$ for every $S \subsetneq N$, which implies that $\bar{w} \in \mathcal{C}(N, v)$. ■

An immediate implication of Proposition 4.2 is that if (N, v) is a non-balanced game, then the economy $\mathcal{E}(v)$ has no Wlarasian equilibrium. We show next that, if firms and consumers are allowed to sign employment contracts contingent on the outcome of a lottery, equilibrium wages always exist and coincide with the aspiration core vectors of the corresponding game v. Aside from establishing a novel connection between equilibria with employment lotteries and the aspiration core, our treatment also extends the analogy between games and direct economies presented in Shapley and Shubik (1975), Garratt and Qin (1997), and Sun, Trockel, and Yang (2008). In particular, we show that not only payoffs, but formed coalitions (in game v) and productive firms (in economy $\mathcal{E}(v)$), coincide. For any coalition $S \in \mathcal{N}$, its balancing weight λ_S in an optimally balanced family $\mathcal{B}$ is linked to the probability that firm S is active in the lottery equilibrium of the corresponding direct production economy.

5 Equilibria with Employment lotteries

Assume that consumers and firms may choose to default on their labor contracts. Consumers may contemplate switching between equally-paying jobs, while firms can layoff workers and get out of business. However, rather than modeling strictly enforceable contracts and punishments for default, we design our model such that the probabilities of default will be embedded in the equilibrium market prices. We assume therefore that consumers and firms trade in *labor* or *employment lotteries* specifying, for each party, a probability of employment termination as described below.

A *labor lottery* for agent i is a vector $p_i \in \Delta_{\mathcal{N}_i}$ such that $\sum_{S \in \mathcal{N}_i} p_i^S = 1$. Thus, p_i^S specifies the probability with which agent i chooses to work for firm S (or, alternatively, $1 - p_S^i$ can be interpreted as the probability that i will terminate his/her contract with firms S, if hired). Given a wage level w_i, agent i chooses a probability distribution over the firms $S \in \mathcal{N}_i$. The utility consumer i derives from choosing the labor lottery p^i and consumption c is

$$U^i(p^i, c) := c + \sum_{S \in \mathcal{N}_i} p_S^i u_i(e^S) = c,$$

for all $(p_i, c) \in \Delta_{\mathcal{N}_i} \times \mathbb{R}_+$.

An *employment lottery* for firm S specifies a probability, $\phi_S \in [0,1]$ of maintaining employment from that firm or, equivalently, a probability $1-\phi_S$ of being laid off. Alternatively, one can interpret ϕ_S as the probability that S remains in business. Each firm S chooses an employment lottery and, contingent on being active, an operating level (labor force size) $k_S \in \mathbb{N}$. Each firm S is assumed to maximize its expected profits and thus it solves

$$\max\left\{\phi_S \cdot k_S\left(v(S) - w(S)\right) \mid \phi_S \in [0,1], k_S \in \mathbb{N}\right\}. \tag{2}$$

As opposed to the standard employment lottery models which assume a continuum of agents (e.g., Rogerson (1988)), our economy is finite and thus we cannot rely on the law of large numbers to ensure labor market clearing. Along the lines of Garratt (1995) we say that a vector of labor/employment lotteries is feasible if its elements are the marginals of some (auctioneer-run) joint lottery on the set of feasible labor contracts. More precisely, we define a (pure) *labor contract* as a vector $x \in \{0,1\}^{\mathcal{N}}$, in which the component x_S is equal to 1 if and only if firm S is active (and thus every consumer $i \in S$ is employed full-time). A labor contract is *feasible* if $[x_S = x_{S'} = 1] \Rightarrow [S \cap S' = \emptyset]$ for all $S \neq S'$.

Note that feasibility of labor contracts only requires that there is no excess demand for labor/human capital. It does *not* require that the labor market clears. For every feasible labor contract x, define $T(x) := \bigcup\{S \mid x_S = 1\}$ as the set of employed agents. At a feasible labor contract, $T(x)$ may be a strict subset of N. Denote the set of all feasible labor contracts by $\mathcal{X}$, and consider an arbitrary probability distribution γ on $\mathcal{X}$. Then, given γ, the probability that firm S is active is $\sum_{\{x|x_S=1\}}\gamma(x)$, while the probability that consumer i is employed is $\gamma_i := \sum_{\{x|T(x)\in\mathcal{N}_i\}}\gamma(x)$.

Definition 5.1 *A set of labor and employment lotteries* $((p_i)_{i\in N}, (\phi_S)_{S\in\mathcal{N}})$ *is feasible if*

1. *There exists* $\gamma \in \Delta_{\mathcal{X}}$ *such that* $\phi_S = \sum_{\{x|x_S=1\}}\gamma(x)$*, for all* $S \in \mathcal{N}$*,*
2. $p_i^S = \frac{\phi_S}{\sum_{T\in\mathcal{N}_i}\phi_T}$*, for every* $S \in \mathcal{N}$ *and every* $i \in S$*, and* $p_i^S = p_j^S$ *if* $i, j \in S$*.*

The first condition is a compatibility condition for labor demand. It requires that the probability of firm S operating coincides with the marginal of a joint probability distribution over the set of feasible labor contracts. The second condition requires that the probability that agent i assigns to working for firm S is exactly the probability of firm S operating, conditional on i being employed. The second part of condition 2 captures the labor complementarities embedded in firms' technologies. Note that the two conditions imply that $\sum_{S \in \mathcal{N}_i} \phi_S = \gamma_i > 0$ and $\gamma_i = \gamma_j$ for all $i, j \in N$.

Definition 5.2 *An employment lottery equilibrium for economy $\mathcal{E}(v)$ is a vector*

$$\big((\bar{w}_i)_i, (\bar{p}_i)_i, (\bar{\phi}_S)_S, (\bar{k}_S)_S\big)$$

such that

1. $\bar{p}_i \in \Delta_{\mathcal{N}_i}$ *for every* $i \in N$
2. $(\bar{\phi}_S, \bar{k}_S)$ *solves* (2) *for every* $S \in \mathcal{N}$
3. $k_S = 1$, *for all* $S \in \mathcal{N}$
4. $((\bar{p}_i)_i, (\bar{\phi}_S)_S)$ *is feasible according to Definition 5.1.*

We can now relate aspiration core allocations and equilibrium wages.

Theorem 5.3 *If $\bar{w} \in \mathcal{AC}(v)$ and $(\lambda_S)_S$ is a system of balancing weights associated with $\mathcal{GC}(\bar{w})$, then*

$$\left[(\bar{w}_i)_{i \in N}, ((\lambda_S)_{S \in \mathcal{N}_i})_{i \in N}, \left(\frac{\lambda_S}{\Lambda}\right)_{S \in \mathcal{N}}, (k_S = 1)_{S \in \mathcal{N}}\right]$$

is an employment lottery equilibrium for the economy $\mathcal{E}(v)$, where $\Lambda := \sum_{S \in \mathcal{N}} \lambda_S$.

Reciprocally, if $[(\bar{w}_i)_{i \in N}, (\bar{p}_i)_{i \in N}, (\bar{\phi}_S)_{S \subseteq N}, (\bar{k}_S)_{S \subseteq N}]$ is a lottery equilibrium for $\mathcal{E}(v)$, then $\bar{w} \in \mathcal{AC}(v)$ and $S \in \mathcal{GC}(\bar{w})$ whenever $\bar{\phi}_S > 0$.

Proof. Let $\bar{w} \in \mathcal{AC}(v)$ and $(\lambda_S)_S$ a system of balancing weights associated with $\mathcal{GC}(\bar{w})$. Define $\bar{p}_i^S := \lambda_S$, $\bar{\phi}_S := \frac{\lambda_S}{\Lambda}$ and $\gamma \in \Delta_{\mathcal{X}}$ such that $\gamma(x) = \bar{\phi}_S$ if $x = e_S$ for some $S \in \mathcal{N}$ and $\gamma(x) = 0$ otherwise. Then $((\bar{p}_i)_{i \in N}, (\bar{\phi}_S)_{S \in \mathcal{N}})$ is feasible, being supported by the joint lottery $\gamma \in \Delta_{\mathcal{X}}$. Moreover, since $\bar{w} \in \mathcal{AC}(v)$, $\bar{w}(S) \geq v(S)$ and thus $(\bar{\phi}_S, 1)$ is an optimal choice for firm S, which generates an expected profit of 0.

Reciprocally, if $[(\bar{w}_i)_{i \in N}, (\bar{p}_i)_{i \in N}, (\bar{\phi}_S)_{S \subseteq N}, (\bar{k}_S)_{S \subseteq N}]$ is a lottery equilibrium for $\mathcal{E}(v)$, then $\bar{w}(S) \geq v(S)$, otherwise firm S would make infinite profits. Profit maximization also dictates that $\bar{\phi}_S > 0$ only if $\bar{w}(S) = v(S)$. On the other hand, feasibility implies that $\sum_{S \in \mathcal{N}_i} \bar{\phi}_S > 0$ and thus, for every $i \in N$ there exists $S \in \mathcal{N}_i$ such that $\bar{\phi}_S > 0$ and $\bar{w}(S) = v(S)$, which implies that $\bar{w}$ is an aspiration. In addition, $\lambda_S := \frac{\phi_S}{\sum_{T \in \mathcal{N}_i} l_T}$ does not depend on i and $\sum_{S \in \mathcal{N}_i} \lambda_S = 1$ for every $i \in N$. This proves that $\mathcal{GC}(\bar{w})$ is balanced and thus $\bar{w} \in \mathcal{AC}(v)$. ∎

An immediate consequence of Theorem 5.3 is that the core of a game v is non-empty if and only if $\mathcal{E}(v)$ has a degenerate lottery equilibrium in which $p_i^N = 1$ for every $i \in N$, $l_N = 1$ and $\phi_S = 0$ for every $S \subsetneq N$. Thus, all consumers are employed by one firm and there is no default in the labor-employment contracts. Each agent receives a wage (and utility) equal to his/her payoff at a core allocation. This is equivalent to saying that the grand coalition forms and its worth is split among agents according to some core vector. Note however that other equilibria might exist as well and, in particular, more than one firm can form if, for some $x \in \mathcal{C}(N, v)$, $\mathcal{GC}(x) \neq \{N\}$.

If the core of the super-additive TU-game (N, v) is empty, then each player faces a positive probability of being unemployed and thus, in every realization of the joint lottery, the labor market is in excess supply. Firms that are active at a particular realization of the equilibrium lottery correspond to elements of the generating collection and wages paid are elements of the aspiration core of the game.

6 Final Remarks

An alternative way of dealing with indivisibilities, extensively used in the literature, consists in the introduction of a source of extrinsic uncertainty –or sunspots– and corresponding markets in which agents can trade in sunspot-contingent contracts (see for example Cass and Shell (1983)). As pointed out originally by Shell and Wright (1993), there is a close connection between lottery and sunspot equilibria since, as the authors showed, equilibrium employment lotteries of Rogerson (1988) can be implemented as sunspot equilibria. A similar result is valid in our model, too. The lottery equilibria presented here can be supported as sunspot equilibria via a standard construction in which the state space is the unit interval $[0, 1]$ with the σ-algebra of its Borel sets and the Lebesgue measure λ. From every lottery equilibrium $[(\bar{w}_i)_{i \in N}, (\bar{p}_i)_{i \in N}, (\bar{\phi}_S)_{S \in \mathcal{N}}, (\bar{k}_S)_{S \in \mathcal{N}}]$, if the induced distribution on the space of feasible labor contracts is γ, then one can construct a sunspot equilibrium as follows. Consider a partition of the interval $[0, 1]$ into Borel sets $(S_x)_{x\in\mathcal{X}}$ such that $\lambda(S_x) = \gamma(x)$. A sunspot equilibrium that induces the same outcome as the lottery equilibrium can be defined by letting, for every feasible labor contract $x \in \mathcal{X}$ and every state $s \in S_x$, spot market wages, individual labor supply and firm-by-firm labor demand be defined as follows:

$$\tilde{w}_i(s) = \begin{cases} \bar{w}_i & \text{if } i \in T(x), \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{l}_i(s) = \begin{cases} 1 & \text{if } i \in T(x), \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{L}_S(s) = \begin{cases} 1 & \text{if } x_S = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Reciprocally, using a line of arguments similar to that of Garratt, Keister, Qin, and Shell (2002), one can also show that, with a continuum, non-atomic state space, for every sunspot equilibrium, there exists another one that generates the same expected utility for consumers and the same total production while also having wages that are constant across employment-equivalent states. In other words, for every sunspot equilibrium, there exists an equivalent one which can be identified with a lottery equilibrium. This shows that the space of sunspot equilibria is richer than that of lottery equilibria and only a weaker version of Theorem 5.3 holds for sunspot equilibria. That is, every element of the aspiration core can be supported as a sunspot equilibrium of the associated direct economy, but only those sunspot equilibria with constant wages across employment-equivalent states can be identified with aspiration core allocations of the original game.

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