Maximum Likelihood Methods
for the Social Sciences
POLS 510 · CSSS 510

Monty Hall
Monte Carlo

Political Science *and* CSSS
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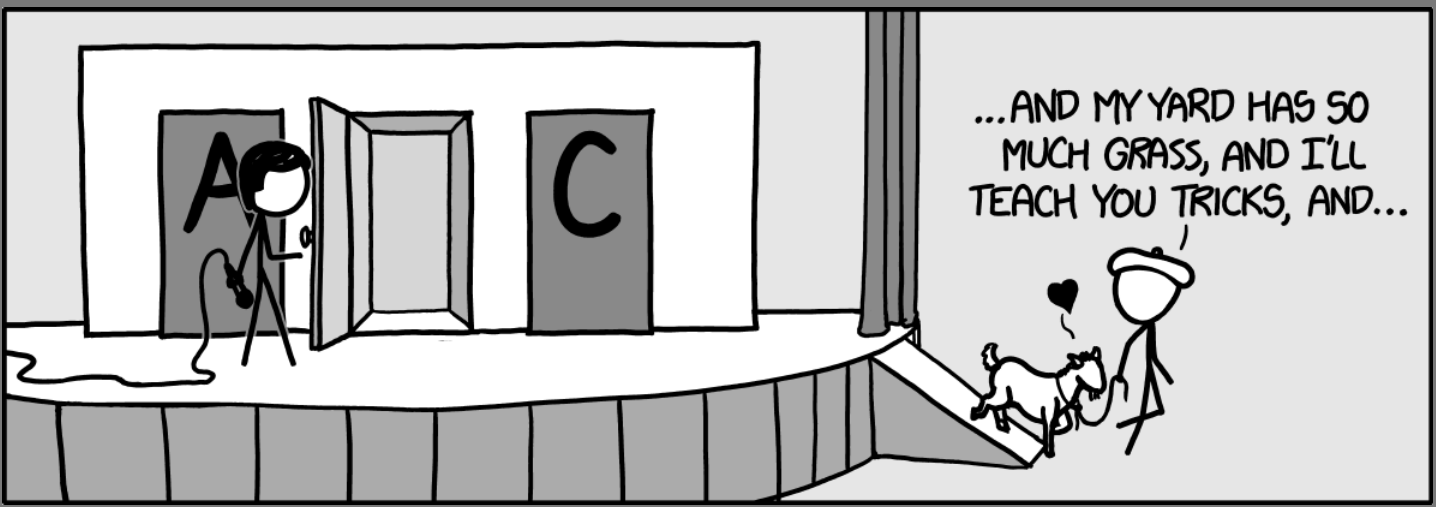

A few minutes later, the goat from
behind door C drives away in the car

Randall Munroe `xkcd.com/1282`

The Monty Hall Problem

On *Let's Make a Deal*, host Monty Hall offers you the following choice:

1. There are 3 doors. Behind one is a car. Behind the other two are goats.
2. You choose a door. It stays closed.
3. Monty picks one of the two remaining doors, and opens it to reveal a goat.
4. Your choice: Keep the door you chose in step 1, or switch to the third door.

What should you do?

A longer example: Monte Hall

What is the probability problem here?

What is the probability of winning a car from staying?

What is the probability of winning a car from switching?

How can we solve the problem?

1. Probability theory: Bayes Rule, also known as “math”

2. Brute force: stochastic simulation, also known as Monte Carlo

The first gets harder for hard problems.

The second doesn’t get any harder but is less general.

A longer example: Monty Hall

Pseudo-code: A sketch of the solution

```
# Set up the doors, goats, and car

# Contestant picks a door

# Monty ``picks'' a remaining door

# Record where the car and goats were

# Do all of the above many many times

# Print the fraction of times a car was found
```

Monty Hall: Easy-to-read solution

```
# Monty Hall Problem
# Chris Adolph
# 1/6/2005

sims <- 10000                   # Simulations run
doors <- c(1,0,0)               # The car (1) and the goats (0)
cars.chosen <- 0                # Save cars from first choice here
cars.rejected <- 0              # Save cars from switching here

for (i in 1:sims) {             # Loop through simulations

  # First, contestant picks a door
  first.choice <- sample(doors, 3, replace=FALSE)

  # Choosing a door means rejecting the other two
  chosen <- first.choice[1]
  rejected <- first.choice[2:3]

  # Monty Hall removes a goat from the rejected doors
  rejected <- sort(rejected)
```

```
  if (rejected[1]==0)
    rejected <- rejected[2]

  # Record keeping:  where was the car?
  cars.chosen <- cars.chosen + chosen
  cars.rejected <- cars.rejected + rejected

}

cat("Probability of a car from staying with 1st door",
    cars.chosen/sims,"\n")

cat("Probability of a car from switching to 2nd door",
    cars.rejected/sims,"\n")
```

Monty Hall: Less code solution

```
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for (i in 1:sims) {
  first.choice <- sample(doors, 3, replace=FALSE)
  cars.chosen <- cars.chosen + first.choice[1]
  cars.rejected <- cars.rejected + sort(first.choice[2:3])[2]
}

cat("Probability of a car from staying with 1st door",
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cat("Probability of a car from switching to 2nd door",
    cars.rejected/sims,"\n")
```

Monty Hall: Faster runtime solution

```
# Monty Hall Problem
# Chris Adolph
# 10/8/2013

sims <- 10000                    # Simulations run
doors <- c(1,0,0)                # The car (1) and the goats (0)

# Faster: avoiding loops with lapply()
result <- lapply (1:sims,
                  function (x, doors) {
                         pick <- sample(doors, 3, replace=FALSE);
                         c(pick[1],max(pick[2:3])) },
                  doors)

# Combine the list of results into a matrix
result <- do.call(rbind, result)

# Take the average of each column
result <- apply(result, 2, mean)
```

Monty Hall: Faster runtime solution

```
cat("Probability of a car from staying with 1st door",
    result[1],"\n")

cat("Probability of a car from switching to 2nd door",
    result[2],"\n")
```

On your own

A sample session to work through will be on the web

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- Collectively, the total probability of car behind door 2 *or* 3 is 2/3
- By revealing a goat, Monty shows us where car must be if it is behind 2 or 3
- In effect, Monty is giving us a choice to take *any* car behind door 1 *or any* car behind doors 2 *and* 3

Monty Hall & Bayes Rule

Solved Monty Hall by brute force. Could we solve by mathematics?

Recall

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\mathrm{P}(a|b) &= \frac{\mathrm{P}(b|a)\mathrm{P}(a)}{\mathrm{P}(b)}
\end{aligned}$$

Bayes Rule.

Monty Hall & Bayes Rule (1)

Recall that we have doors A, B, and C.

Ex ante, the probability the car is behind each of these doors is just

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{3}$$

Because the contestant picks a door $\mathcal{D}$ at random,

$$\Pr(\mathcal{D}) = \frac{1}{3}$$

For the sake of argument, suppose the contestant chooses $\mathcal{D} = A$.

Monty Hall & Bayes Rule (2)

Now, Monty Hall picks a door $\mathcal{E}$ to show a goat.

We want to know the probability that the remaining door $\mathcal{F}$ hides a car *given Monty's exposure of door* $\mathcal{E}$.

For the sake of argument, suppose that Monty picks $\mathcal{E} = B$.

We need to calculate $\Pr(\mathcal{F}|\mathcal{E} = B)$. Use Bayes' Rule:

$$\Pr(a|b) = \frac{\Pr(b|a) \times \Pr(a)}{\Pr(b)}$$

or in our case,

$$\Pr(\mathcal{F}|\mathcal{E} = B) = \frac{\Pr(\mathcal{E} = B|\mathcal{F}) \times \Pr(\mathcal{F})}{\Pr(\mathcal{E} = B)}$$

Monty Hall & Bayes Rule (3)

The problem we need to solve:

$$\Pr(\mathcal{F}|\mathcal{E} = B) = \frac{\Pr(\mathcal{E} = B|\mathcal{F}) \times \Pr(\mathcal{F})}{\Pr(\mathcal{E} = B)}$$

We need $\Pr(\mathcal{E} = B|\mathcal{F})$, the probability Monty would open door B if the remaining door $\mathcal{F}$ actual held the car. By the rules of the game, this must be 1: Monty never shows the car.

We have the marginal probability that $\mathcal{F}$ holds the car; it's $1/3$.

Finally, we have the probability that Monty would "choose" to open B rather than C. This, of course, is $1/2$.

Substituting into Bayes' Rule, we find

$$\Pr(\mathcal{F}|\mathcal{E} = B) = \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$