CSSS/SOC/STAT 321
Case-Based Statistics I

Random Variables & Probability Distributions I: Discrete Distributions

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Two examples

Flipping coins In the play *Rosenscratz and Guildenstern are dead*, R & G flip a coin repeatedly, and get heads 89 times. How likely is this outcome? In general, how likely is it get X *or more* heads out of N flips? Can you think of real-world situations like this one?

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Anticipating a landslide A Democrat-controlled state legislature redistricts a 55% Democratic state so that they have a 55% majority in every district. They say they expect to win all 20 Congressional seats in the state. “Majority rules, after all.” Are they right?

Sample spaces & sets

One way to understand sample spaces is to list every outcome

These lists are *sets*, or collections of elements, which could be numbers

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a_1 is an element of $\mathcal{A}$, which we write $a_1 \in \mathcal{A}$
A set may also be empty, e.g., $\mathcal{B} = \emptyset = \{\}$

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Suppose we toss a coin twice and record the results.

We can use a set to record this complex event.

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The *sample space*, or universe of possible results, is in this case a set of sets:

$$\Omega = \{\{H,H\},\{H,T\},\{T,H\},\{T,T\}\}$$

Note that our sample space has separate entries for every *ordering* of heads or tails we could see.

Sample space for two dice

Now suppose that we roll two dice. The sample space is:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

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The sum of the dice rolls for each event:

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Note that many sums repeat

Sample spaces and complex events

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What are the odds?

Outcome	2	3	4	5	6	7
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Outcome	2	3	4	5	6	7
Frequency	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$
Probability						

Outcome	8	9	10	11	12
Frequency	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
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Each event (a, b) is equally likely. But each sum, $a + b$, is not.

Outcome	2	3	4	5	6	7
Frequency	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$
Probability	0.028	0.056	0.115	0.111	0.139	0.167

Outcome	8	9	10	11	12
Frequency	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
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The trouble with sample spaces

For most processes we could study, the sample space of all events is huge:

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This mapping can produce discrete or continous variables,
and each will have a different distribution of probabilities

Probability for random variables

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Five things we'd like to know about the theoretical distribution of X:

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We start by defining a *probability distribution*, or mathematical function which the random variable should follow in theory

Let

$$\Pr(X = x) = f(X = x)$$

Note that we haven't yet spelled out what $f(\cdot)$ looks like.

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For coin flips, the obvious parameter is the probability a single flipped coin comes out as heads, or π

$$\Pr(X|\pi) = f(X, \pi)$$

The Bernoulli distribution

The probability of a coin flip is described by a simple probability function known as the Bernoulli distribution:

$$f_{\text{Bern}}(X|\pi) = \begin{cases} 1-\pi & \text{if } X = 0 \\ \pi & \text{if } X = 1 \end{cases}$$

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If we are clever, we can write it much more conveniently:

$$f_{\text{Bern}}(X|\pi) = \pi^X(1-\pi)^{1-X}$$

Another name for a probability distribution function is *pdf*,
so this is also called the Bernoulli pdf

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$\Pr(X \leq x)$ What is the probability that X is *at least* equal to some specific value, like $x = 1$? Sum up the pdf over all $X \leq x$.

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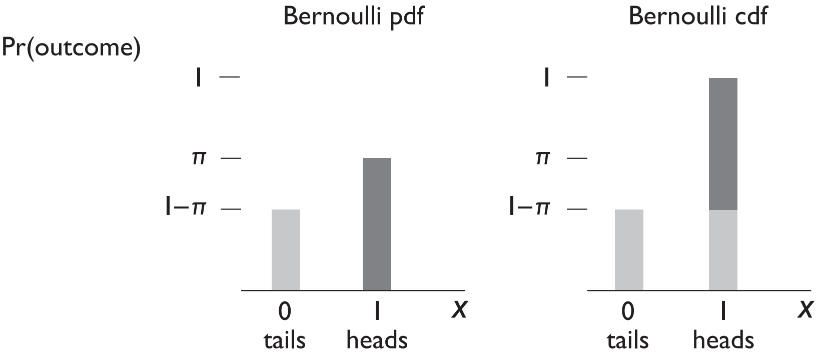
The sum of a pdf over a range of values is the *cumulative density*

We can write a cumulative density function (cdf) for any discrete distribution as $F(X)$:

$$F(X = x) = \sum_{\forall X \leq x} f(x)$$

The Bernoulli distribution



For the Bernoulli, the pdf and cdf are simple functions

Note that both have *support* on the values $0, 1$ only –
the probability of X is defined as 0 for any other X beside 0 and 1.

The Bernoulli distribution

A distribution has an expected value and variance, just like a sample does, though they are now *theoretical*, pre-existing any data:

$$\mathrm{E}(X) \quad = \quad \sum_{\forall i} X_i f_{\mathrm{Bern}}(X_i|\pi)$$

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Note that this is maximized at $\pi = 0.5$. Why?

The binomial distribution

Now let's go back to our original question:
what is the distribution of the sum of M coin flips?

Real world applications are many:

- rainy days in a month
- students our of a class who pass
- votes from a known population of voters (elections, Congressional votes)

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Key assumption:
each trial is identically and independently distributed (iid) Bernoulli
For the moment, take this to mean that each trial has the same π of success

The binomial distribution

How do we come up with a pdf for these assumptions?
Back to sample spaces for a moment

Suppose we wanted to calculate the probability that e get two heads out of a total of three flips: $\Pr(X = 2|\pi, M = 3)$

There are three events in the sample space that meet these criteria:

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How do we come up with a pdf for these assumptions?
Back to sample spaces for a moment

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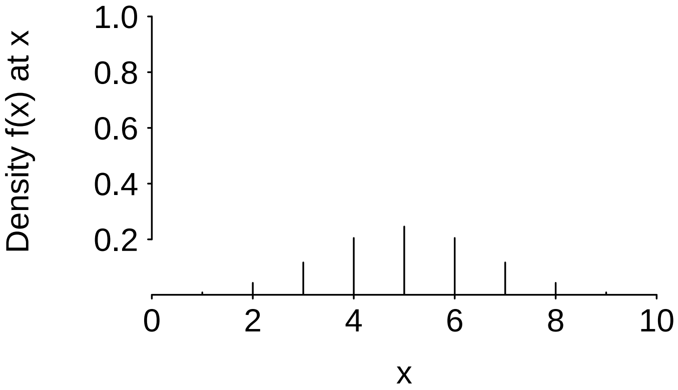
$$\mathrm{E}(x) = M\pi \qquad \mathrm{Var}(x) = M\pi(1-\pi)$$

The binomial distribution: PDF

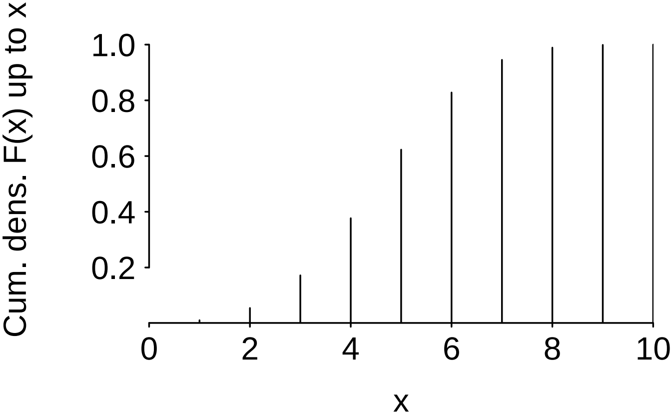
This graph shows the probability of each outcome X



The pdf of binomial sums over 10 trials,
with each trial having an 0.5 probability of success

The binomial distribution: CDF

This graph shows the probability of seeing an outcome *less than or equal to* X



The cdf of binomial sums over 10 trials,
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Example: Rosencrantz and Guildenstern are still dead

Rosencrantz and Guildenstern are minor characters in Shakespeare's *Hamlet*

Claudius has killed his brother, the King, and married his sister-in-law, the Queen, to assume the throne

He dispossesses Prince Hamlet, without ever explicitly acknowledging the fact

Hamlet becomes depressed and sullen. Expecting Hamlet to eventually challenge him, Claudius summons two of Hamlet's childhood friends, R & G, to spy on the prince

Claudius tricks R & G into delivering Hamlet to England for execution. Hamlet tricks the (none-too-bright) R & G into being executed in his place.

Tom Stoppard's play *Rosencrantz and Guildenstern are Dead* retells this story through R & G's perspective, emphasizing their powerlessness and lack of real choice

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At the start of Stoppard's play, Rosenstern and Guildencrantz have flipped a coin 89 times, getting heads every time

How unlikely is this event?

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Working with large factorials might "break" your calculator.

$89!$ has 136 digits before the decimal, for example

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Or a 1 in 618,000,000,000,000,000,000,000,000 chance

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Or one chance in 618 million trillion trillion tries

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If a billion people each flippled a set of 89 coins once a second
– repeating this action *every* second since time began –
we would expect approximately *one* person to get a single set of all heads.

Example: Rosencrantz and Guildenstern are still dead

If you see such a freak occurrence in life you must ask yourself:

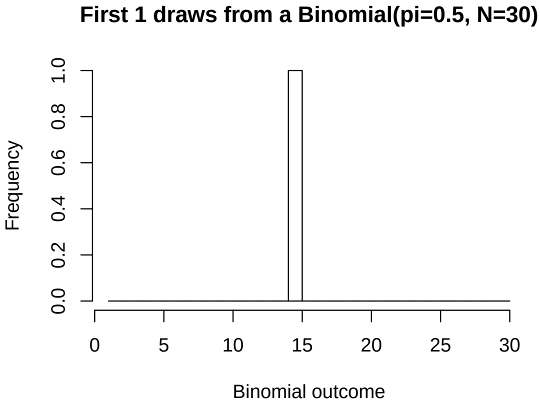
"Is it more likely that I saw 89 fair coins come up heads by chance, *or* that someone has changed the probability of a heads to something close to 1?"

Clearly, something is manipulating the coins,
just like Claudius & Hamlet are manipulating R & G

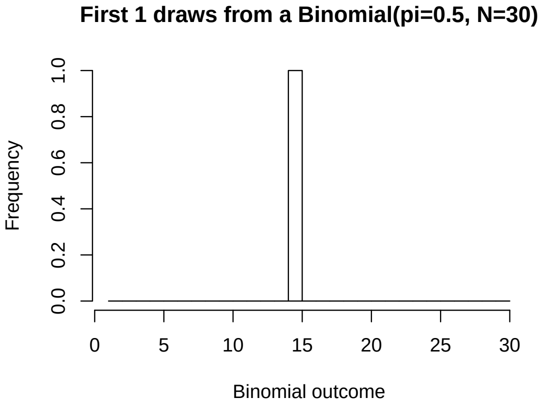
One way to use probability distributions:
How likely is an event to have been "mere chance"?

Random samples from the binomial

Another way to get a handle on pdfs and cdfs
is to just draw repeatedly from the distribution of interest



Let's draw repeatedly from the Binomial with 30 trials, each with probability 0.5 of success, and save our results in a histogram



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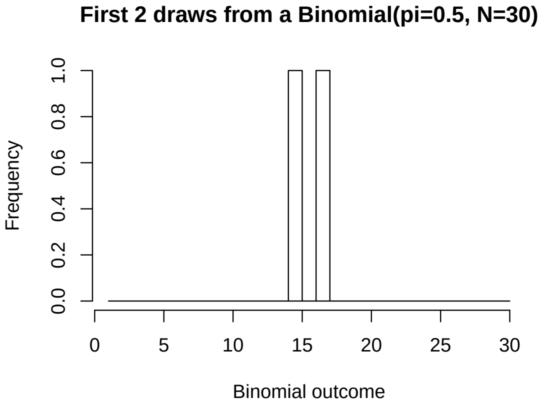

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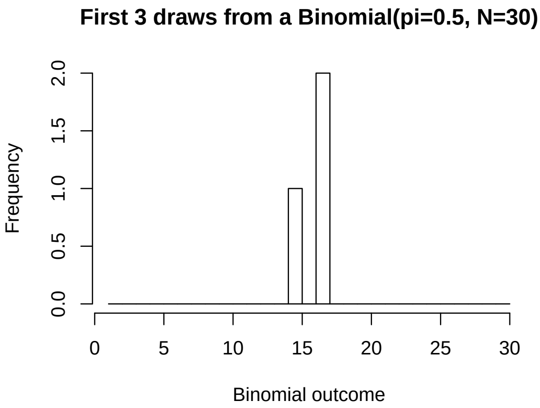

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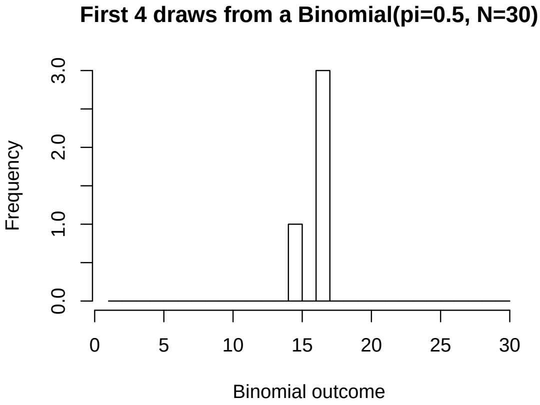

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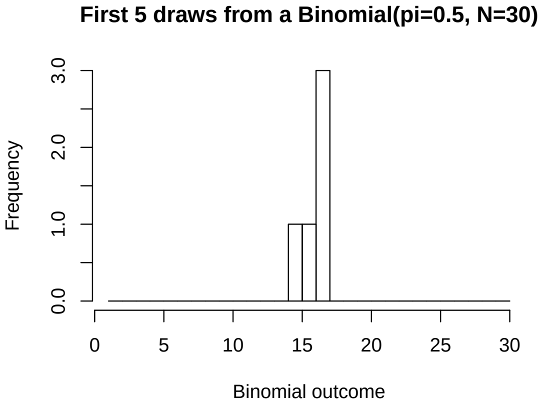

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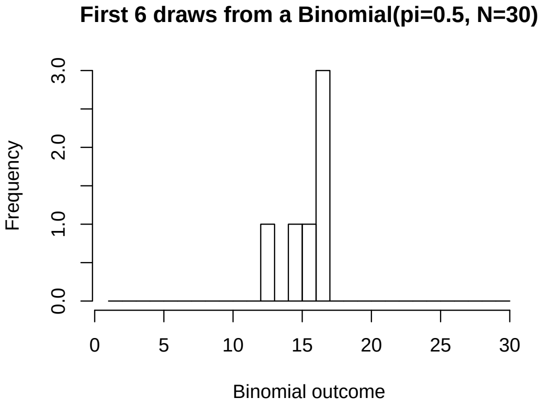

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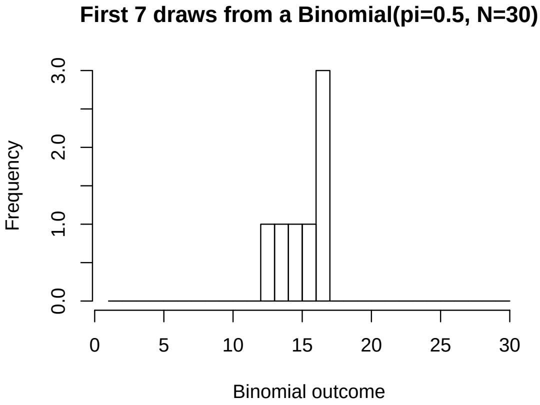

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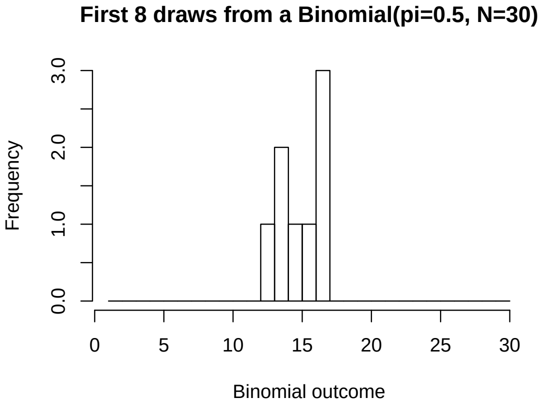

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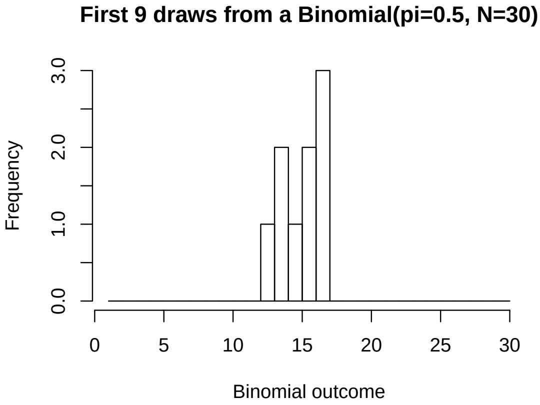

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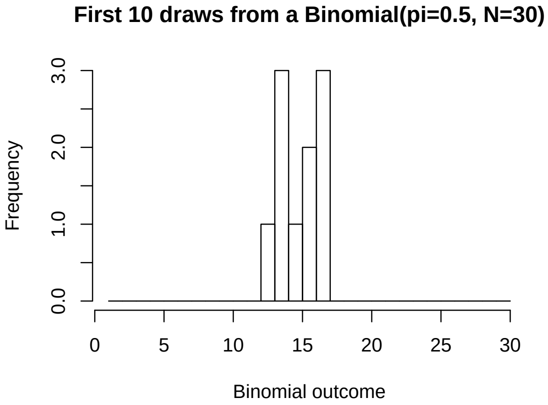

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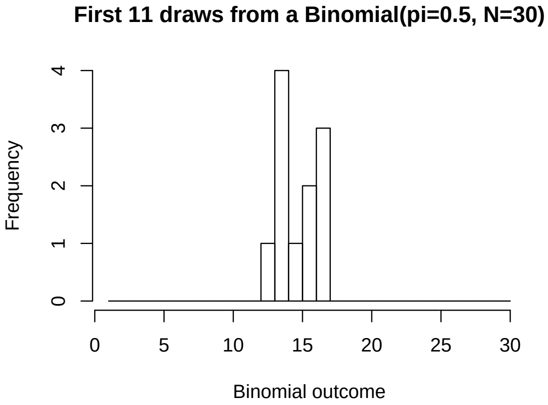

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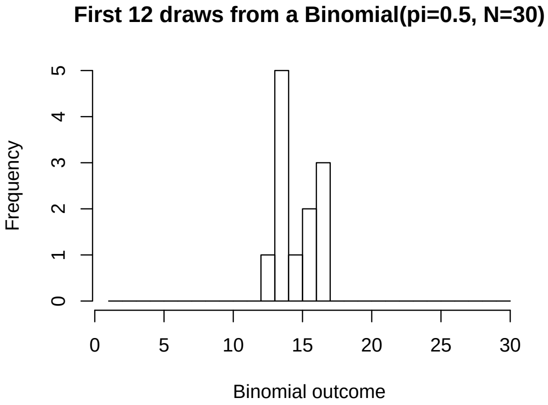

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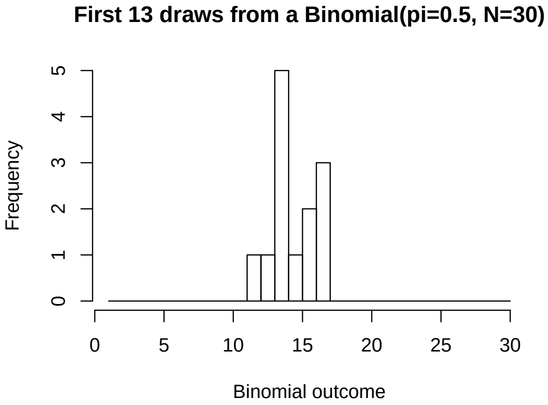

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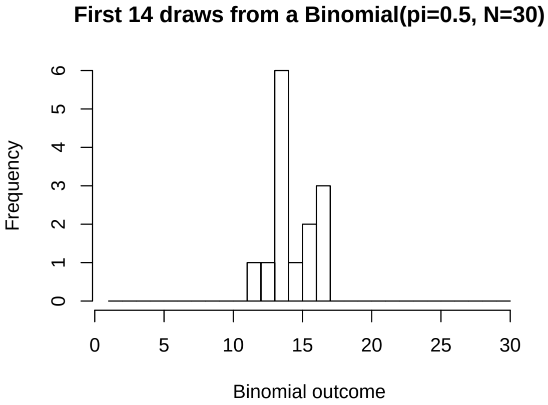

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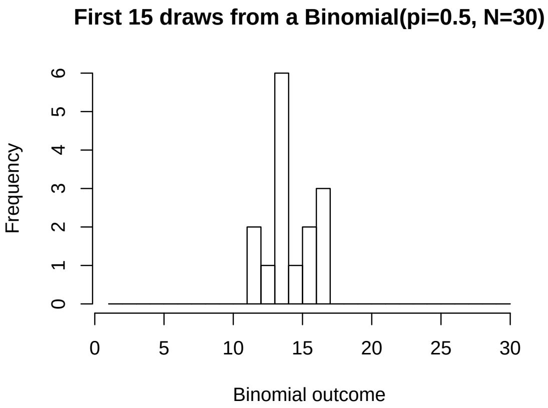

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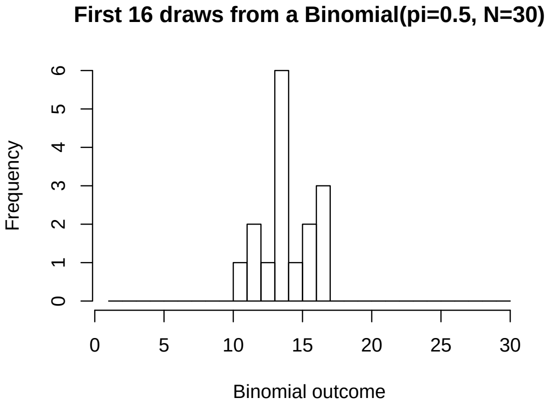

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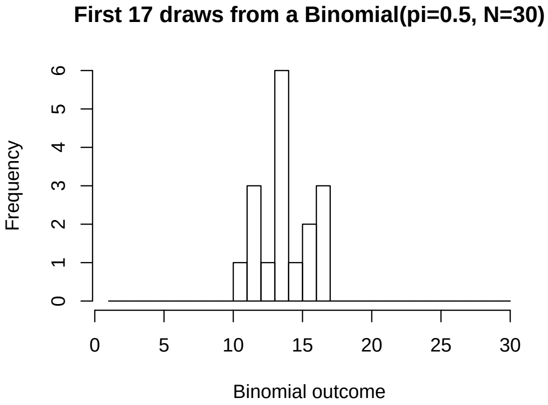

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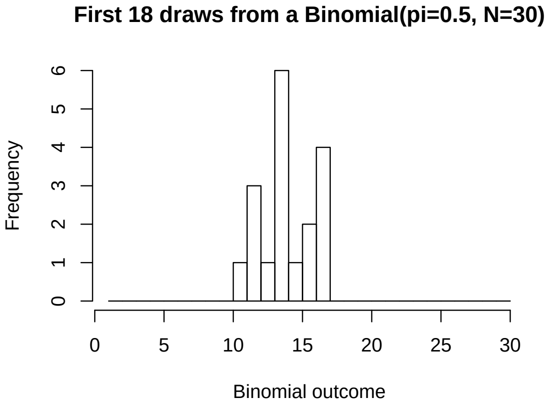

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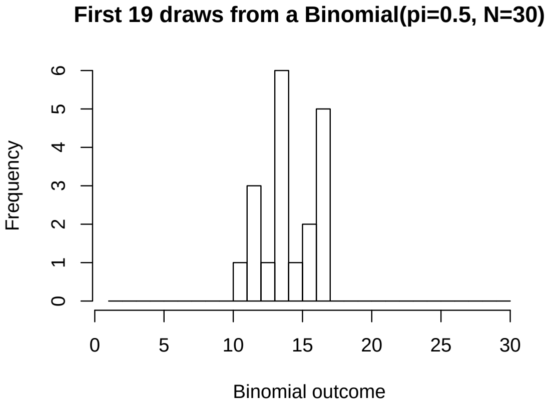

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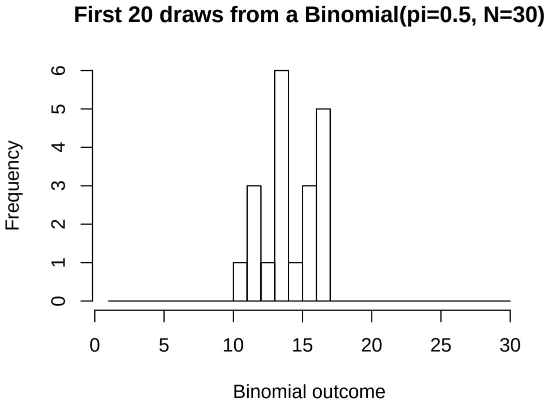

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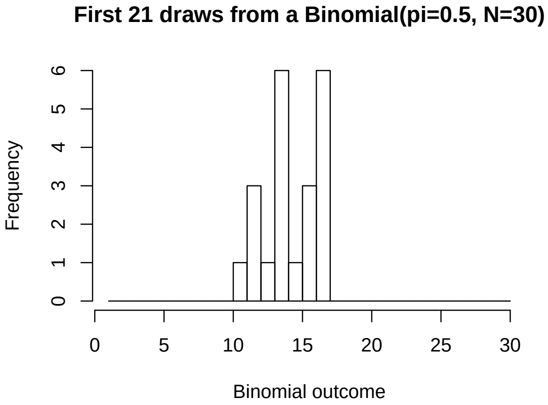

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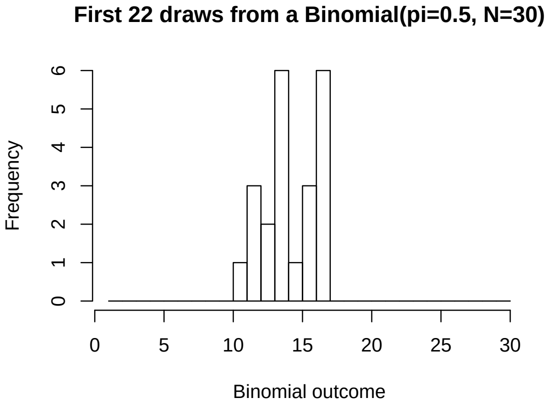

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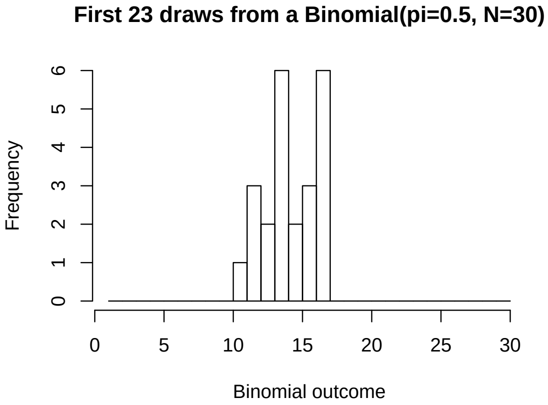

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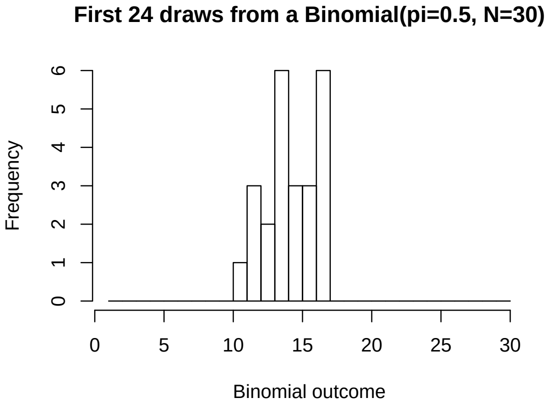

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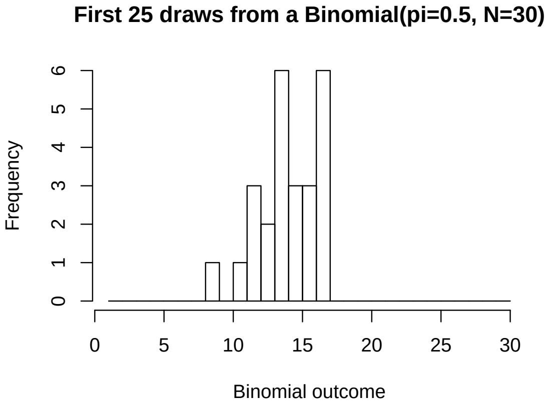

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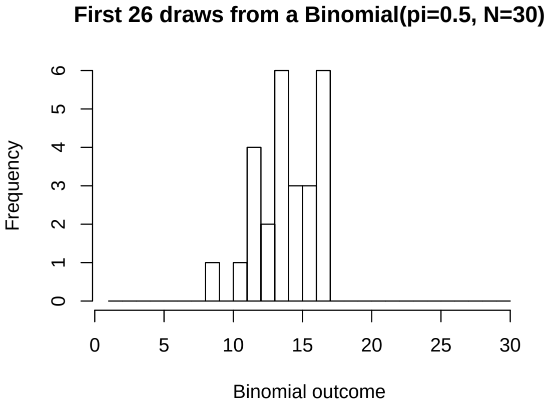

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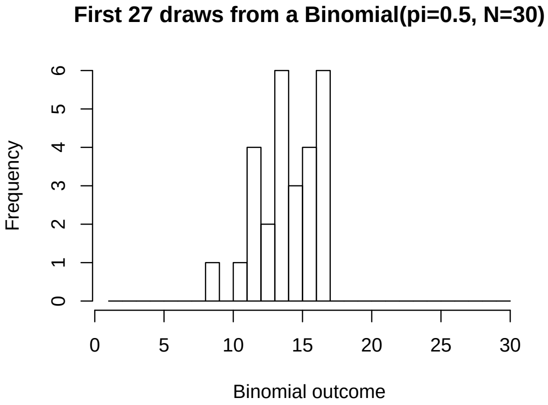

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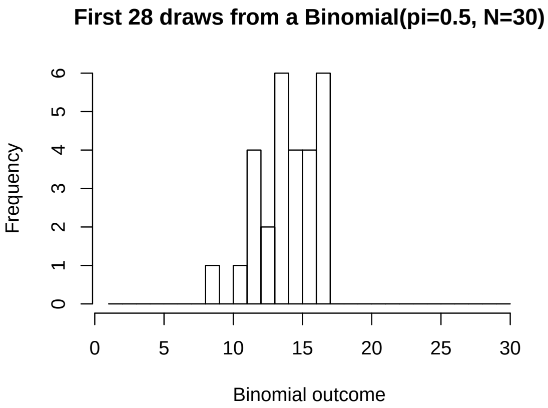

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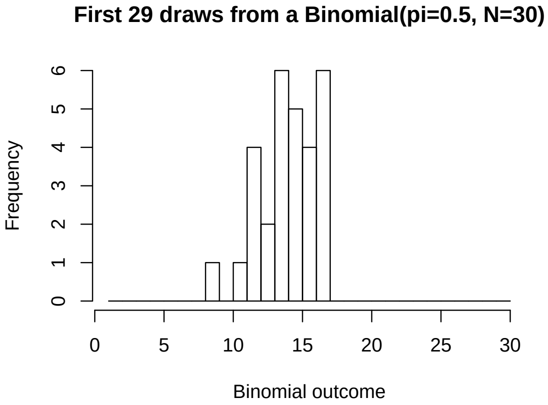

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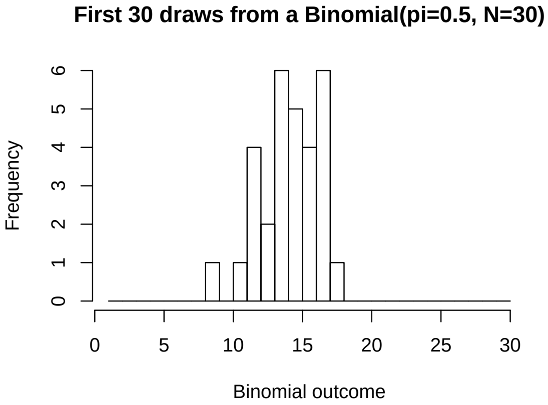

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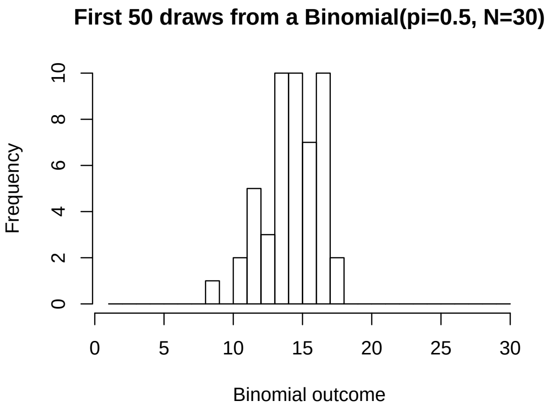
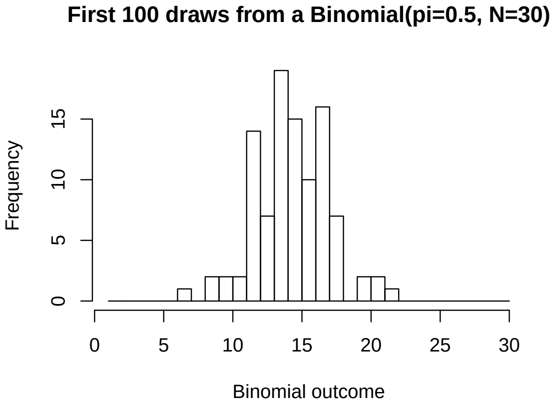


Now that we have the basic idea, let's add more than one draw to the histogram at a time

Now we're up to 50 draws



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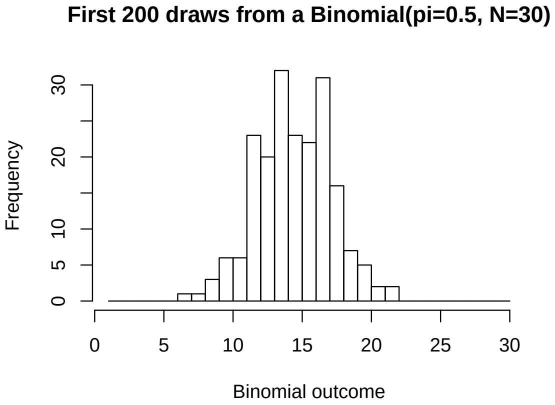
Now we're up to 100 draws



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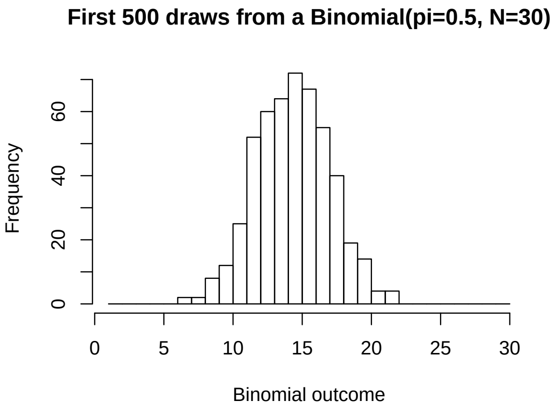

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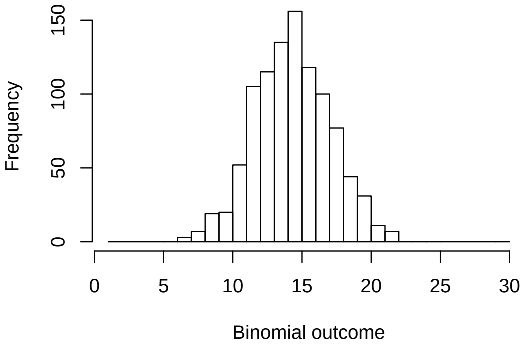
Now we're up to 500 draws

First 1000 draws from a Binomial(pi=0.5, N=30)



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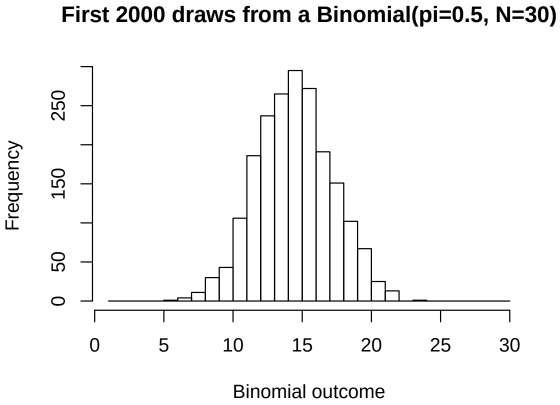
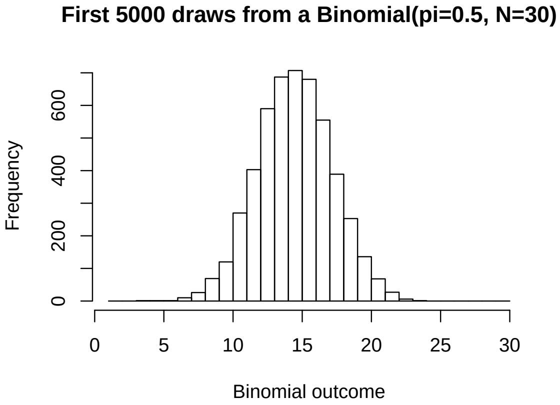
Now we're up to 1,000 draws



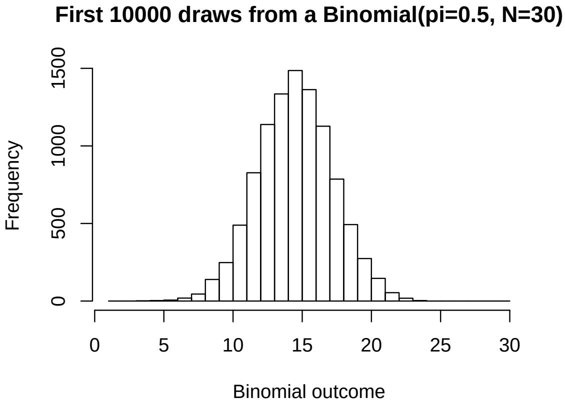
Now that we have the basic idea, let's add more than one draw to the histogram at a time

Now we're up to 2,000 draws



Now that we have the basic idea, let's add more than one draw to the histogram at a time

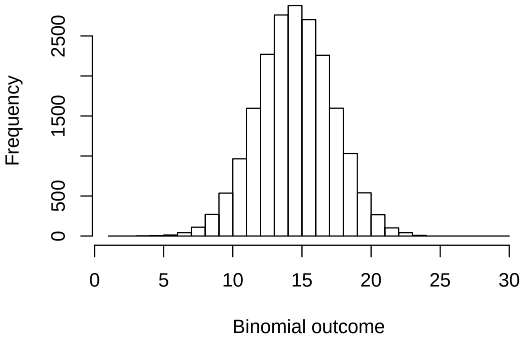
Now we're up to 5,000 draws



Now we're up to 10,000 draws

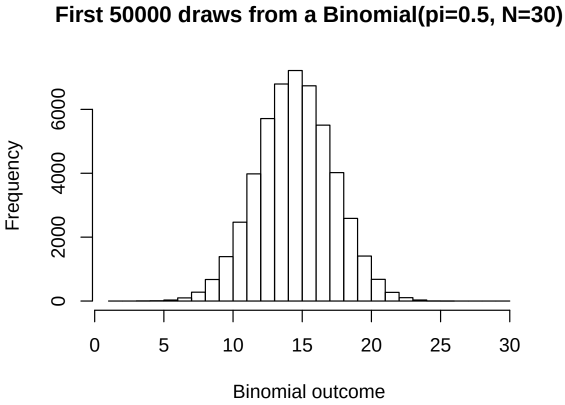
Does this distribution remind you of any other distribution you've read about?

First 20000 draws from a Binomial(pi=0.5, N=30)



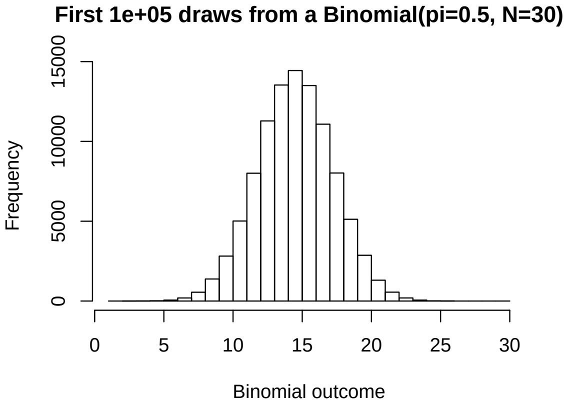
Now we're up to 20,000 draws

As the number of draws from the binomial gets large, it starts to look like a Normal, or bell-shaped distribution



Now we're up to 50,000 draws

Won't ever be exactly normal: it's still discrete (you can't get 10.5, for example), and it only has support on [0,30]



Now we're up to 100,000 draws

(Does this remind you of anything?)

Example: Anticipating a landslide

Every ten years, the states redraw the lines of Congressional districts

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Why we have a decennial census: how many districts should each state get, and how should they be drawn to have equal numbers of residents

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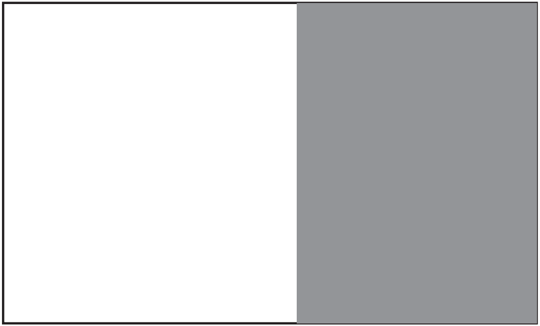
Why we have a decennial census: how many districts should each state get, and how should they be drawn to have equal numbers of residents

Political parties use redistricting to strategically draw district boundaries that achieve their goals

Key (conflicting!) goals: incumbent protection and maximizing total seats

Called political gerrymandering. Currently quite legal.

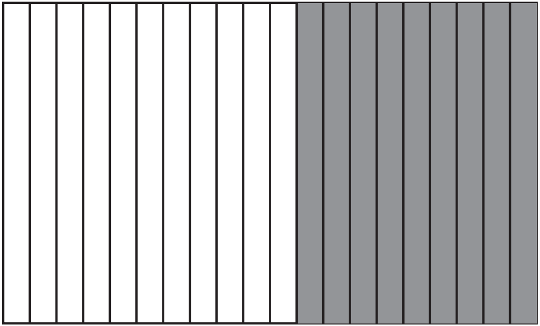
Example: Anticipating a landslide



Imagine a state of 10 million people where the population is 55% Democratic, and 45% Republican.

Suppose that every Democrat lives on the west side of the state, and every Republican on the east.

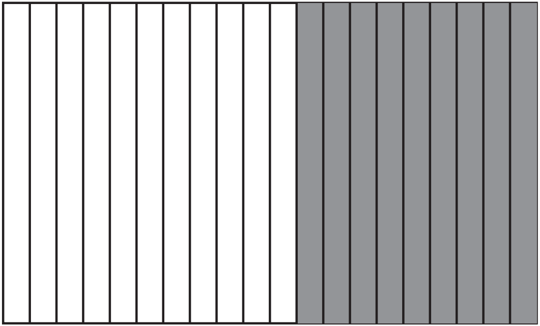
Example: Anticipating a landslide



Currently, there are 20 districts of 500,000 people each, draw in North-South strips.

Each district has homogenous D or R population

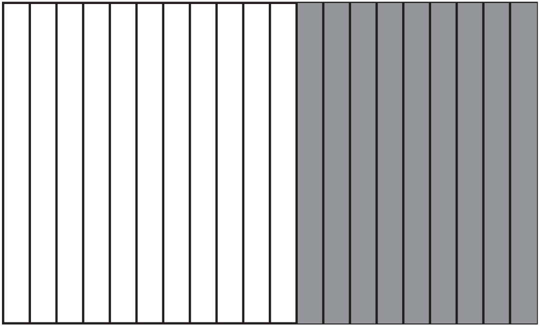
Example: Anticipating a landslide



Normally in an election, the side that can get the most voters to turnout wins (approximately half of the eligible voters actual go to the polls)

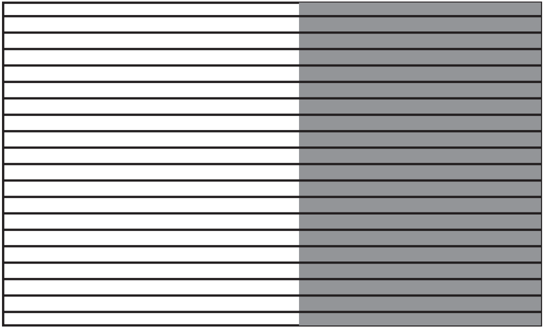
But if each district is homogenous, this doesn't matter: Every election, regardless of turnout, the D's win 11 seats, and R's win 9 seats

Example: Anticipating a landslide



The Democrats also control the state legislature, and have a clever idea.

Example: Anticipating a landslide

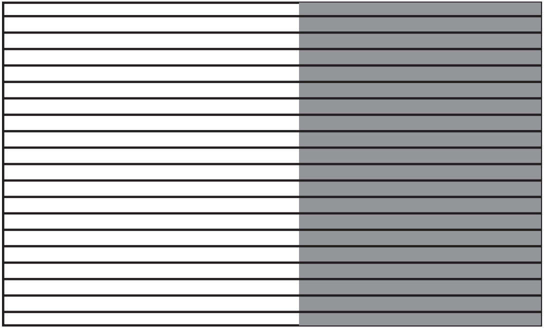


"Let's redraw the map this year as a series of East-West districts, each 55% Democratic.

We'll have a majority in every district, so we should expect to win 20 seats!"

Are these legislators correct? How many seats should they expect to win?

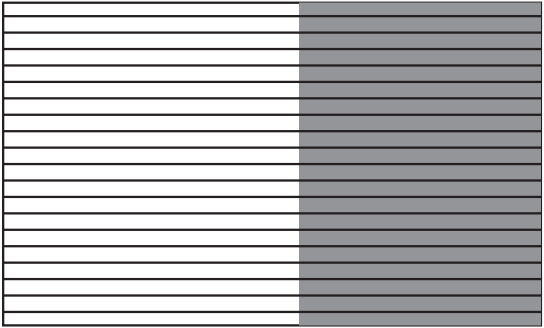
Example: Anticipating a landslide



The legislators have assumed the probability of victory is 100% in a district that is 55% Democratic and 45% Republican

Not a good assumption. Let's instead assume the probability of victory is a healthy 75%

Example: Anticipating a landslide



The Democrats will usually win, but 1 in 4 years, the Republicans will manage to get enough voters to the polls to overwhelm the D's registration advantage

Under the new map, what is the expected number of Democratic seats across a large number of elections?

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Example: Anticipating a landslide

Under the new map, what is the expected number of Democratic seats across a large number of elections?

$$\begin{aligned} \mathrm{E}(\text{seats}) &= M\pi \\ &= 20 \times 0.75 \\ &= 15 \end{aligned}$$

The Demcrats should expect to add 4 seats, not 9 seats

Failure to understand binomial probability has lead to a massive overestimate of the effects of redistricting

A common mistake in redistricting and in news articles about it

Example: Anticipating a landslide

Gains from redistricting are not a sure thing.

How much should the Democrats expect their winning to vary on average from 15?

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How much should the Democrats expect their winning to vary on average from 15?

$$\begin{aligned}
\mathrm{sd}(x) &= \sqrt{M\pi(1-\pi)} \\
&= \sqrt{20 \times 0.75 \times 0.25} \\
&= \sqrt{3.75} \\
&= 1.94
\end{aligned}$$

How would you explain all this to a Democratic strategist?

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How much should the Democrats expect their winning to vary on average from 15?

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How would you explain all this to a Democratic strategist?
If you redistrict for maximum gain, you may expect to win 15 seats out of 20 on average, but you'll typically overshoot or undershoot by 2 seats. So don't be surprised if you only pick up two new seats, for a total of 13!

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What is the probability the Democrats will actually get all 20 seats from their gerrymander?

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To calculate probabilities of ranges of outcomes, we need to use the binomial cdf, which in `R` is given by `pbinom()`

Desired prob	`R` command	Result
$\Pr(20 \mid \pi = 0.75, M = 20)$	`dbinom(20,size=20,prob=0.75)`	0.003

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$\Pr(11 \mid \pi = 0.75, M = 20)$	`dbinom(11,size=20,prob=0.75)`	0.027

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$\Pr(11\|\pi = 0.75, M = 20)$	`dbinom(11,size=20,prob=0.75)`	0.027
$\Pr(< 11\|\pi = 0.75, M = 20)$	`pbinom(10,size=20,prob=0.75)`	0.014

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Pr$(11 \vert \pi = 0.75, M = 20)$	`dbinom(11,size=20,prob=0.75)`	0.027
Pr$(< 11 \vert \pi = 0.75, M = 20)$	`pbinom(10,size=20,prob=0.75)`	0.014
Pr$(> 11 \vert \pi = 0.75, M = 20)$	`1-pbinom(11,size=20,prob=0.75)`	0.959