

**UW CSSS/POLS 512:  
Time Series and Panel Data for the Social Sciences**

**Modeling Nonstationary Time Series**

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# What we're doing today

Next steps:

- Learn some (weak) techniques for identifying non-stationary time series (from previous lecture slides)
- Analyze non-stationary series using differences
- Analyze non-stationary series using cointegration

## Differences & Integrated time series

Define  $\Delta^d y_t$  as the  $d$ th difference of  $y_t$

For the first difference ( $d = 1$ ), we write

$$\Delta y_t = y_t - y_{t-1}$$

For the second difference ( $d = 2$ ), we write

$$\Delta^2 y_t = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$$

or the difference of two first differences

or the difference in the difference

## Differences & Integrated time series

For the third difference ( $d = 3$ ), we write

$$\Delta^3 y_t = ((y_t - y_{t-1}) - (y_{t-1} - y_{t-2})) - (y_{t-1} - y_{t-2}) - (y_{t-2} - y_{t-3})$$

or the difference of two second differences

or the difference in the difference in the difference

This gets perplexing fast.

Fortunately, we will rarely need  $d > 1$ , and almost never  $d > 2$ .

## Differences & Integrated time series

What happens if we difference a stationary AR(1) process ( $|\phi_1| < 1$ )?

$$y_t = y_{t-1}\phi_1 + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

## Differences & Integrated time series

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$$y_t = y_{t-1}\phi_1 + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

$$y_t - y_{t-1} = y_{t-1}\phi_1 - y_{t-1} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

## Differences & Integrated time series

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$$y_t - y_{t-1} = y_{t-1}\phi_1 - y_{t-1} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

$$\Delta y_t = (1 - \phi)y_{t-1} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

We still have an AR(1) process,  
*and* we've thrown away some useful information – the levels in  $y_t$  –  
that our covariates  $x_t$  might have explained

## Differences & Integrated time series

What happens if we difference a random walk?

$$y_t = y_{t-1} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$



## Differences & Integrated time series

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## Differences & Integrated time series

What happens if we difference a random walk?

$$\begin{aligned}y_t &= y_{t-1} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t \\y_t - y_{t-1} &= y_{t-1} - y_{t-1} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t \\ \Delta y_t &= \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t\end{aligned}$$

The result is AR(0), and stationary –  
we could analyze it using ARMA(0,0), which is just LS regression!

When a single differencing removes non-stationarity from a time series  $y_t$ ,  
we say  $y_t$  is *integrated* of order 1, or I(1).

A time series that does not need to be differenced to be stationary is I(0).

This differencing trick comes at a price:  
we can only explain changes in  $y_t$ , *not* levels,  
and hence not the long-run relationship between  $y_t$  and  $\mathbf{x}_t$ .

## Differences & Integrated time series

What happens if we difference an AR(2) unit root process?

$$y_t = 1.5y_{t-1} - 0.5y_{t-2} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

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## Differences & Integrated time series

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$$y_t - y_{t-1} = 1.5y_{t-1} - y_{t-1} - 0.5y_{t-2} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

$$\Delta y_t = 0.5y_{t-1} - 0.5y_{t-2} + \mathbf{x}_t\boldsymbol{\beta} + \varepsilon_t$$

We get a stationary AR(2) process.

We could analyze this new process with ARMA(2,0).

We say that the original process is ARIMA(2,1,0),  
or an integrated autoregressive process of order 2, integrated of order 1.

## Differences & Integrated time series

Recall our GDP & Democracy example

$$\text{GDP}_t = \phi_1 \text{GDP}_{t-1} + \beta_0 + \beta_1 \text{Democracy}_t + \varepsilon_t$$

$$\text{GDP}_t = 0.9 \times \text{GDP}_{t-1} + 10 + 2 \times \text{Democracy}_t + \varepsilon_t$$

At year  $t$ ,  $\text{GDP}_t = 100$  and the country is a non-democracy  $\text{Democracy}_t = 0$ , and we were curious what would happen to GDP if in  $t + 1$  to  $t + k$ , the country becomes a democracy.

But now suppose  $\phi_1$  is 1,  $\beta_0$  is 0, and we want to model the first difference,  $\Delta \text{GDP}_t$ , instead of the level of  $\text{GDP}_t$ .

## Differences & Integrated time series

At year  $t$ ,  $GDP_t = 100$  and the country is a non-democracy  $Democracy_t = 0$ , and we are curious what would happen to GDP if in  $t + 1$  to  $t + k$ , the country becomes a democracy.

$$GDP_t = GDP_{t-1} + \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

## Differences & Integrated time series

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$$GDP_t = GDP_{t-1} + \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

$$GDP_t - GDP_{t-1} = GDP_{t-1} - GDP_{t-1} + \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$



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$$GDP_t = GDP_{t-1} + \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

$$GDP_t - GDP_{t-1} = GDP_{t-1} - GDP_{t-1} + \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

$$\Delta GDP_t = \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

## Differences & Integrated time series

At year  $t$ ,  $GDP_t = 100$  and the country is a non-democracy  $Democracy_t = 0$ , and we are curious what would happen to GDP if in  $t + 1$  to  $t + k$ , the country becomes a democracy.

$$GDP_t = GDP_{t-1} + \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

$$GDP_t - GDP_{t-1} = GDP_{t-1} - GDP_{t-1} + \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

$$\Delta GDP_t = \beta_0 + \beta_1 Democracy_t + \varepsilon_t$$

$$\Delta GDP_t = 0 + 2 \times Democracy_t + \varepsilon_t$$

## Differences & Integrated time series

At year  $t$ ,  $GDP_t = 100$  and the country is a non-democracy  $Democracy_t = 0$ , and we are curious what would happen to GDP if in  $t + 1$  to  $t + k$ , the country becomes a democracy.

$$\Delta GDP_t = 0 + 2 \times Democracy_t + \varepsilon_t$$

Works just as before – but gives only the one period change in GDP

Iterating, we get the cumulative change

We have to supply external information on the *levels* in order to get predictions of the level

The model doesn't know them

Moreover, the impact of lagged  $\varepsilon$ 's here doesn't ever diminish over time

So long predictions are very unreliable

## ARIMA(p,d,q) models

An ARIMA(p,d,q) regression model has the following form:

$$\begin{aligned}\Delta^d y_t &= \Delta^d y_{t-1} \phi_1 + \Delta^d y_{t-2} \phi_2 + \dots + \Delta^d y_{t-p} \phi_p \\ &\quad + \varepsilon_{t-1} \rho_1 + \varepsilon_{t-2} \rho_2 + \dots + \varepsilon_{t-q} \rho_q \\ &\quad + \mathbf{x}_t \boldsymbol{\beta} + \varepsilon_t\end{aligned}$$

This just an ARMA(p,q) model applied to differenced  $y_t$

The same MLE that gave us ARMA estimates still estimates  $\hat{\phi}$ ,  $\hat{\rho}$ , and  $\hat{\beta}$

We just need to choose  $d$  based on theory, ACFs and PACFs, and unit root tests (ugh)

## ARIMA( $p,d,q$ ) models

Mechanically,  
conditional forecasting and in-sample counterfactuals work just as before

Same code from last time will work; just change the  $d$  term of the ARIMA order to 1

But we need to be careful about forecasting too far into the future. . .

## Example: Presidential Approval

We have data on the percent ( $\times 100$ ) of Americans supporting President Bush, averaged by month, over 2/2001–6/2006.

Our covariates include:

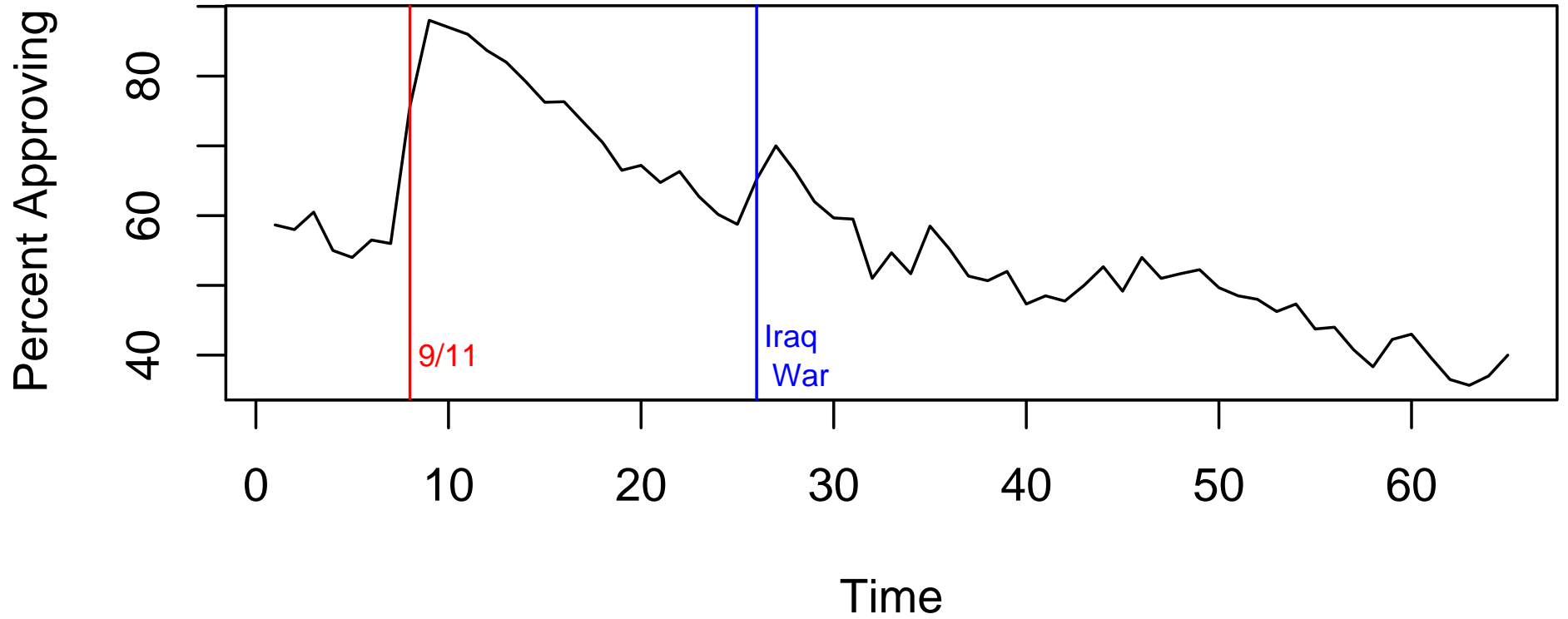
The average price of oil per month, in \$/barrel

Dummies for September and October of 2001

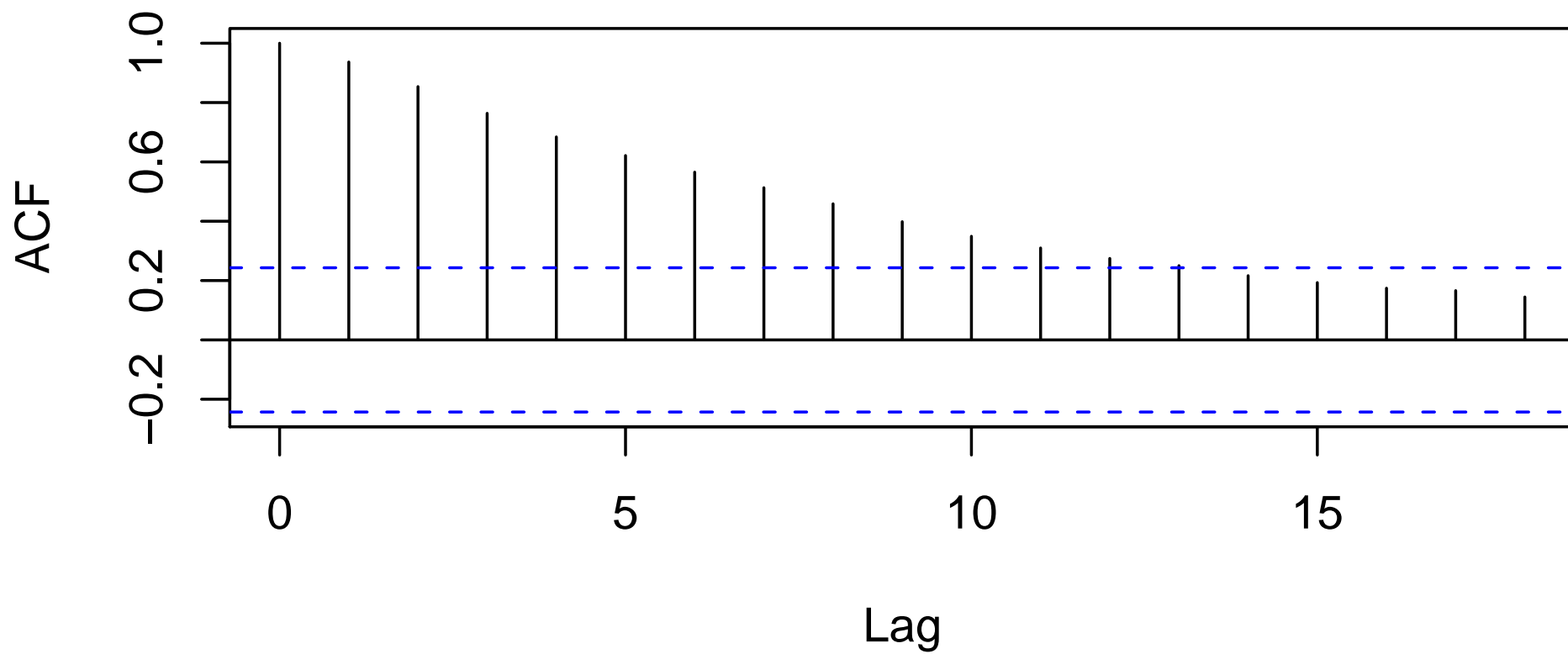
Dummies for first three months of the Iraq War

Let's look at our two continuous time series

# US Presidential Approval

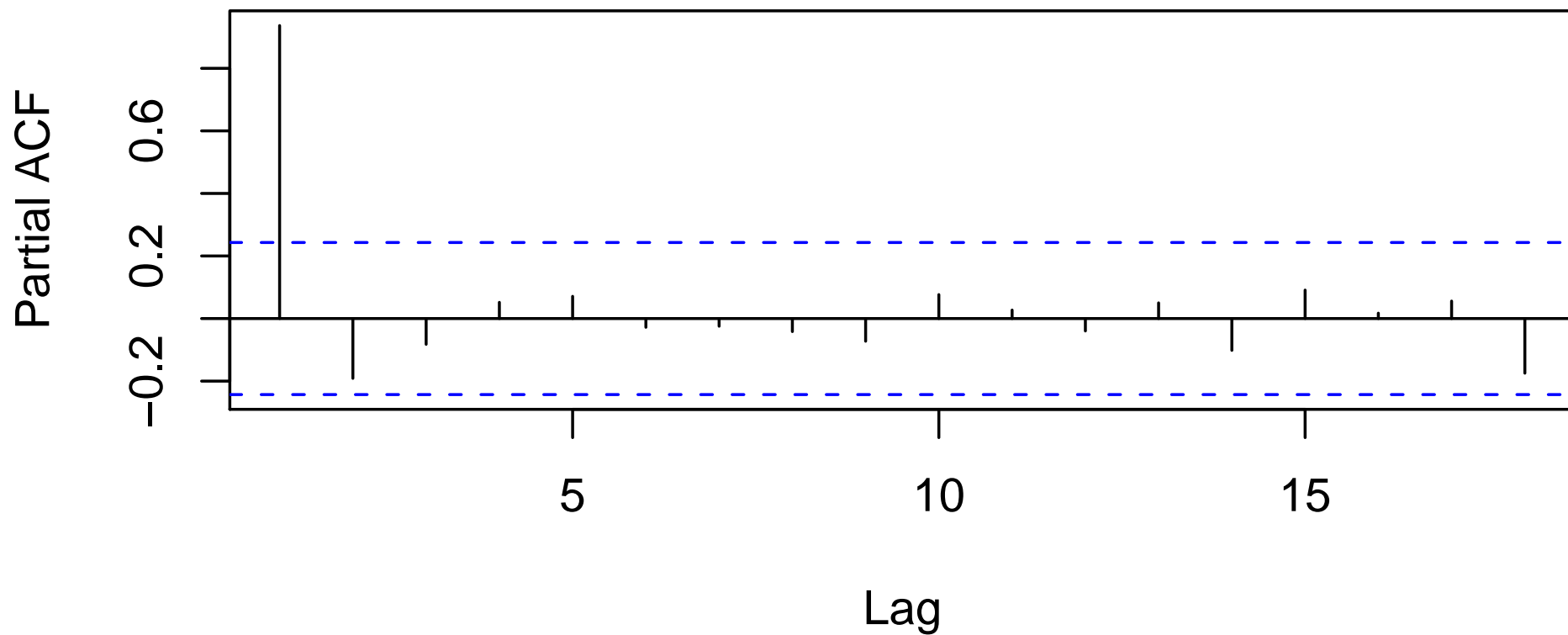


# Series approve

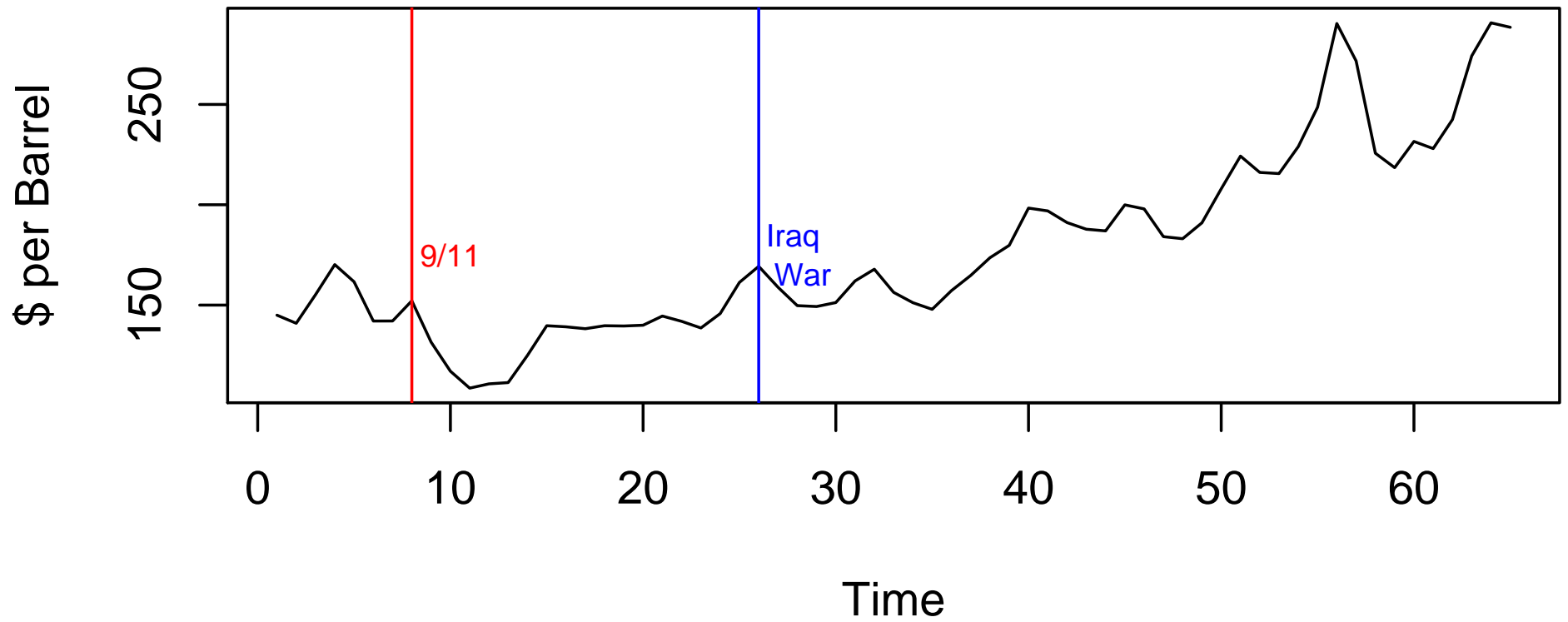




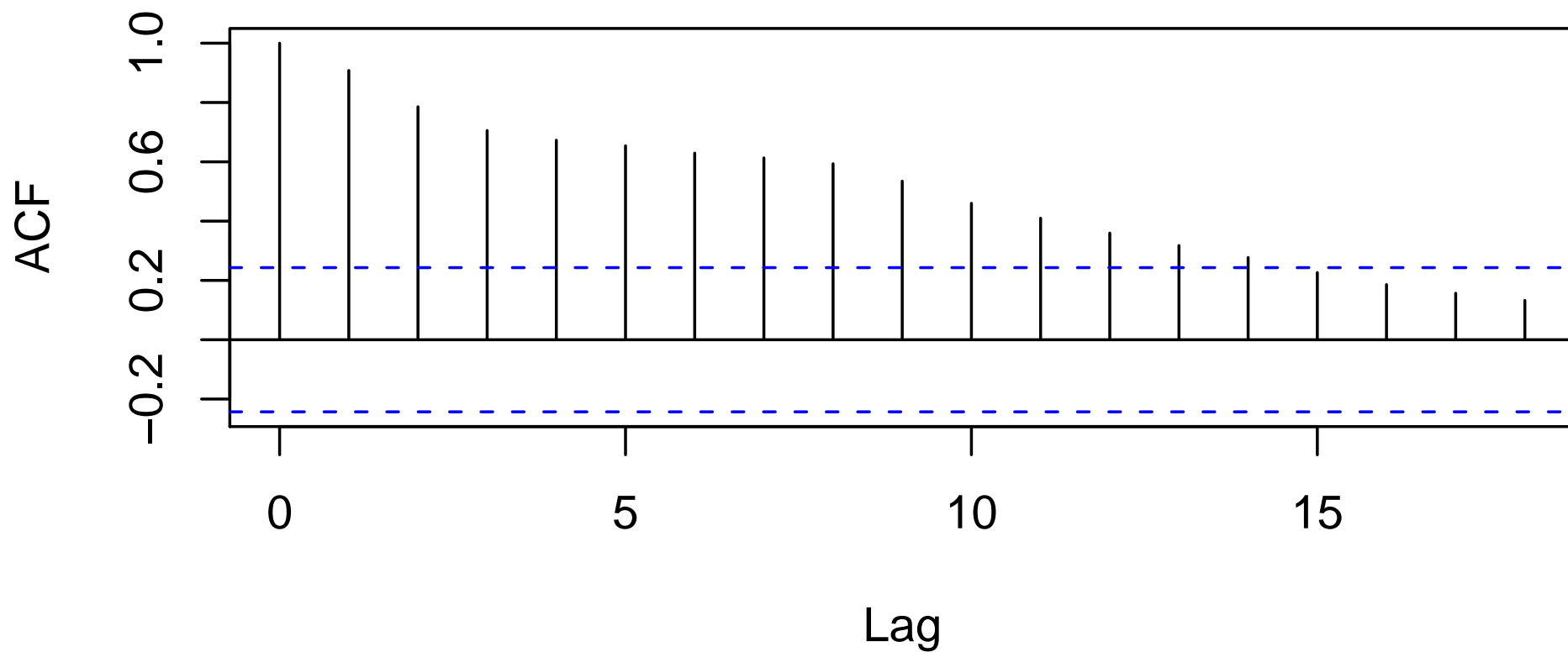
# Series approve



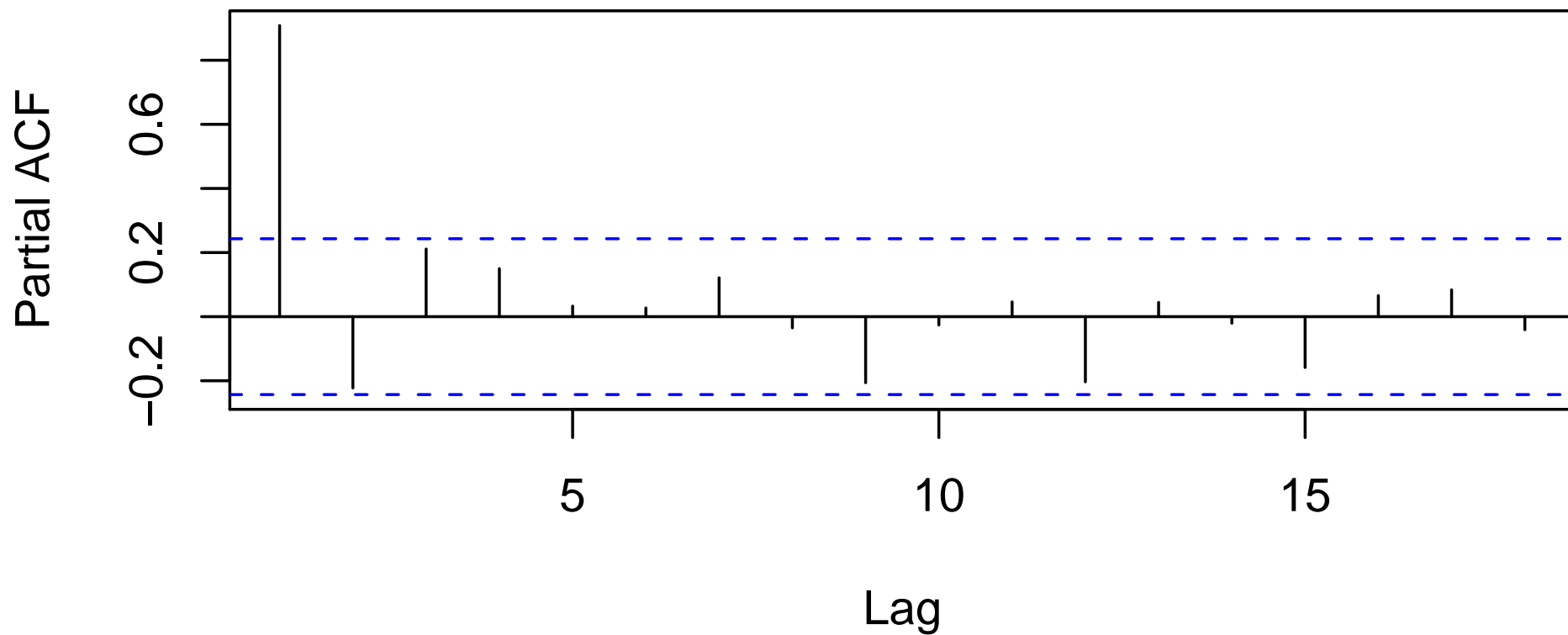
# Average Price of Oil



# Series avg.price

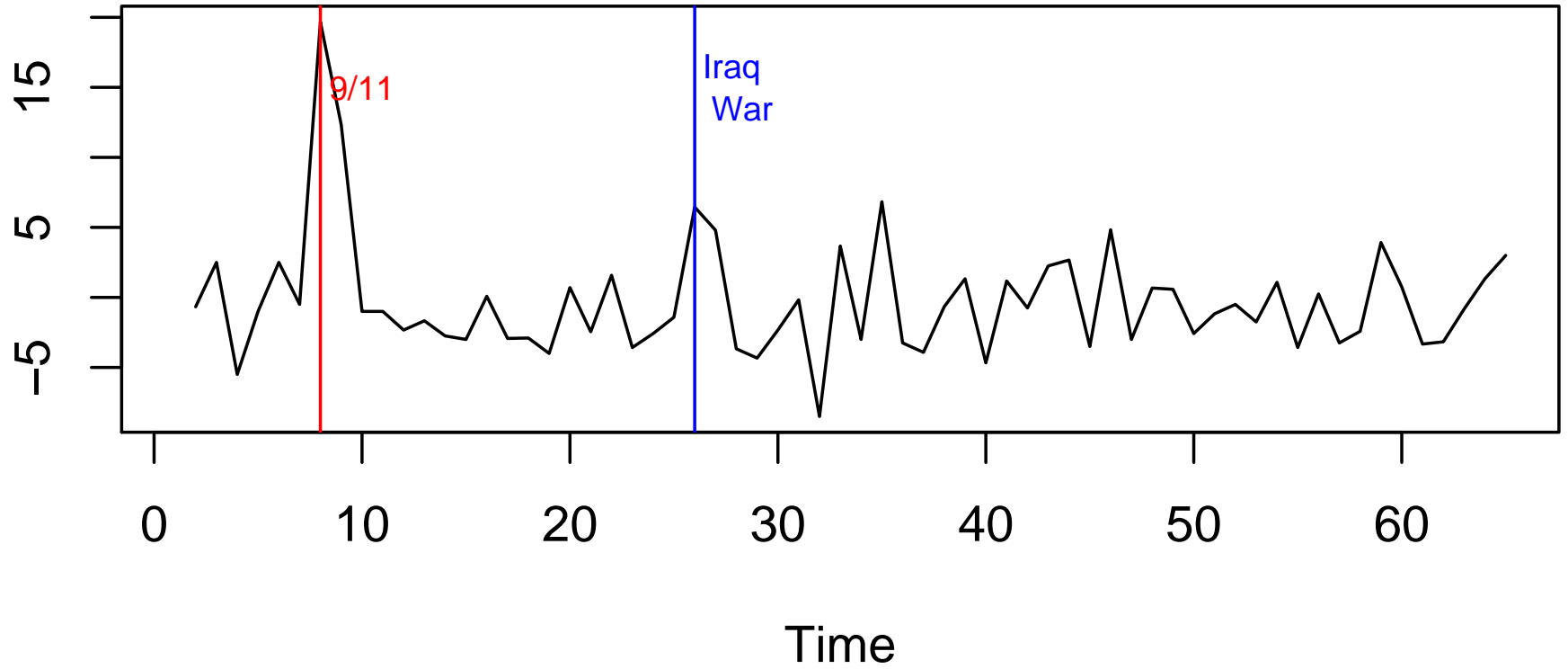


# Series avg.price

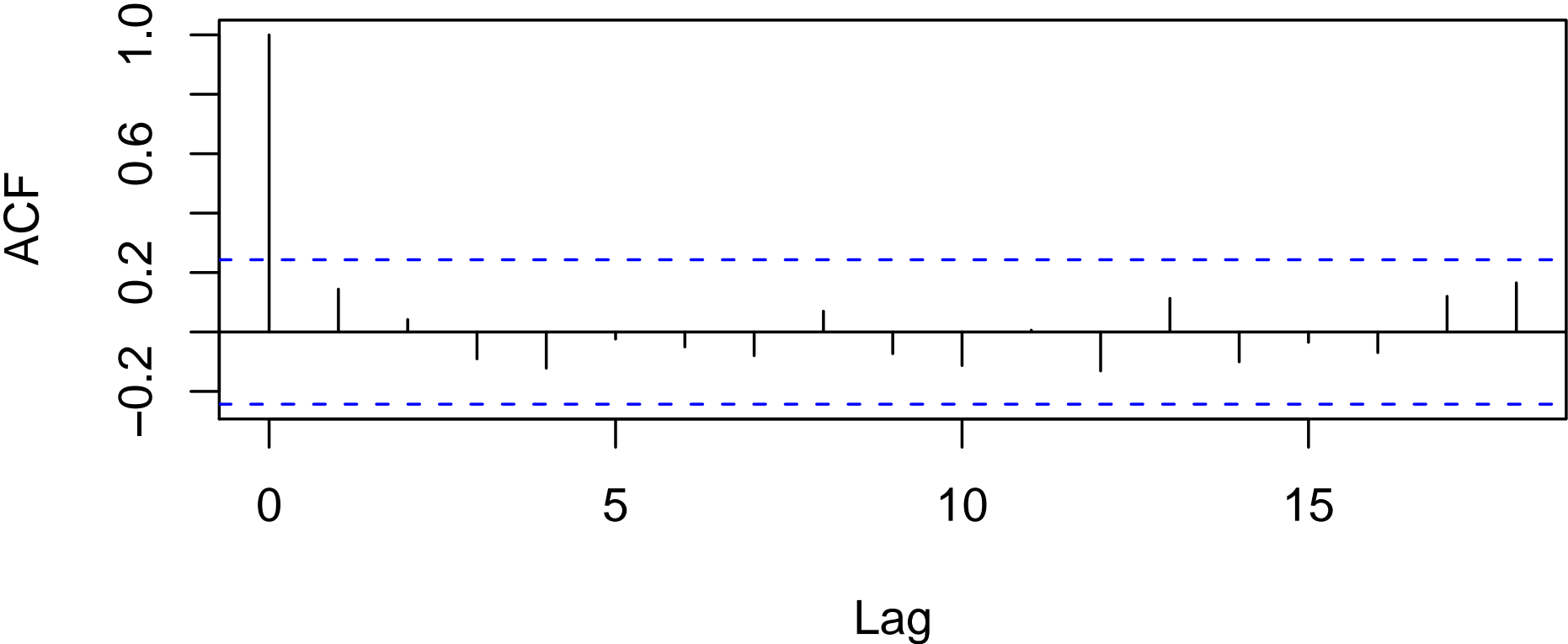


# US Presidential Approval

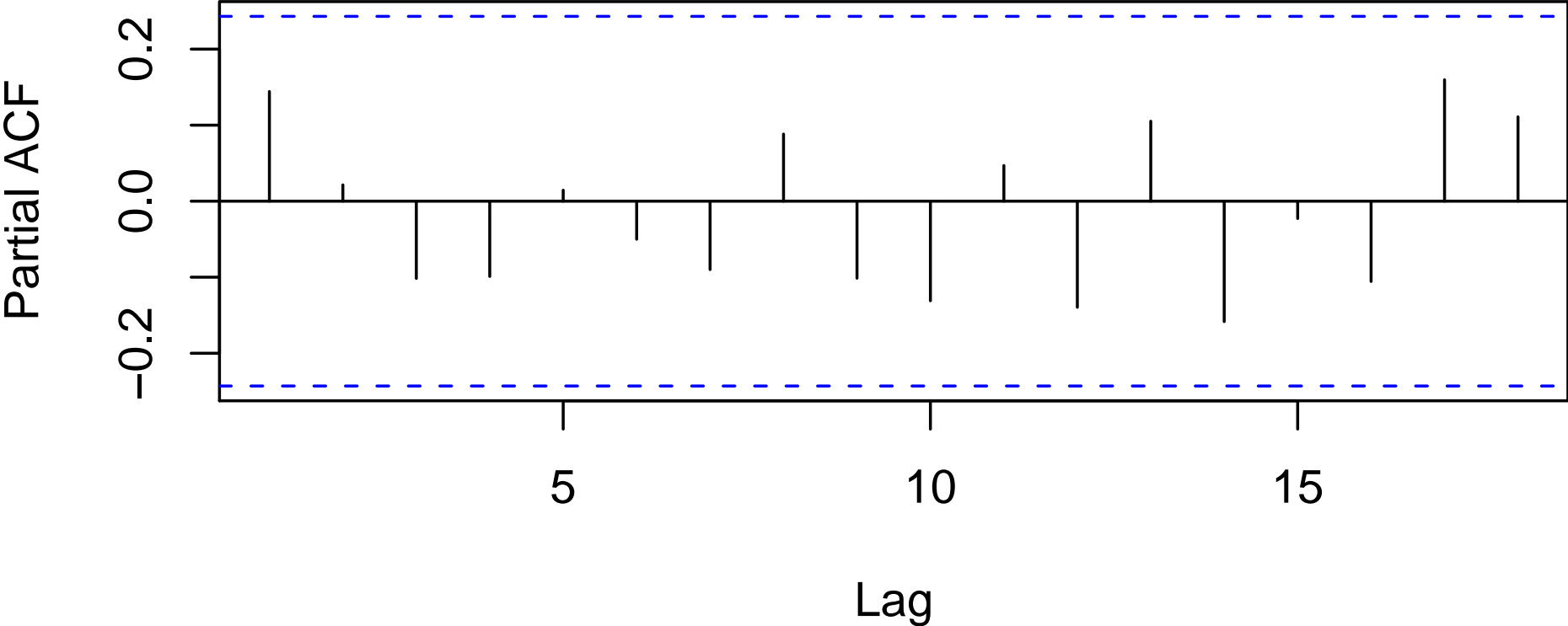
Change in Percent Approving



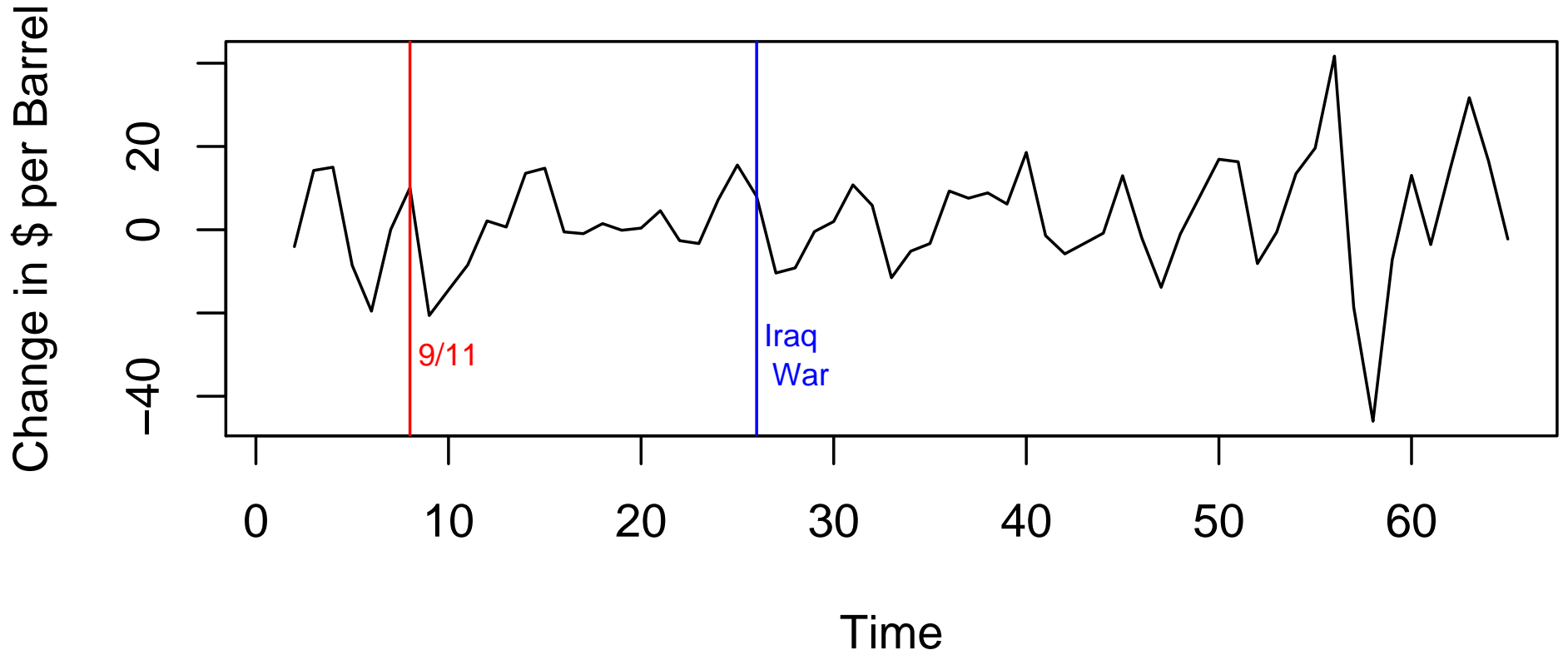
# Series approveDiff



# Series approveDiff

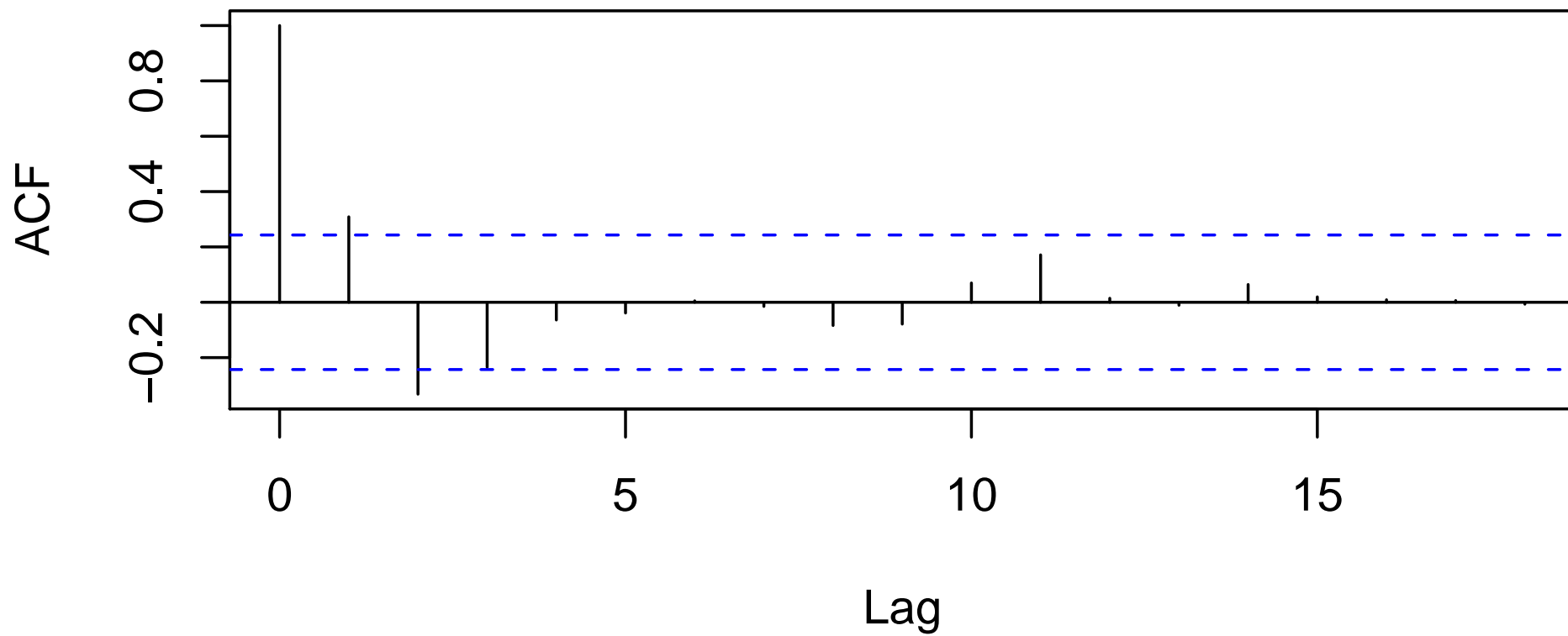


# Average Price of Oil

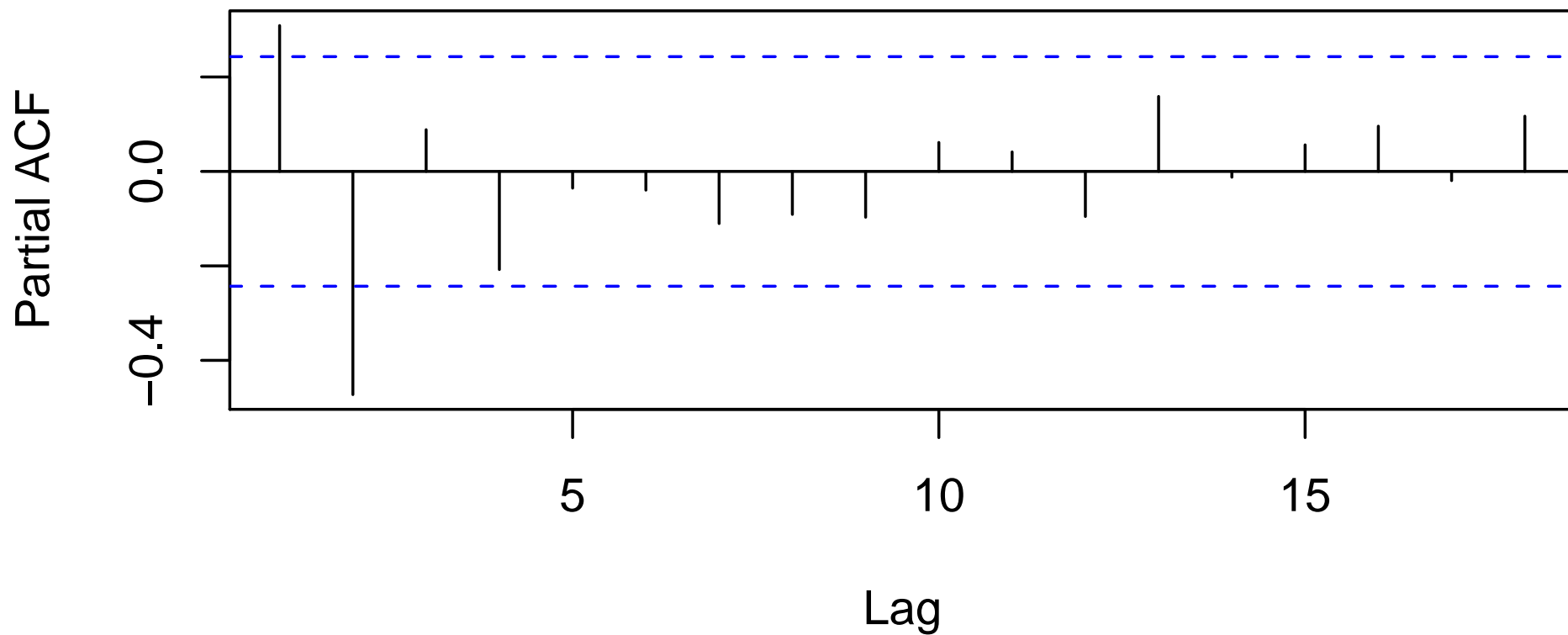




# Series avg.priceDiff



# Series avg.priceDiff



## Example: Presidential Approval

Many suspect `approve` and `avg.price` are non-stationary processes

Theoretically, what does this mean? Could an approval rate drift anywhere?

Note a better dependent variable would be the logit transformation of `approve`,  $\log(\text{approve}/(1 - \text{approve}))$ , which is unbounded and probably closer to the latent concept of support

And extending `approve` out to  $T = \infty$  would likely stretch the concept too far for a democracy with regular, anticipated elections

We'll ignore this to focus on the time series issues

## Example: Presidential Approval

*To a first approximation*, we suspect approve and avg.price may be non-stationary processes

We know that regressing one  $I(1)$  process on another risks spurious correlation

How can we investigate the relationship between these variables?

Strategy 1: ARIMA(0,1,0), first differencing

## Example: Presidential Approval

We load the data, plot it, with ACFs and PACFs

Then perform unit root tests

```
> PP.test(approve)
```

```
Phillips-Perron Unit Root Test
```

```
data: approve
```

```
Dickey-Fuller = -2.839, Truncation lag parameter = 3, p-value = 0.2350
```

```
> adf.test(approve)
```

```
Augmented Dickey-Fuller Test
```

```
data: approve
```

```
Dickey-Fuller = -3.957, Lag order = 3, p-value = 0.01721
```

```
alternative hypothesis: stationary
```

## Example: Presidential Approval

```
> PP.test(avg.price)
```

```
Phillips-Perron Unit Root Test
```

```
data: avg.price
```

```
Dickey-Fuller = -2.332, Truncation lag parameter = 3, p-value = 0.4405
```

```
> adf.test(avg.price)
```

```
Augmented Dickey-Fuller Test
```

```
data: avg.price
```

```
Dickey-Fuller = -3.011, Lag order = 3, p-value = 0.1649
```

```
alternative hypothesis: stationary
```

## Example: Presidential Approval

We create differenced versions of the time series, and repeat

```
> adf.test(na.omit(approveDiff))
```

```
Augmented Dickey-Fuller Test
```

```
data: na.omit(approveDiff)
Dickey-Fuller = -4.346, Lag order = 3, p-value = 0.01
alternative hypothesis: stationary
```

```
> adf.test(na.omit(avg.priceDiff))
```

```
Augmented Dickey-Fuller Test
```

```
data: na.omit(avg.priceDiff)
Dickey-Fuller = -5.336, Lag order = 3, p-value = 0.01
alternative hypothesis: stationary
```

## Example: Presidential Approval

We estimate an ARIMA(0,1,0), which fit a little better than ARIMA(2,1,2) on the AIC criterion

Call:

```
arima(x = approve, order = c(0, 1, 0),  
      xreg = xcovariates, include.mean = TRUE)
```

Coefficients:

	sept.oct.2001	iraq.war	avg.price
	11.207	5.690	-0.071
s.e.	2.519	2.489	0.034

sigma<sup>2</sup> estimated as 12.4: log likelihood = -171.2, aic = 350.5



## Example: Presidential Approval

To interpret the model, we focus on historical counterfactuals

What would Bush's approval have looked like if 9/11 hadn't happened?

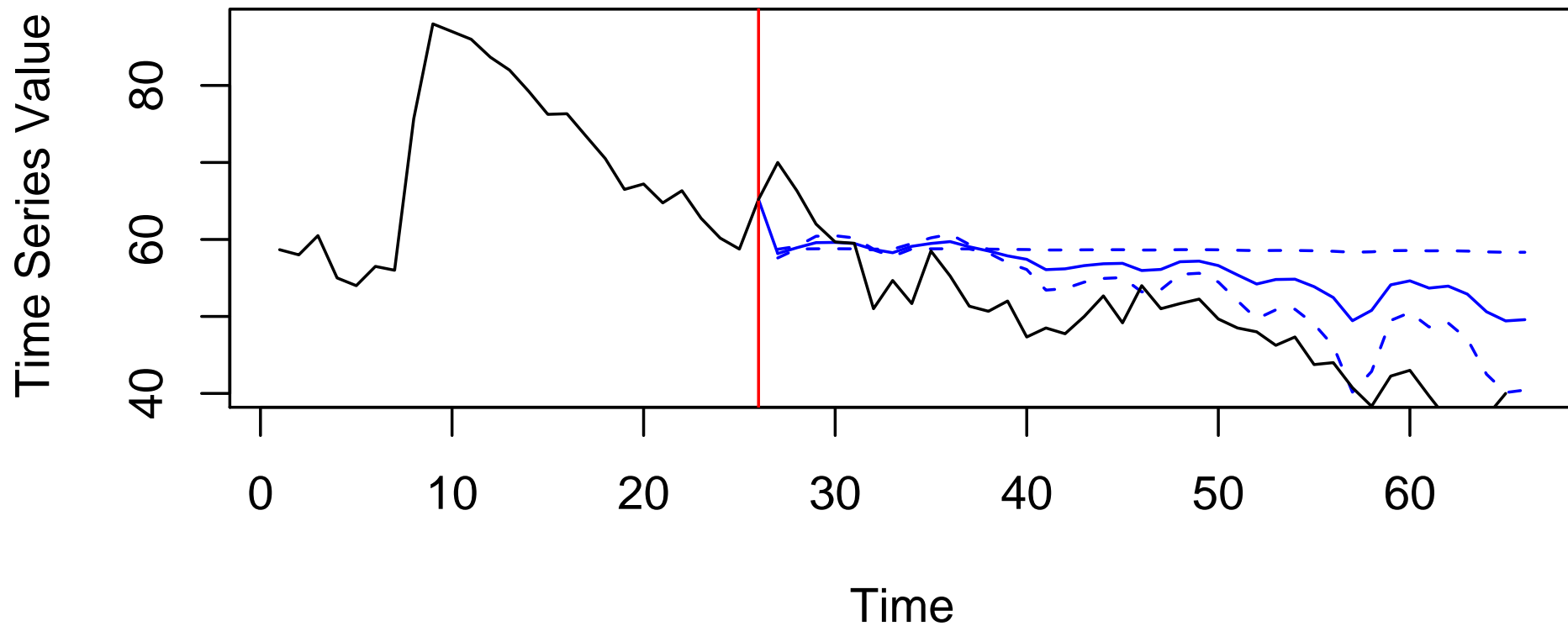
What if Bush had not invaded Iraq?

What if the price of oil had remained at pre-war levels?

Naturally, we only trust our results so far as we trust the model

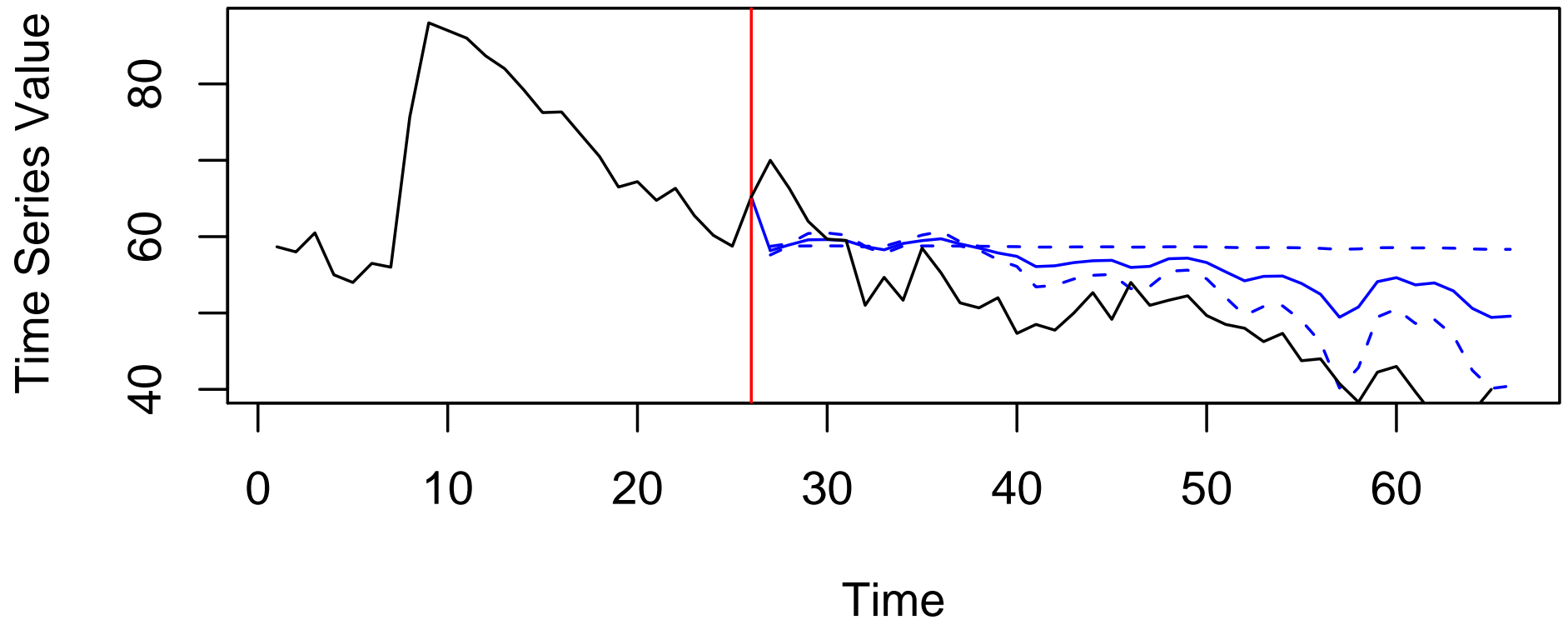
(which is not very much – we've left out a lot, like unemployment, inflation, boundedness of approve, . . . )

We simulate counterfactual approval using Zelig's implementation of ARIMA



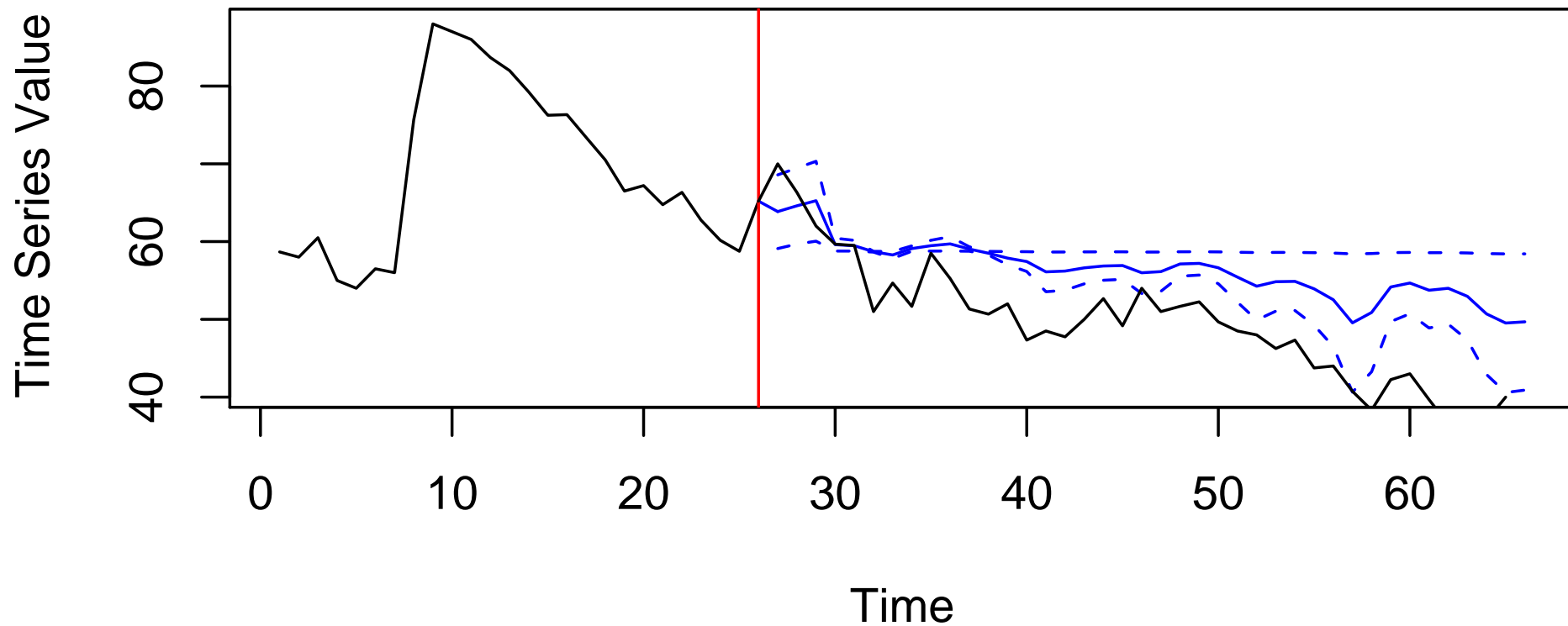
In blue: Predicted Bush approval without Iraq

In black: Actual approval



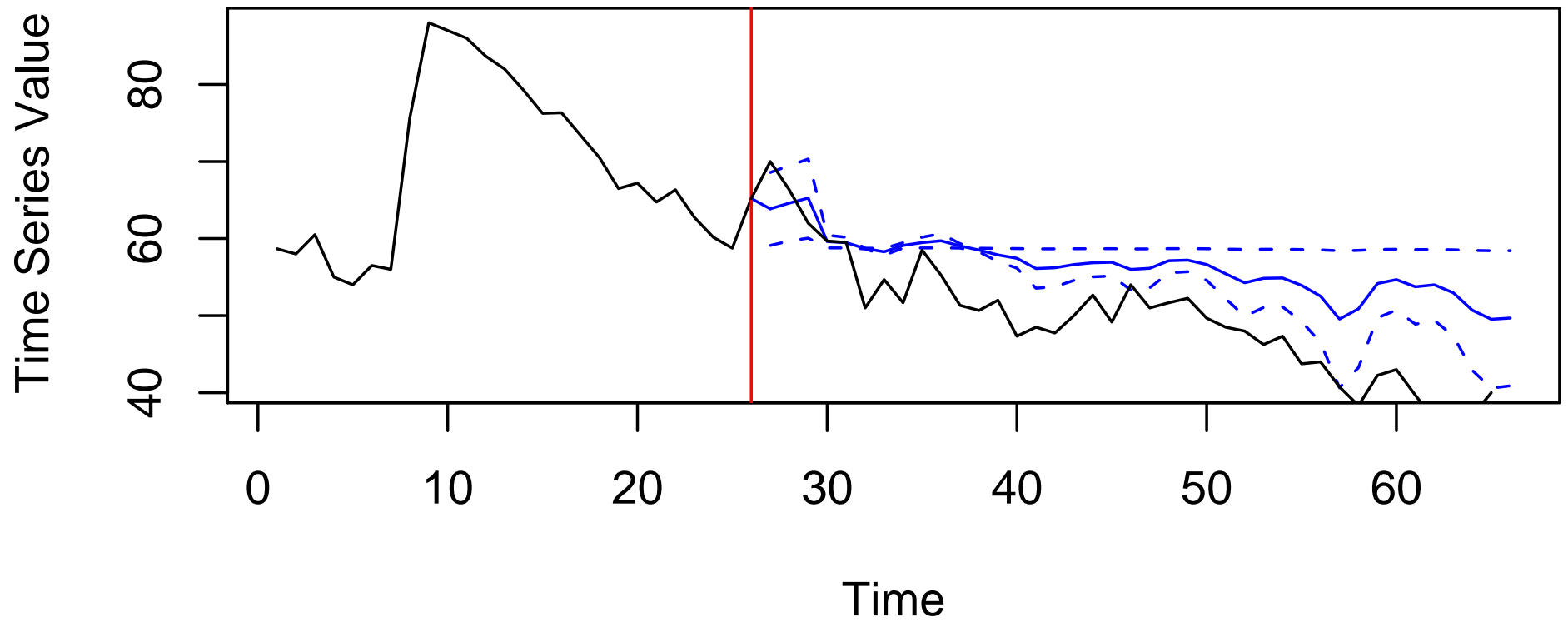
At first, starting the war in Iraq appears to help Bush's popularity

Then, it hurts – a lot. Sensible result. So are we done?

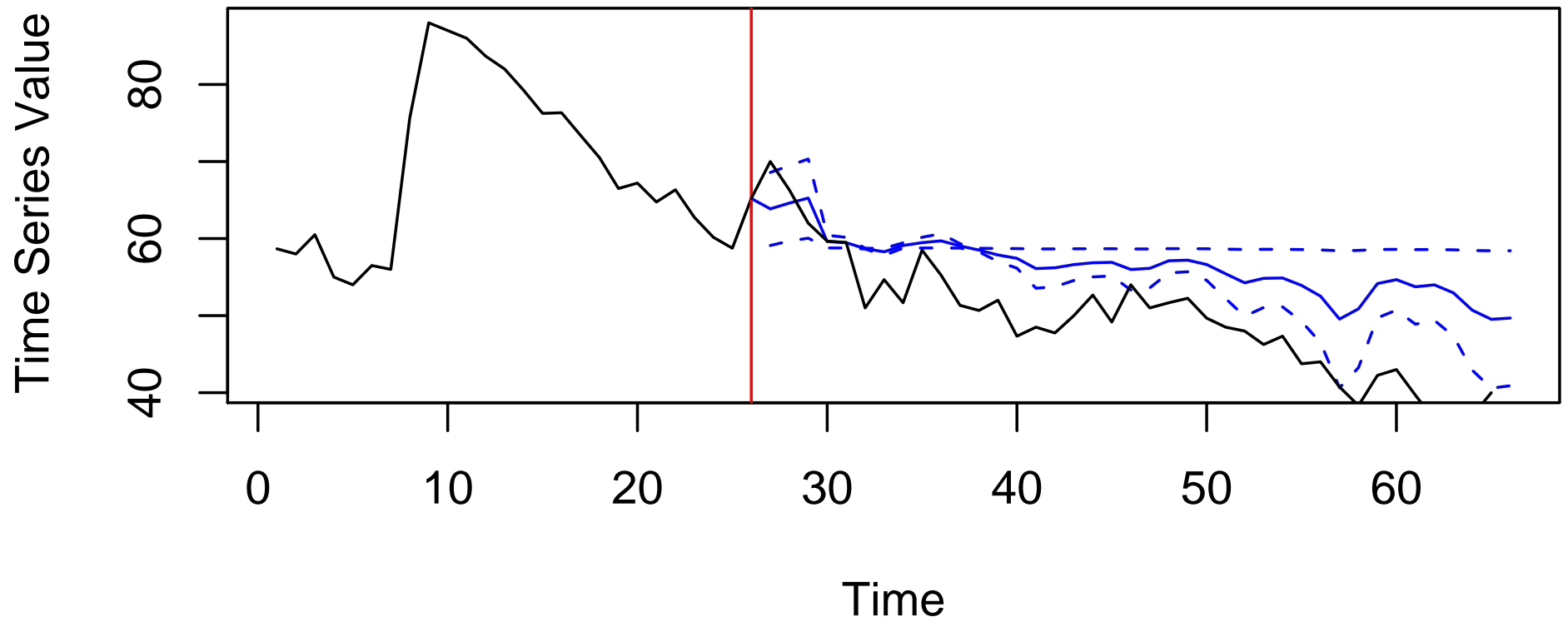


In blue: Predicted Bush approval with Iraq war

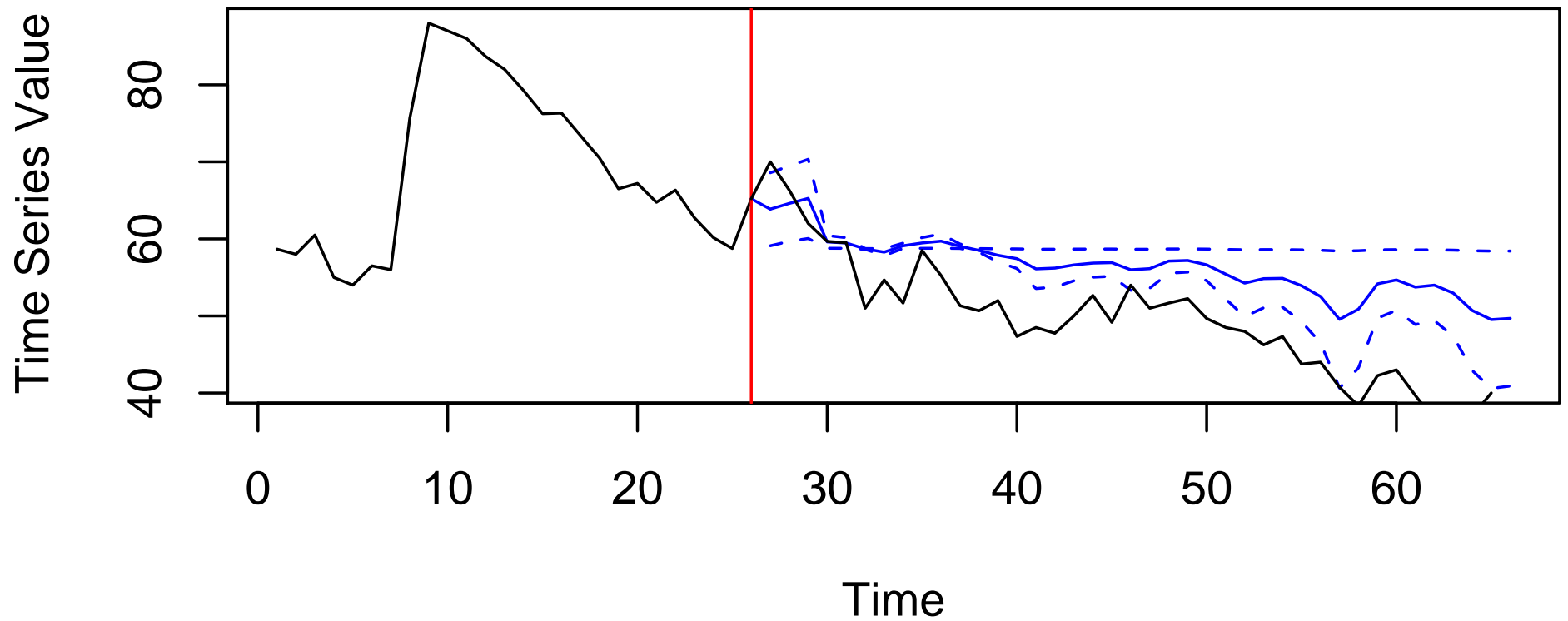
In black: Actual approval



Wait – can the model predict the long run approval rate?



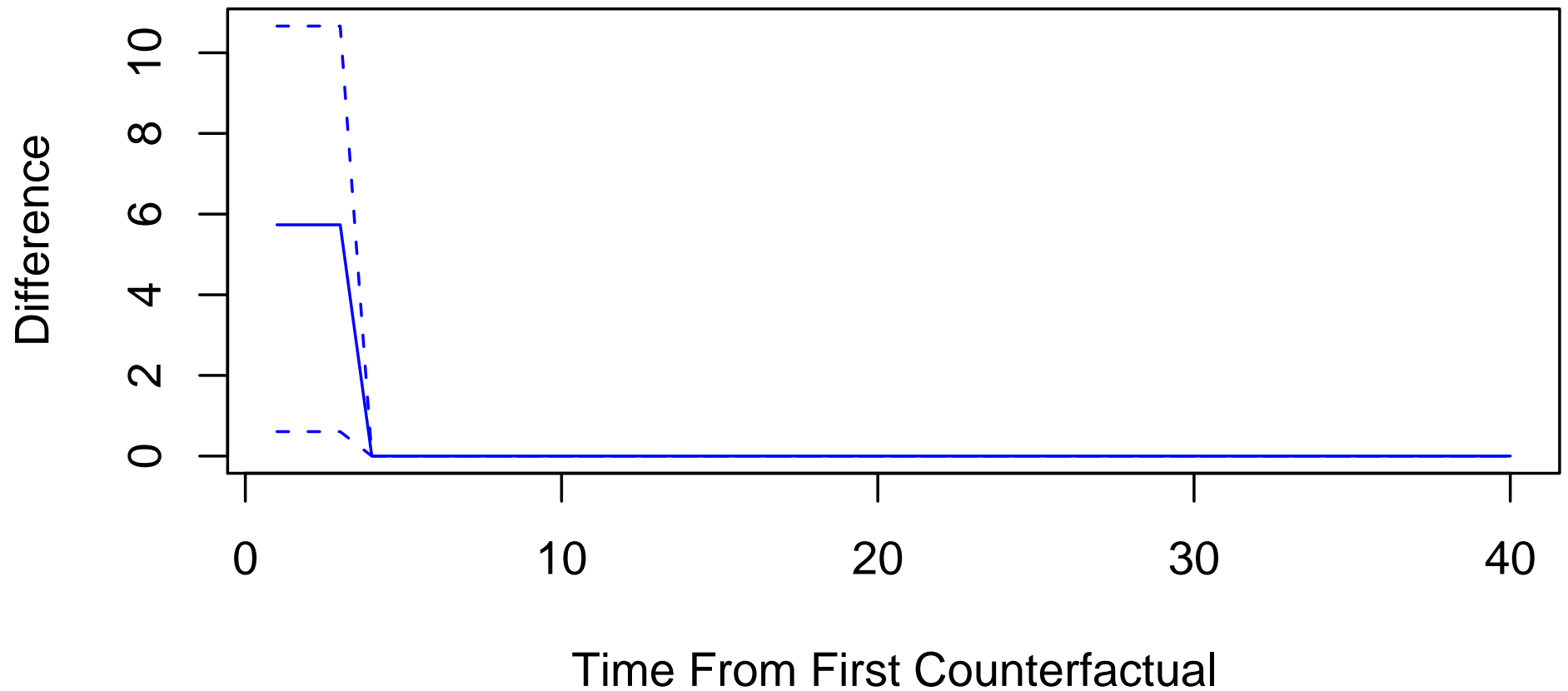
Wait – can the model predict the long run approval rate? Not even close



The model fit well for the first few months, then stays close to the ex ante “mean” approval

But reality (which is  $I(1)$ ) drifts off into the cellar

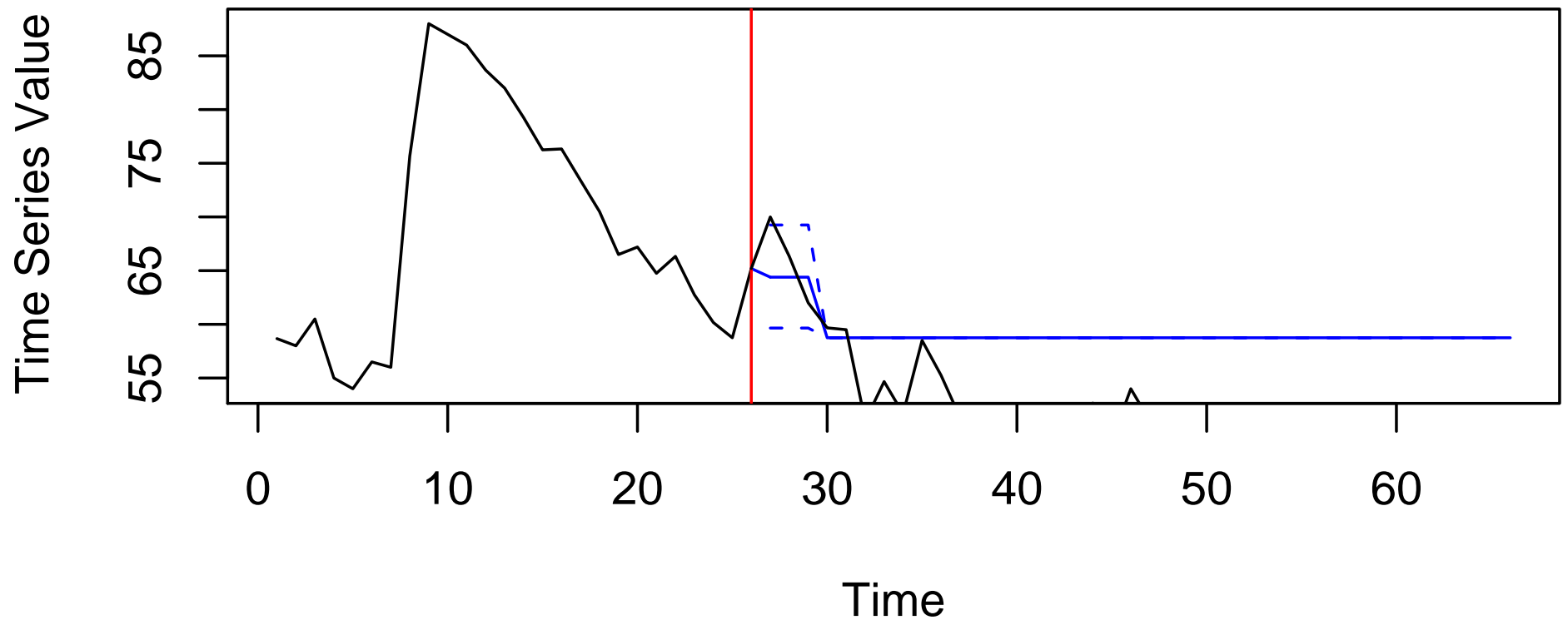
$$E[Y|X1] - E[Y|X]$$



First differences show that all the action is in the short-run

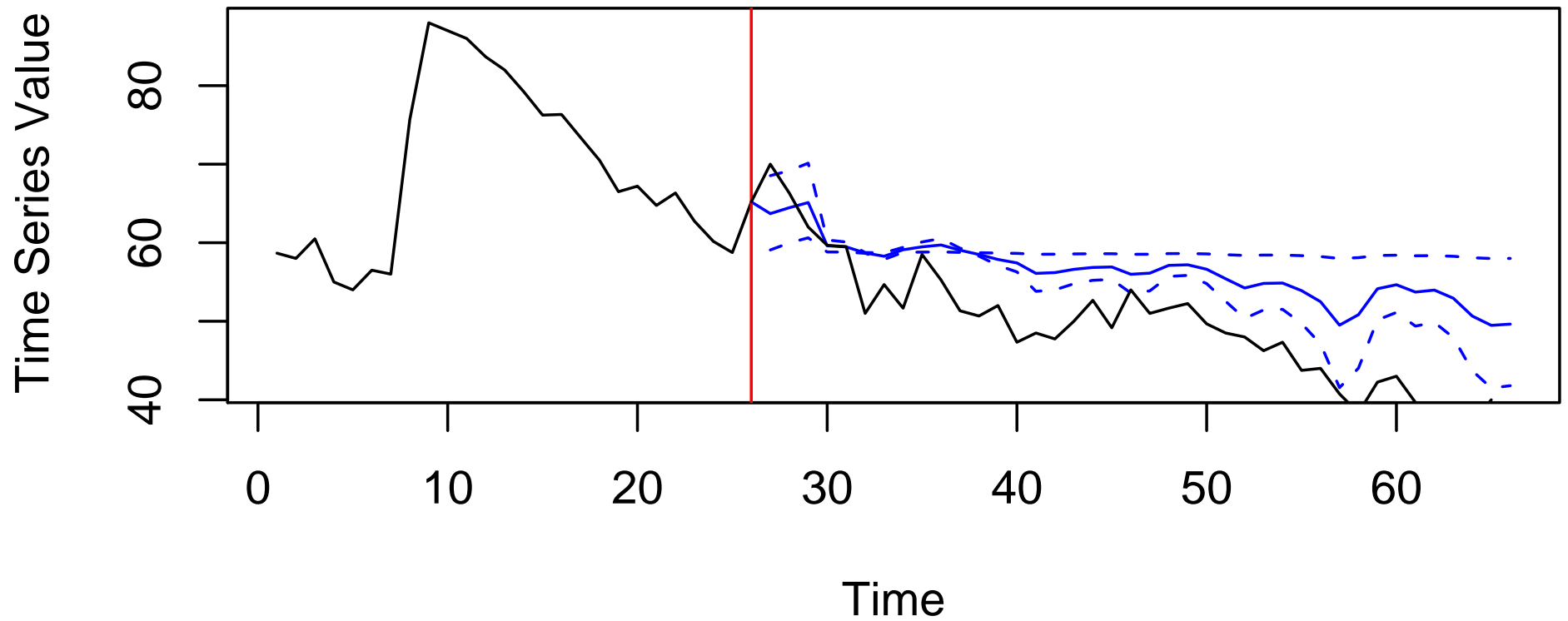
Long-run predictions are not feasible with unit root processes





Suppose Oil had stayed at its pre-war price of \$161/barrel

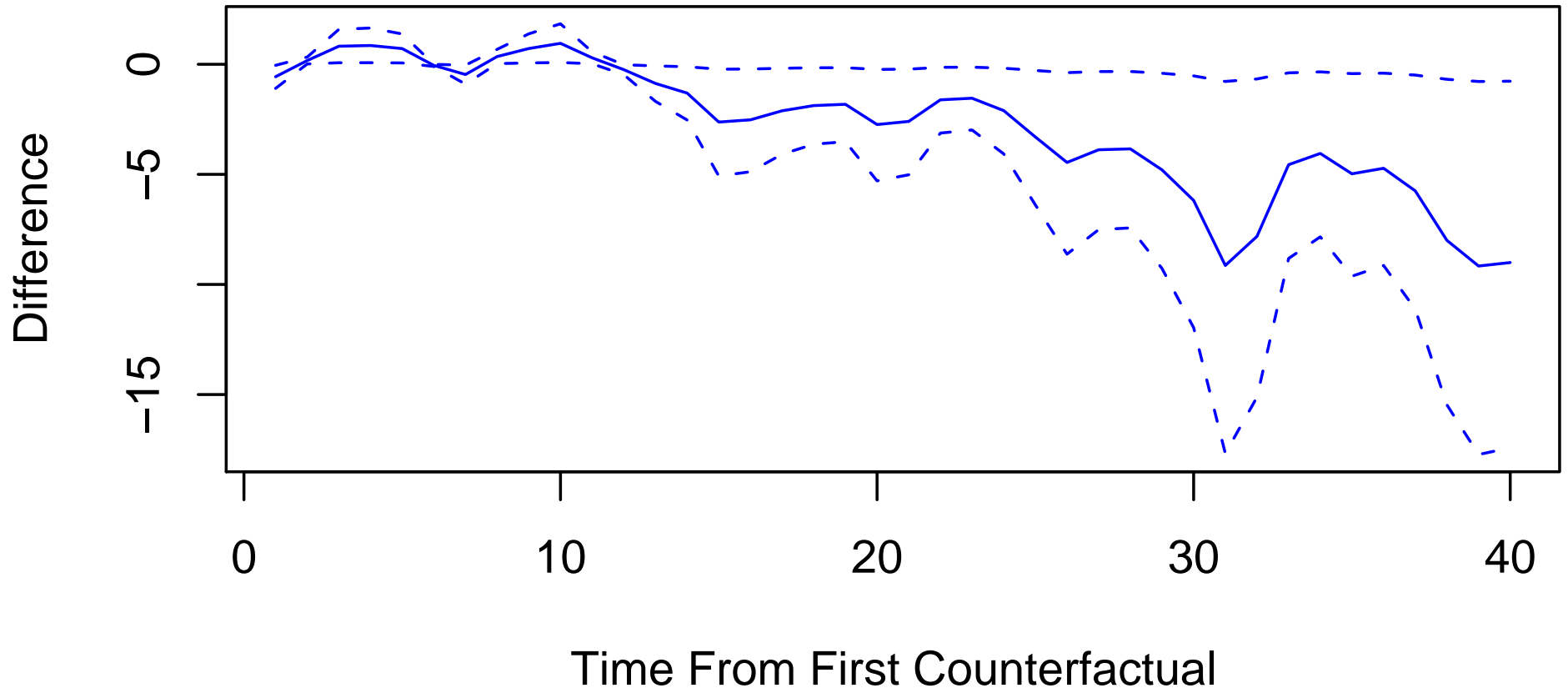
Then Bush's predicted popularity looks higher than the data



But wait – here are the factual “predictions” under the actual oil price

Miss the data by a mile

$$E[Y|X1] - E[Y|X]$$



## Limits of ARIMA

ARIMA(p,1,q) does a good job of estimating the short run movement of stationary variables

But does a terrible job with long-run levels

No surprise: The model includes no level information

While the observed level could drift anywhere

## Limits of ARIMA

Using  $\Delta y_t$  as our response has a big cost

Purging all long-run equilibrium relationships from our time series

These empirical long-run relationships may be spurious (why we're removing them)

But what if they are not? What if  $y_t$  and  $x_t$  really move together over time?

Then removing that long-run relationship removes theoretically interesting information from our data

Since most of our theories are about long-run levels of our variables, we have usually just removed the *most* interesting part of our dataset!

## Aside: Multiple Time Series

In our stationary time series example (road accidents), we had a single continuous time series and a binary covariate

In our nonstationary example (approval), we have two continuous time series

We chose to model approval as a function of oil prices, but we could have reversed this, and modeled oil prices as a function of approval

Why didn't we?

## Aside: Multiple Time Series

In our stationary time series example (road accidents), we had a single continuous time series and a binary covariate

In our nonstationary example (approval), we have two continuous time series

We chose to model approval as a function of oil prices, but we could have reversed this, and modeled oil prices as a function of approval

Why didn't we? We had a theory-based model: we don't think oil markets are driven by American opinions about their president

Theory informs the specification of "structural models"

Structural models could, of course, encompass multiple equations, as in SEM

But what if we don't have (or trust) a single theory about temporal relationships among multiple continuous variables?

## Aside: Multiple Time Series

There are atheoretical approaches to multiple continuous time series

Consider the following Vector Autoregression (VAR):

$$\text{approval}_t = \beta_0 + \sum_{j=0}^J \beta_{1j} \text{oil}_{t-j} + \sum_{k=1}^K \beta_{2k} \text{approval}_{t-k} + \varepsilon_t$$



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$$\text{oil}_t = \delta_0 + \sum_{j=0}^J \delta_{1j} \text{approval}_{t-j} + \sum_{k=1}^K \delta_{2k} \text{oil}_{t-k} + \eta_t$$

## Aside: Multiple Time Series

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$$\text{oil}_t = \delta_0 + \sum_{j=0}^J \delta_{1j} \text{approval}_{t-j} + \sum_{k=1}^K \delta_{2k} \text{oil}_{t-k} + \eta_t$$

This setup allows the price of oil to affect many things:

the future price of oil, through  $\delta_{2k}$ ,  $k = 1, \dots, K$ ;

the current approval rate, through  $\beta_{10}$ ; and

the future approval rating, through  $\beta_{1j}$ ,  $j = 1, \dots, J$

A parallel set of effects is possible for approval ratings

## Aside: Multiple Time Series

$$\text{approval}_t = \beta_0 + \sum_{j=0}^J \beta_{1j} \text{oil}_{t-j} + \sum_{k=1}^K \beta_{2k} \text{approval}_{t-k} + \varepsilon_t$$

$$\text{oil}_t = \delta_0 + \sum_{j=0}^J \delta_{1j} \text{approval}_{t-j} + \sum_{k=1}^K \delta_{2k} \text{oil}_{t-k} + \eta_t$$

We can use this system of equations to model the short run effects of shocks in any variable on all other variables

Those shocks should gradually die out in stationary series

Note the absence of MA terms. We could add them, making a VARMA model

Note the absence of binary covariates. We could add them, too.

Of course, this VAR assumes stationarity of both oil and approval

That's a problem – so we need, as before, to difference these variables first

## Aside: Multiple Time Series

$$\Delta \text{approval}_t = \psi_0 + \sum_{j=1}^J \psi_{1j} \Delta \text{oil}_{t-j} + \sum_{k=1}^K \psi_{2k} \Delta \text{approval}_{t-k} + u_t$$

$$\Delta \text{oil}_t = \zeta_0 + \sum_{j=1}^J \zeta_{1j} \Delta \text{approval}_{t-j} + \sum_{k=1}^K \zeta_{2k} \Delta \text{oil}_{t-k} + v_t$$

Now this is a VAR on differenced time series  
(I've changed parameters to emphasize this)

Now the recent changes in each variable can influence  
subsequent changes in all variables

*This model still does not have anything to say about the long run*

If you want to know more about VAR models, start with Box-Steffensmeier Ch. 4;  
there's a huge literature in econometrics

# Cointegration

Consider two time series  $y_t$  and  $x_t$ :

$$x_t = x_{t-1} + \varepsilon_t$$

$$y_t = y_{t-1} + 0.6x_t + \nu_t$$

where  $\varepsilon_t$  and  $\nu_t$  are (uncorrelated) white noise

$x_t$  and  $y_t$  are both: AR(1) processes, random walks, non-stationary, and I(1).

They are not spuriously correlated, but genuinely causally connected

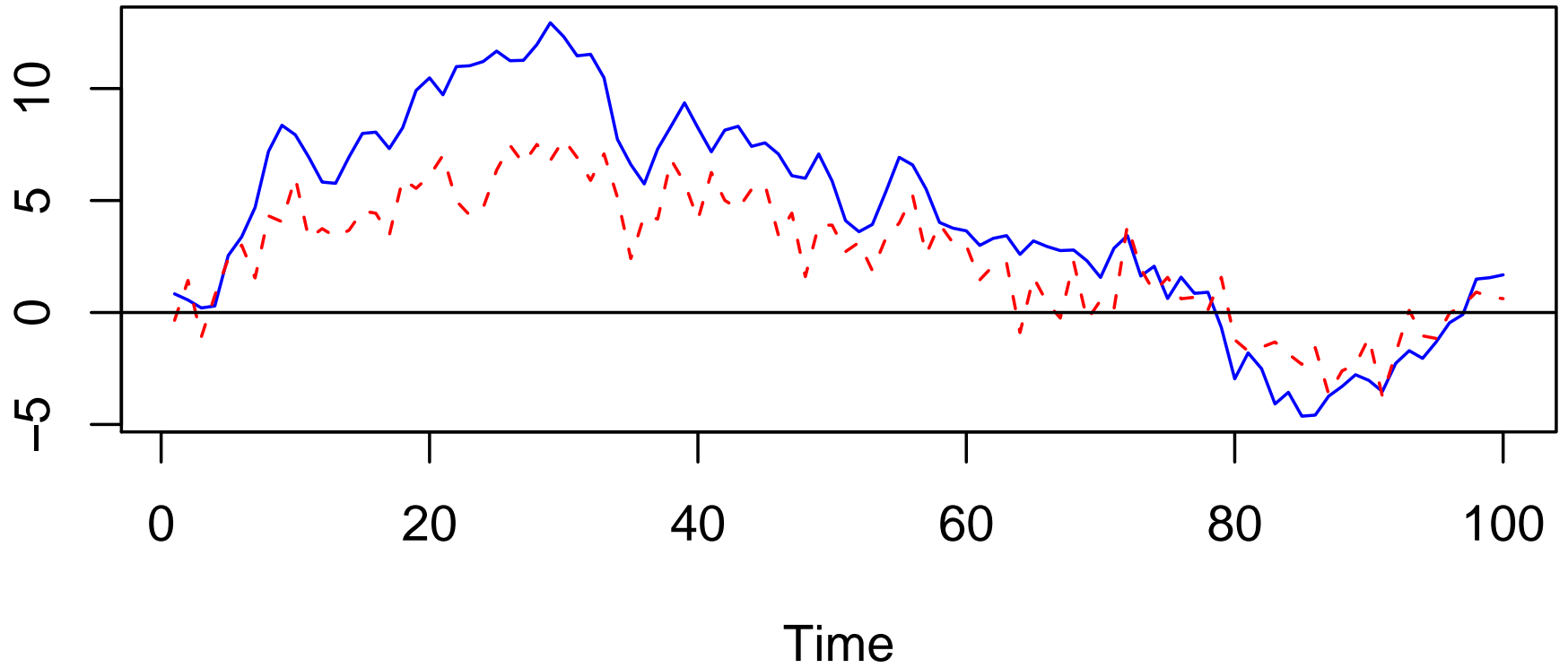
Neither tends towards any particular level, but each tends towards the other

A particularly large  $\nu_t$  may move  $y_t$  away from  $x_t$  briefly, but eventually,  $y_t$  will move back to  $x_t$ 's level

As a result, they will move together through  $t$  indefinitely

$x_t$  and  $y_t$  are said to be *cointegrated*

# Cointegrated I(1) variables



# Cointegration

Any two (or more) variables  $y_t, x_t$  are said to be cointegrated if

1. each of the variables is  $I(d)$ ,  $d \geq 1$

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Any two (or more) variables  $y_t, x_t$  are said to be cointegrated if

1. each of the variables is  $I(d)$ ,  $d \geq 1$  . . . usually, both are assumed to be  $I(1)$
2. there is some *cointegrating vector*  $\alpha$  such that

$$z_t = [y_t, x_t]' \alpha$$
$$z_t \sim I(0)$$

or in words, there is some linear combination of the non-stationary variables which is stationary

There may be many cointegrating vectors;  
the cointegration rank  $r$  gives their number

## Cointegration: Engle-Granger Two Step

Several ways to find the cointegration vector(s) and use it to analyze the system

Simplest is Engle-Granger Two Step Method

Just repeated application of linear regression!

Works best if cointegration rank is  $r = 1$  and

serial correlation is  $ARIMA(p,d,0)$  with clearly established  $p$

that is, more complex or uncertain serial correlation can cause bias,  
as with any other least square time series model

Fancier estimation techniques will address this limitation

## Cointegration: Engle-Granger Two Step

**Step 1:** Estimate the cointegration vector by least squares with no constant:

$$y_t = \alpha_1^* x_{t-1} + \alpha_2^* x_{t-2} + \dots + \alpha_K^* x_{t-K} + z_t$$

This gives us the cointegration vector  $\alpha = (1, -\alpha_1^*, -\alpha_2^*, \dots, -\alpha_K^*)$

and the long-run equilibrium path of the cointegrated variables,  $\hat{z}_t$

We can test for cointegration by checking that  $\hat{z}_t$  is stationary

Note that the usual unit root tests work, but with different critical values

This is because the  $\hat{\alpha}$ 's are very well estimated: “super-consistent”  
(converge to their true values very fast as  $T$  increases)

## Cointegration: Engle-Granger Two Step

### Step 2: Estimate an Error Correction Model

After obtaining the equilibrium  $\hat{z}_t$ 's and confirming they are  $I(0)$ , we can estimate a particularly useful specification known as an *error correction model*, or ECM

ECMs simultaneously estimate long- and short-run effects for a system of cointegrated variables

Better than  $ARIMA(p,d,0)$  because we don't throw away level information

ECMs are simple generalizations of VARs in the differences of our time series

Interestingly, ECMs can also be estimated with least squares

## Cointegration: Engle-Granger Two Step

For a bivariate system of  $y_t, x_t$ , two equations describe how this cointegrated process evolves over time:

$$\Delta y_t = \psi_0 + \gamma_1 \hat{z}_{t-1} + \sum_{j=1}^J \psi_{1j} \Delta x_{t-j} + \sum_{k=1}^K \psi_{2k} \Delta y_{t-k} + u_t$$

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These equations are the “error correction” form of the model

Like the VAR on which it is based (the one we saw earlier!), it shows the short-run relationships across all our time series

Unlike a VAR, an error correction model (ECM) also captures how  $y_t$  and  $x_t$  respond to deviations from their long run relationship

(Technically, the above is a Vector ECM or VECM model, which is the cointegrated generalization of VAR)



## Cointegration: Engle-Granger Two Step

Let's focus on the evolution of  $\Delta y_t$  as a function of its lags, lags of  $\Delta x_t$ , and the "error" in the long-run equilibrium,  $\hat{z}_{t-1}$ :

$$\Delta y_t = \psi_0 + \gamma_1 \hat{z}_{t-1} + \sum_{j=1}^J \psi_{1j} \Delta x_{t-j} + \sum_{k=1}^K \psi_{2k} \Delta y_{t-k} + u_t$$

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$\gamma < 0$  must hold for at least one  $\gamma$ :

This is the speed of adjustment back to equilibrium;  
larger negative values imply faster adjustment

This is the central assumption of cointegration:

In the long run,  $y_t$  and  $x_t$  cannot diverge too much

So short-run differences must be made up later by convergence

For example,  $y_t$  (or  $x_t$ ) must *eventually reverse course* after a big shift away from  $x_t$

A negative  $\gamma_1$  shows how quickly  $y_t$  reverses back to  $x_t$

## Cointegration: Engle-Granger Two Step

Recall our cointegrated time series,  $y_t$  and  $x_t$ :

$$x_t = x_{t-1} + \varepsilon_t$$

$$y_t = y_{t-1} + 0.6x_t + \nu_t$$

To estimate the Engle-Granger Two Step for these data, we do the following in R:

```
set.seed(123456)

# Generate cointegrated data
e1 <- rnorm(100)
e2 <- rnorm(100)
x <- cumsum(e1)
y <- 0.6*x + e2

# Run step 1 of the E-G two step
coint.reg <- lm(y ~ x -1)
coint.err <- residuals(coint.reg)
```

```
# Check for stationarity of the cointegration vector
punitroot(adf.test(coint.err)$statistic, trend="nc")

# Make the lag of the cointegration error term
coint.err.lag <- coint.err[1:(length(coint.err)-2)]

# Make the difference of y and x
dy <- diff(y)
dx <- diff(x)

# And their lags
dy.lag <- dy[1:(length(dy)-1)]
dx.lag <- dx[1:(length(dx)-1)]

# Delete the first dy, because we are missing lags for this obs
dy <- dy[2:length(dy)]

# Estimate an Error Correction Model with LS
ecm1 <- lm(dy ~ coint.err.lag + dy.lag + dx.lag)
summary(ecm1)
```

Call:

```
lm(formula = x ~ y - 1)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-3.4565	-0.7754	0.3567	1.7542	5.7091

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
y	1.43472	0.05568	25.77	<2e-16 ***

---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

Residual standard error: 1.737 on 99 degrees of freedom

Multiple R-squared: 0.8702, Adjusted R-squared: 0.8689

F-statistic: 663.9 on 1 and 99 DF, p-value: < 2.2e-16

```
> punitroot(adf.test(coint.err)$statistic, trend="nc")
```

Dickey-Fuller

6.551997e-05

Call:

```
lm(formula = dy ~ coint.err.lag + dy.lag + dx.lag)
```

Residuals:

Min	1Q	Median	3Q	Max
-2.9553	-0.5375	0.1538	0.7042	2.3240

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	0.02267	0.10381	0.218	0.828
coint.err.lag	-0.96617	0.15864	-6.090	2.45e-08 ***
dy.lag	-1.05776	0.10848	-9.751	6.21e-16 ***
dx.lag	0.81035	0.11223	7.221	1.33e-10 ***

---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

Residual standard error: 1.026 on 94 degrees of freedom

Multiple R-squared: 0.5456, Adjusted R-squared: 0.5311

F-statistic: 37.62 on 3 and 94 DF, p-value: 4.624e-16

## Cointegration: Johansen estimator

Alternatively, we can use the `urca` package, which handles unit roots and cointegration analysis:

```
# Create a matrix of the cointegrated variables
cointvars <- cbind(y,x)
```

```
# Perform cointegration tests
```

```
coint.test1 <- ca.jo(cointvars,
                    ecdet = "const",
                    type="eigen",
                    K=2,
                    spec="longrun"
                    )
```

```
summary(coint.test1)           # Check the cointegration rank here
```

```
# Using the output of the test, estimate an ECM
```

```
ecm.res1 <- cajorls(coint.test1,
                   r = 1,           # Cointegration rank
                   reg.number = 1) # which variable(s) to put on LHS
                                   # (column indexes of cointvars)
```

```
summary(ecm.res1$rlm)
```

## Cointegration: Johansen estimator

```
#####  
# Johansen-Procedure #  
#####
```

Test type: maximal eigenvalue statistic (lambda max) , without linear t

Eigenvalues (lambda):

```
[1] 3.105e-01 2.077e-02 -1.400e-18
```

Values of teststatistic and critical values of test:

	test	10pct	5pct	1pct
r <= 1	2.06	7.52	9.24	12.97
r = 0	36.44	13.75	15.67	20.20

Eigenvectors, normalised to first column:

(These are the cointegration relations)

	y.l2	x.l2	constant
y.l2	1.00000	1.00	1.000
x.l2	-0.58297	10.13	-1.215



constant -0.02961 -50.24 -38.501

Weights W:

(This is the loading matrix)

	y.l2	x.l2	constant
y.d	-0.967715	-0.001015	-1.004e-18
x.d	0.002461	-0.002817	-2.899e-19

## Cointegration: Johansen estimator

Call:

```
lm(formula = substitute(form1), data = data.mat)
```

Residuals:

Min	1Q	Median	3Q	Max
-2.954	-0.536	0.150	0.712	2.318

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
ect1	-0.968	0.158	-6.13	2.0e-08	***
y.d11	-1.058	0.108	-9.82	4.1e-16	***
x.d11	0.809	0.112	7.26	1.1e-10	***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.02 on 95 degrees of freedom

Multiple R-squared: 0.546, Adjusted R-squared: 0.532

F-statistic: 38.1 on 3 and 95 DF, p-value: 2.97e-16

## Example: Approval

Return to our Bush approval example, and estimate an ECM equivalent to the ARIMA(0,1,0) model we chose:

Residuals:

Min	1Q	Median	3Q	Max
-7.140	-1.675	-0.226	1.643	5.954

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
ect1	-0.1262	0.0301	-4.20	9.4e-05	***
sept.oct.2001	19.5585	2.1174	9.24	5.4e-13	***
iraq.war	5.0187	1.6243	3.09	0.0031	**
approve.d11	-0.3176	0.0945	-3.36	0.0014	**
avg.price.d11	-0.0505	0.0259	-1.95	0.0561	.

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 '1'

Residual standard error: 2.67 on 58 degrees of freedom

Multiple R-squared: 0.63, Adjusted R-squared: 0.598

F-statistic: 19.8 on 5 and 58 DF, p-value: 1.91e-11

## Cointegration: Final Thoughts

Cointegration and ECMs give us a way to cope with nonstationary time series without throwing away levels information

They provide information on short-run effects *and* long-run tendencies towards equilibrium

They *do not* tell us exact long-run destinations, because for nonstationary series there isn't one

Could you use ECM to talk about long-run equilibria in stationary time series?

Many methodologists think this is possible and useful