CSSS/SOC/STAT 536:
Logistic Regression and Log-linear Models

Introduction to Contingency Tables

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But we will develop special techniques—and a different language—for tabular data

These methods are used often in medicine and sociology

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Today we introduce tables for count data, called *contingency tables*

Main goal today is to get a handle on the language, to aid your reading

The language may seem very different from what you are accustomed to, but connections will emerge

Next week, we'll talk about regression models for contingency tables

Contingency tables

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Sum	$n_{.1}$	$n_{.2}$	n

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The sum over any given row, i, is $\sum_j n_{ij} = n_{i.}$

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The grand sum is simply n

Note we haven't said anything about independent and dependent variables

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If we assume each of the cells is made up of iid draws from some discrete distribution, we get . . .

2×2 Tables: Rows and Columns free

	Y		Sum
X	**1**	**2**	
1	π_{11}	π_{12}	$\pi_{1.}$
2	π_{21}	π_{22}	$\pi_{2.}$
Sum	$\pi_{.1}$	$\pi_{.2}$	1.0

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Marginal distributions are sums over rows or columns

$$\pi_{i.} = \sum_j \pi_{ij} \qquad \pi_{.j} = \sum_i \pi_{ij}$$

The conditional distribution—probability of falling in j given being in i—is

$$pi_{j|i} = \pi_{ij}/\pi_{i.}$$

Probability distributions for contingency tables

What would be an appropriate probability distribution for Table 1?

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The Poisson! We'll talk about this at greater length next week

Now let's consider another case

2×2 Tables: Rows fixed, Columns free

	Y		Sum
X	**1**	**2**	
1	$\pi_{1\|1}$	$\pi_{1\|2}$	1.0
2	$\pi_{2\|1}$	$\pi_{2\|2}$	1.0
Sum	$\pi_{.1}$	$\pi_{.2}$	1.0

Suppose the row marginals of our table are fixed ahead of time

We might be conducting an experiment, placing half our subjects in a control group, and half in a treatment group

In this case, Y is a response variable

Cell entries are conditional probabilities, $\pi_{j|i}$, and sum to 1 by rows

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In fact, you could rewrite any such table to be an MNL, by breaking each cell up into its constituent individuals

In tabular form, each cell is a sum of trials, but there could be more than two outcomes in a row (or table, if only n is fixed, and not $n_{i.}$)

So we'll need something like the binomial, but able to handle more than just two outcomes

Probability distributions for contingency tables

In this case, an appropriate distribution is the multinomial, which is a generalization of the binomial to k categories

Recall the binomial

$$f_{Bin}(y|\pi) = \binom{n}{y}\pi^y(1-\pi)^{N-y}, \quad y = 0, 1, 2, \ldots, n$$

$$\mathrm{E}(y) = n\pi \qquad \mathrm{Var}(y) = n\pi(1-\pi)$$

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The multinomial is quite similar, but allows different π's for each category

$$f_{Multin}(y_j|\pi_j) = \binom{n}{y_1 \cdots y_k}\pi_1^{y_1} \cdots \pi_k^{y_k}, \quad y_j = 0, 1, 2, \ldots, n, \quad \sum_j y_j = n$$

$$\mathrm{E}(y_j) = n\pi_j \qquad \mathrm{Var}(y_j) = n\pi_j(1-\pi_j)$$

Independence of X and Y

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We'll use it to consider one simple hypotheses:

That Y does not depend on X, or vice versa

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The conditional prob that $Y = j$ is just the marginal probability that $Y = j$; $X = i$ is irrelevant

Testing Independence

Suppose we have a k-vector drawn from the multinomial distribution.

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$$H_0 : \pi_i = \pi_{i0}$$

which will try to reject in favor of any other vector of probabilities

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$$\text{Pearson } X^2 = \sum_{i=1}^{k} \frac{(n_i - \mu_i)^2}{\mu_i}$$

where μ_i is our expectation under the null, $\mu_i = n\pi_{i0}$

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$\rightarrow$ we reject independence for large X^2.

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G^2 is often called the *deviance*

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We plug this into the formulas for X^2 and G^2.

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In the 2×2 case, $\mathrm{d.f.} = 4 - 2 - 1 = 1$

(We've estimated one row probability and one column probability)

Testing Independence

If we apply these methods to the a dataset on cancer

	Eventual result		
PCR reading	Relapse	No relapse	Sum
Traces of cancer	30	45	75
No cancer	8	95	103
Sum	38	140	178

Leukemia patients were tested for precursors of relapse using polymerase chain reaction (PCR); three years later, their health was recorded

$n = 178$ is fixed, so we assume the cells follow a multinomial

Research question: Did the PCR predict cancer status? Or are they independent?

(Source: Simonoff 2003, *Analyzing Categorical Data*; an accessible supplement to your readings in Agresti)

Testing Independence

We're assuming independence, so forget the cell entries; we just need the marginals

	Eventual result		
PCR reading	Relapse	No relapse	Sum
Traces of cancer			75
No cancer			103
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Testing Independence

We need the marginal probabilities, so divide by $n = 178$

	Eventual result		
PCR reading	Relapse	No relapse	Sum
Traces of cancer			0.42
No cancer			0.58
Sum	0.21	0.79	1

Testing Independence

Under independence, the MLE of the cells is the product of the marginal probabilities

	Eventual result		
PCR reading	Relapse	No relapse	Sum
Traces of cancer	0.09	0.33	0.42
No cancer	0.12	0.46	0.58
Sum	0.21	0.79	1

Testing Independence

Finally, we multiply by $n = 178$ to get expected cell counts under independence

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Traces of cancer	16.01	58.99	75
No cancer	21.99	81.01	103
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Compare to the data:

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No; the deviance test rejects independence.

PCR did predict cancer recurrence

But then, one could tell just looking at the data and predictions . . .

Testing Independence

Similar tests exist for tables with no fixed sums, or with fixed rows.

But to check for independence in tables with fixed rows and columns, we'll need a different method.

The Lady Tasting Tea

R.A. Fisher, the famous statistician, had a friend who claimed to be able to tell by taste whether tea had been added to milk, or milk to tea

Fisher ran an experiment, with 8 cups of tea, to see if this was true

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Suppose the result was

	Guess		
Truth	Tea first	Milk first	Sum
Tea first	3	1	4
Milk first	1	3	4
Sum	4	4	8

(what is fixed in this table? How many free variables are there?)

Fisher's Exact Test

To see whether his friend's choices were more unusual than picking at random, Fisher proposed a test of *independence*.

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writing out the combinatorics, we get

$$\mathrm{P}(n_{11} = x) = \frac{n_{1.}!n_{2.}!n_{.1}!n_{.2}!}{n!n_{11}!n_{12}!n_{21}!n_{22}!}$$

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Note this doesn't depend on any random parameters, just n, which is known

That makes this an "exact" test

The p-value we calculate is valid even for tiny samples;
it does not depend on asymptotic assumptions

Tends to be conservative

Fisher's Exact Test

So how unlikely was the taster's performance? What is the probability it would have happened by chance?

$$\mathrm{P}(n_{11} \geq 3) \quad = \quad \mathrm{P}(n_{11} = 3) + \mathrm{P}(n_{11} = 4)$$

Fisher's Exact Test

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$$\begin{aligned} \mathrm{P}(n_{11} \geq 3) &= \mathrm{P}(n_{11} = 3) + \mathrm{P}(n_{11} = 4) \\ &= \frac{\binom{4}{3}\binom{4}{1}}{\binom{8}{4}} + \frac{\binom{4}{4}\binom{4}{0}}{\binom{8}{4}} \end{aligned}$$

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(It's unknown how many cups Fisher's friend correctly classified;
we do know Fisher was convinced, given his test)

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Monte Carlo is one solution

1. randomly sample tables

2. compute their probability under the multiple hypergeometric distribution

3. and rank your table against the sorted draws

The need for models of contingency tables

So why all the emphasis on independence?

Independence not usually an interesting question

An exception: controlled experiments

But generally, it's more interesting to know

- effect sizes
- contextual effects
- predicted values

The usual quantities of interest we get from modeling.

Modeling will tend to involve larger and/or higher dimensional tables

Like any modeling exercise, models of tabular data may be misleading if misspecified

3+ Dimensional Tables: $I \times J \times K \times \ldots$

Tables can have more than two dimensions.

We can "nest" rows or columns to show the further dimensions

Or we can collapse over dimensions to reduce back to an $I \times J$ table

It is customary to warn students of the dangers of this move.

. . . Though as social scientists, I suspect the hazards will come as no surprise

An example: Discrimination?

Suppose the (fictional) University of Tlon is sued for discriminatory hiring

Both sides stipulate that

- the best candidate can be determined uniquely
- should always be hired
- is equally likely to be male or female

The case turns on whether the University hired male and female candidates at the same rate

An example: Discrimination?

Here is the data for the university's "eclectic" departments

	Men		Women	
Departments	Hired	Applied	Hired	Applied

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Both departments	46%	$<$	54%

What's going on here?

Simpson's Paradox

The Departments are different. Perhaps AEA has much less funding that NC, and can make fewer offers.

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It's no different from the OVB you already know;
with contingency tables we must still condition on confounding variables

(The linear regression parallel to Simpson's Paradox:
treating a 2-D scatterplot of Y on X as sufficient for all data analysis)

Presentation and EDA

One hazard of working with tabular data is that it encourages lazy presentation

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- Reporting exact data for *replication*
  (Mostly obsolete in the electronic era, but . . .)

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Always consider whether tables are the right choice

Sometimes all you need is a good paragraph

Titanic Example

A 4-D table groups all persons on the *Titanic* by gender, age, class, and survival

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	Adult				Child			
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	Died	Lived	Died	Lived	Died	Lived	Died	Lived
1st class	118	57	4	140	0	5	0	1
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What graphical alternatives to this table could we try?

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(How to: `mosaicplot` in the `R` library `graphics`)

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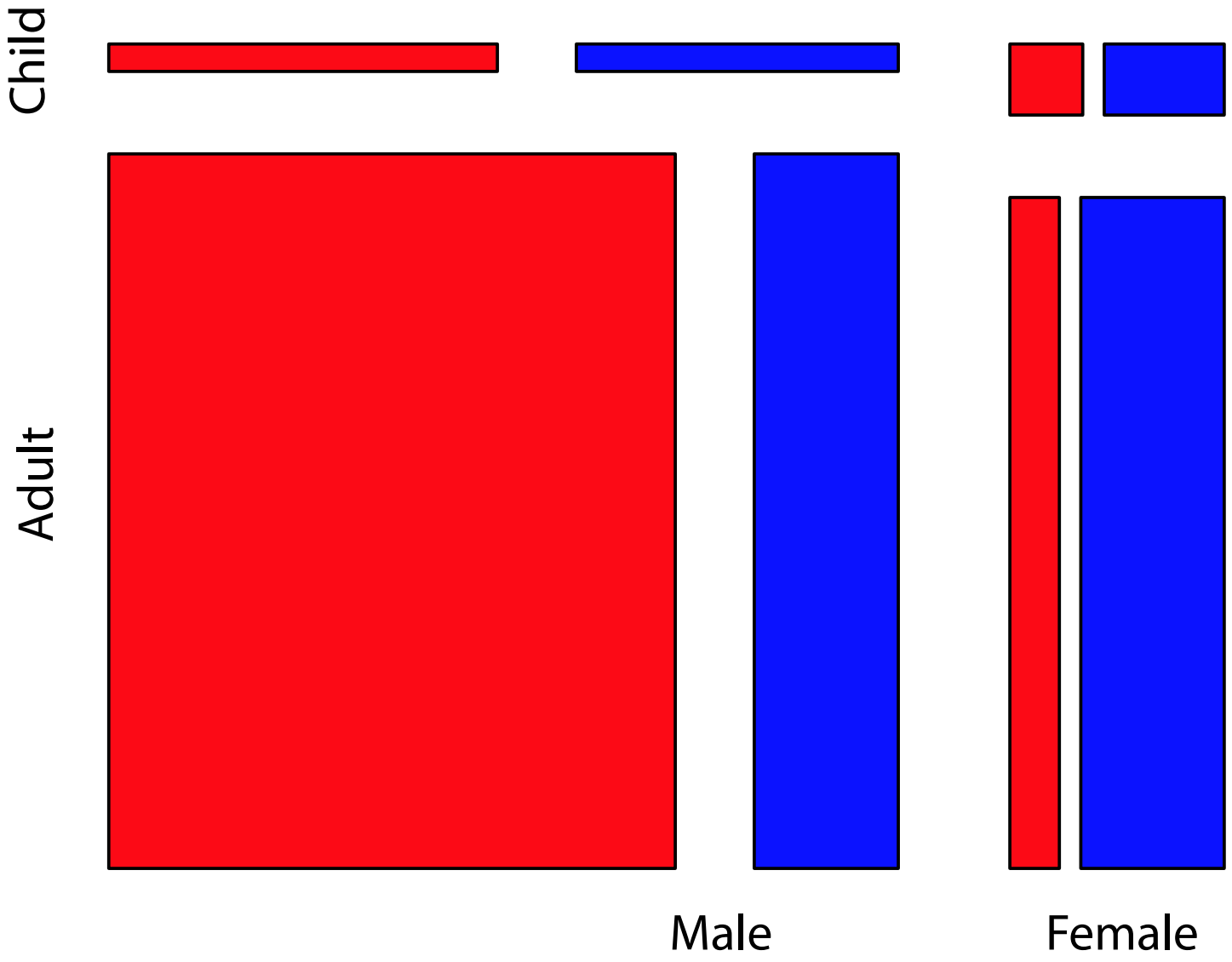
Titanic Survival Proportions: Deaths vs Survivors



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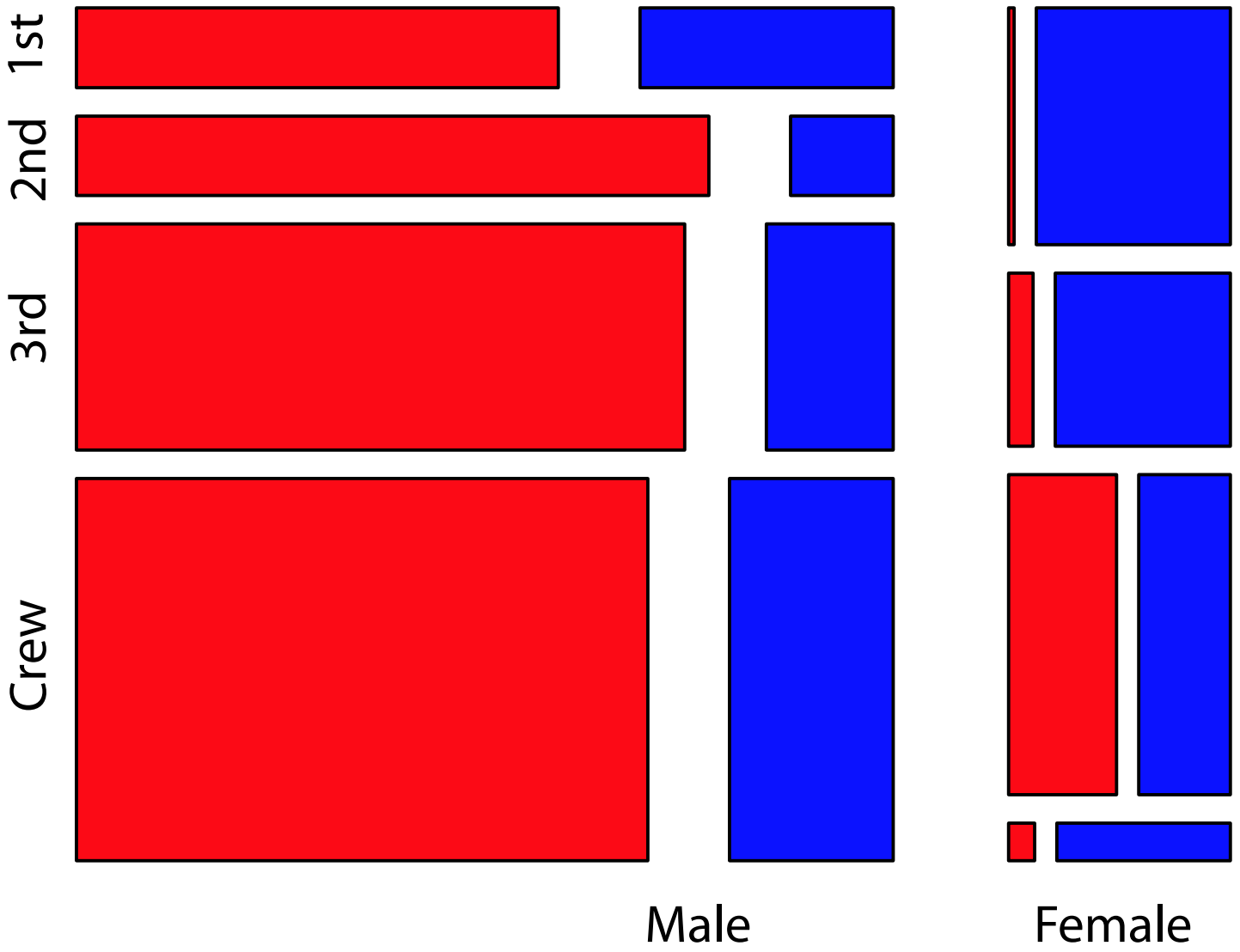

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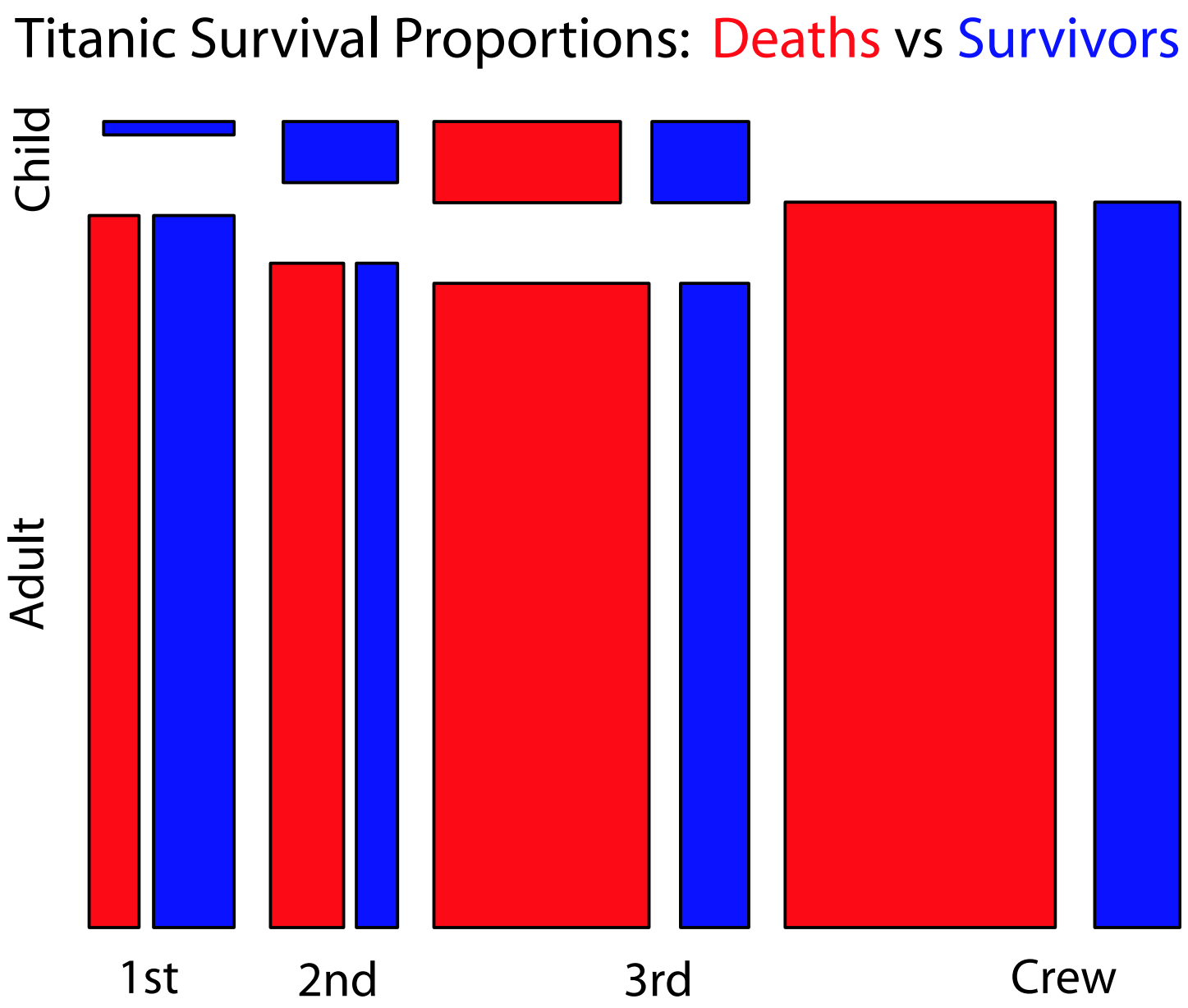

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