

# MULTIPLE REGRESSION

## part 1

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## Motivating Example: Cross-national determinants of fertility

We have cross-national data from several sources:

**Fertility** The average number of children born per adult female, in 2000  
(United Nations)

**Education Ratio** The ratio of girls to boys in primary and secondary education, in 2000 (World Bank Development Indicators)

**GDP per capita** Economic activity in thousands of dollars, purchasing power parity in 2000 (Penn World Tables)

**Agricultural Labor** Percentage of the labor force working in agriculture in 2000  
(International Labor Organization)

Note the addition of a fourth variable

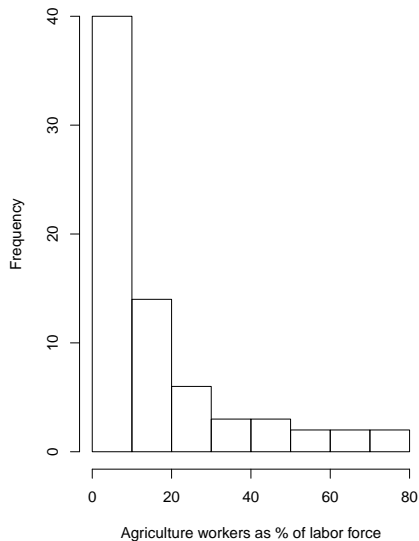
## Motivating Example: Cross-national determinants of fertility

All three independent variables might cause the fertility rate

More agricultural nations may have more children to bolster the labor force on family farms

Let's look at the univariate summaries & bivariate regression results for this new covariate

# Summary of Univariate Distribution: Agricultural Labor

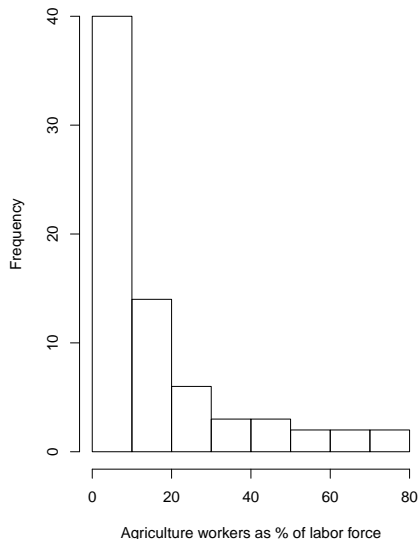


Median = 8.1%

Mean = 16.0 %

std dev = 17.9%

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How would you describe  
this distribution?

## Regression of Fertility on Agricultural Labor

Variable	Estimates	se	t-stat	p-value
Intercept	1.83	(0.15)	12.34	<0.001
Agricultural Labor	0.02	(0.01)	3.52	<0.001
$N$	72			
$R^2$	0.15			
RMSE	0.93			

How do we read this table?

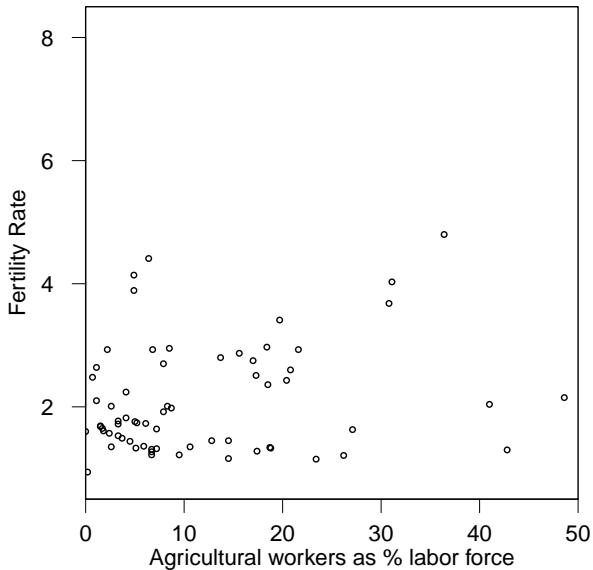
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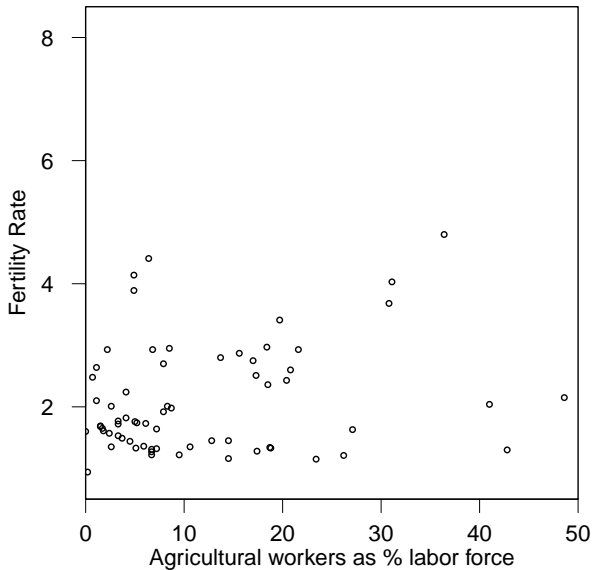
Note the reduction in *N*: lots of cases are missing data on agricultural labor

Any cases missing any covariates need to be deleted from the data before using regression (listwise deletion)



What looks different about this scatterplot?

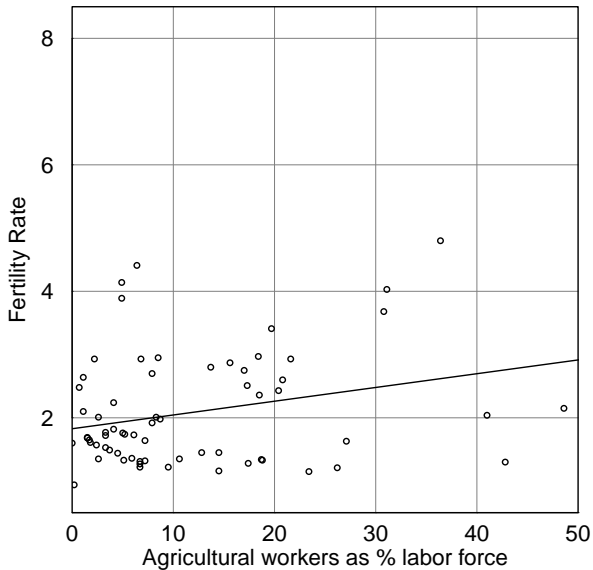




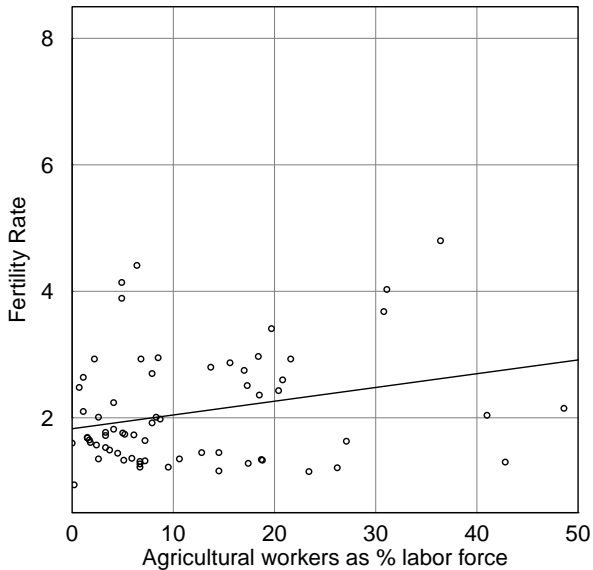
What looks different about this scatterplot?

The high fertility cases seem to be missing (deleted due to missing data)



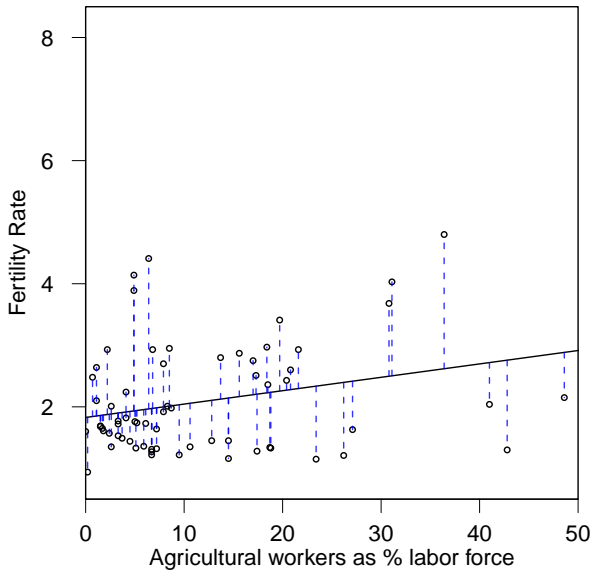


Is this a strong relationship?

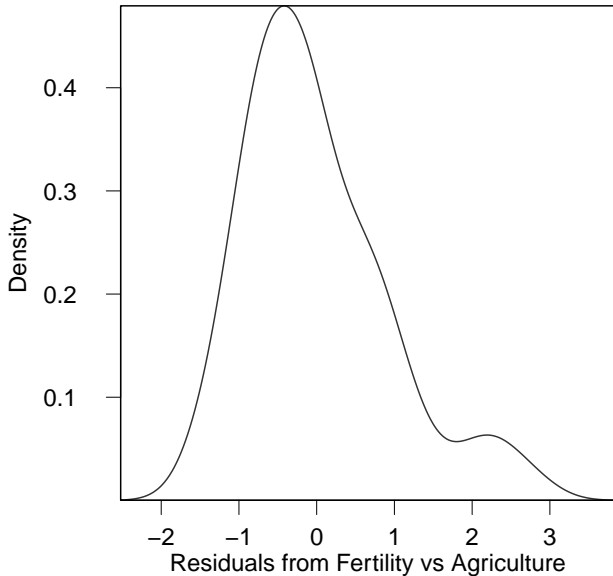


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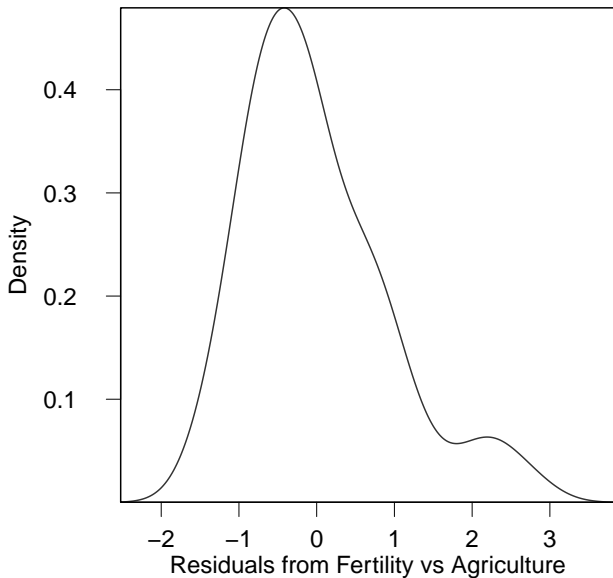
How many datapoints would have to move to reduce the slope to 0?



Which are larger, the residuals or the explained variance?

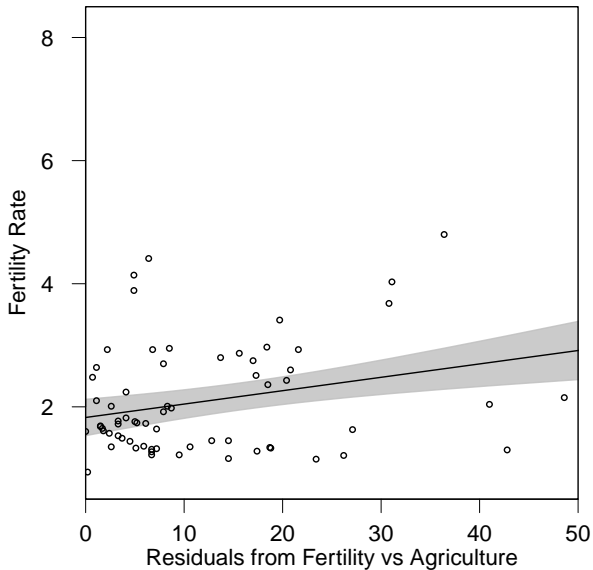


What is the standard deviation of this distribution called?



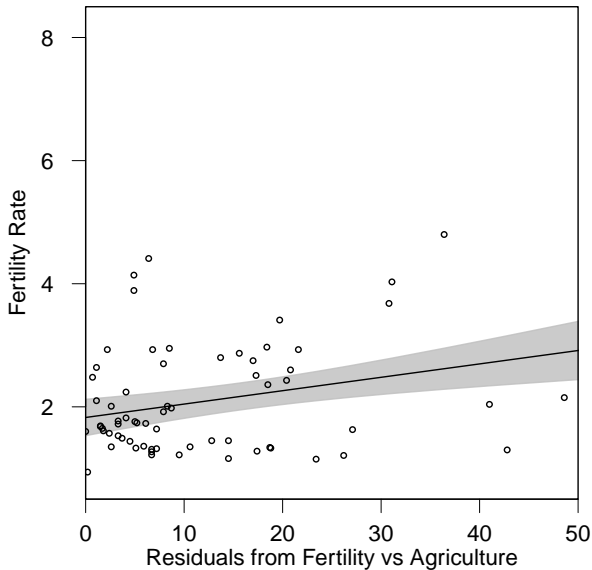
What is the standard deviation of this distribution called?

The RMSE, or standard error of the regression: how much predictions from this model tend to miss by



How confident are we that this line has a positive slope?





How confident are we that this line has a positive slope?

Are we as confident as we were for the other models?

## Confounders and Omitted Variable Bias

Which (if any) of the three models we've looked at are right?

Do Education, GDP, and Ag Labor all affect Fertility?

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If we regress Fertility on Education, and Education is correlated with GDP and Ag, might it proxy all three variables?

Yes: if countries which educate women also tend to be rich and have few ag workers, then the bivariate results will blur all three relationships

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(What do these numbers mean?)

**Omitted variable bias:** Leaving any of these variables out of our model could lead to misleading estimates of the effects of any variables we do include

## The linear regression model, redux

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_k x_{ki} + \varepsilon_i$$

Our dependent variable likely depends on many covariates

For example,

$x_1$  might be the education ratio,

$x_2$  might be GDP per capita,

and so on for as many covariates as we have, up to our  $k$ th covariate

This leads to the above model, with multiple *partial slopes*  $\beta_1, \beta_2, \dots, \beta_k$

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This model is still a linear regression model.

Sometimes called this is called a “multiple” regression model to distinguish it from a “bivariate” regression, but mathematically, they are equivalent

Henceforth, we will assume a linear regression can have many covariates

How many covariates are allowed?

Up to  $N - 1$ , where  $N$  is the number of observations

Each covariate added uses up a degree of freedom;  
once they are gone, there is nothing left for an additional covariate to explain

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How do we interpret the  $\beta$ 's? Just as before.

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The  $\beta$ 's are still slopes,  
or the amount  $y_i$  changes on average for a 1 unit increase in  $x$ , *all else held equal*

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We've finally found a way to control for confounders using observational data!

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The  $\beta$ 's are partial derivatives with respect to the  $x$  they multiply



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$$\frac{\partial y}{\partial x_2} = \frac{\partial(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)}{\partial x_2}$$

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What is the effect of a tiny change in  $x_2$  on  $y$ , holding other  $x$ 's constant?

$$\begin{aligned}\frac{\partial y}{\partial x_2} &= \frac{\partial(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)}{\partial x_2} \\ &= \beta_2\end{aligned}$$

This makes  $\beta_k$  a very useful summary of the "effect" of  $x_k$  on  $y$

## Multiple regression: just like bivariate

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- 5 We can still summarize goodness of fit using such measures as RMSE and  $R^2$

# Fertility as function of Education and GDP per capita

Let's start small: a model with two covariates:

$$\widehat{\text{Fertility}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{EduRatio}_i + \hat{\beta}_2 \text{GDPpc}_i$$
$$\widehat{\text{Fertility}}_i = 11.24 - 0.08 \times \text{EduRatio}_i - 0.05 \times \text{GDPpc}_i$$

We can present this result in several ways:

- 1 In a table by itself
- 2 In a table compared to other models
- 3 Through graphics

## Regression of Fertility on Education Ratio & GDP

Variable	Estimates	se	t-stat	p-value
Intercept	11.25	(0.73)	15.46	<0.001
Education Ratio	-0.08	(0.01)	-9.93	<0.001
GDP per capita (\$k)	-0.05	(0.01)	-5.32	<0.001
<i>N</i>	130			
$R^2$	0.64			
RMSE	1.01			

How do we interpret the above?

## Three regression models of fertility

Variable	Model		
	1	2	3
Intercept	12.59 (0.75)	4.13 (0.17)	11.25 (0.73)
Education Ratio	-0.10 (0.01)		-0.08 (0.01)
GDP per capita		-0.10 (0.01)	-0.05 (0.01)
<i>N</i>	130	130	130
$R^2$	0.55	0.35	0.64
RMSE	1.12	1.35	1.01

Standard errors in parentheses

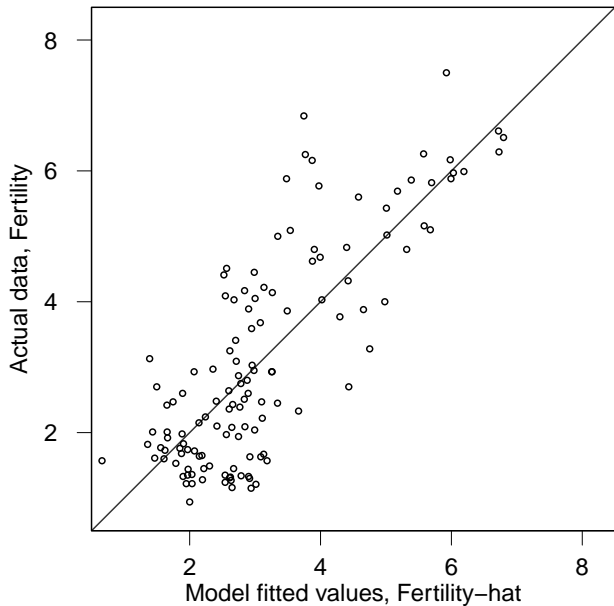
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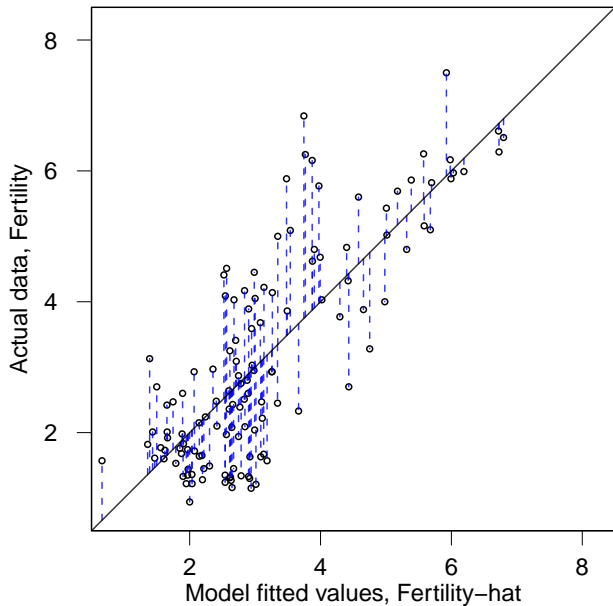
Variable	Model		
	1	2	3
Intercept	12.59 [11.11, 14.08]	4.13 [3.80, 4.46]	11.25 [9.81, 12.69]
Education Ratio	-0.10 [-0.12, -0.08]		-0.08 [-0.10, -0.06]
GDP per capita		-0.10 [-0.12, -0.08]	-0.05 [-0.07, -0.03]
$N$	130	130	130
$R^2$	0.55	0.35	0.64
RMSE	1.12	1.35	1.01

95% confidence intervals in brackets

This table presents the same information, but is easier to digest

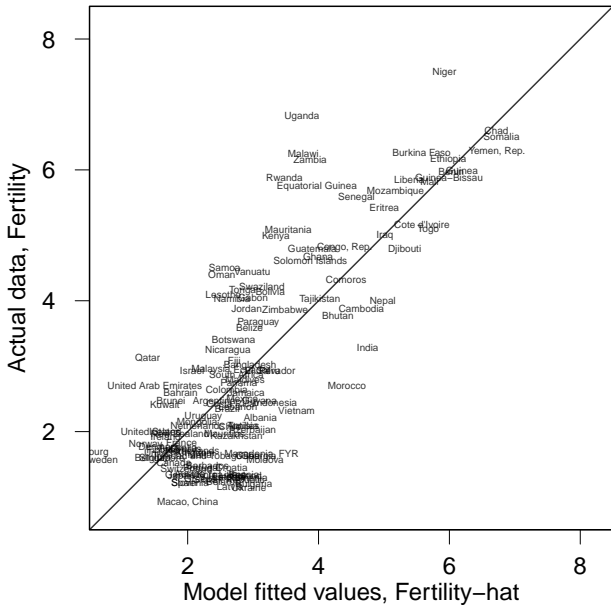


To see the residuals, compare the model fit with reality

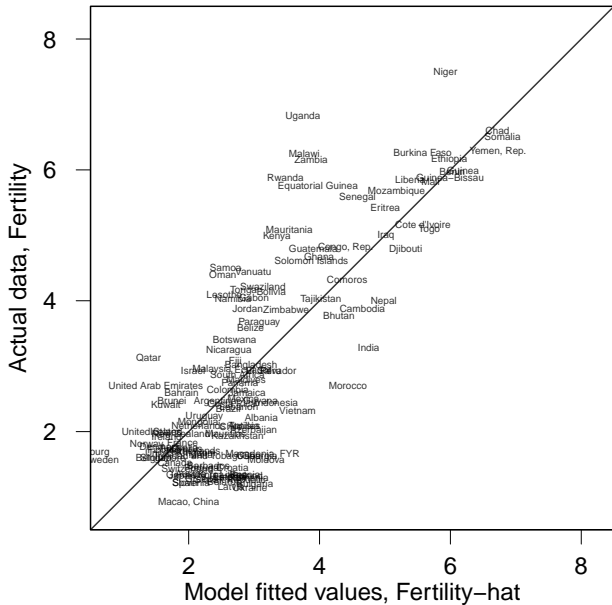


Note that in the multivariate case, we need to plot against  $\hat{y}_i$ , not  $x_i$ , because there is more than one  $x_i$



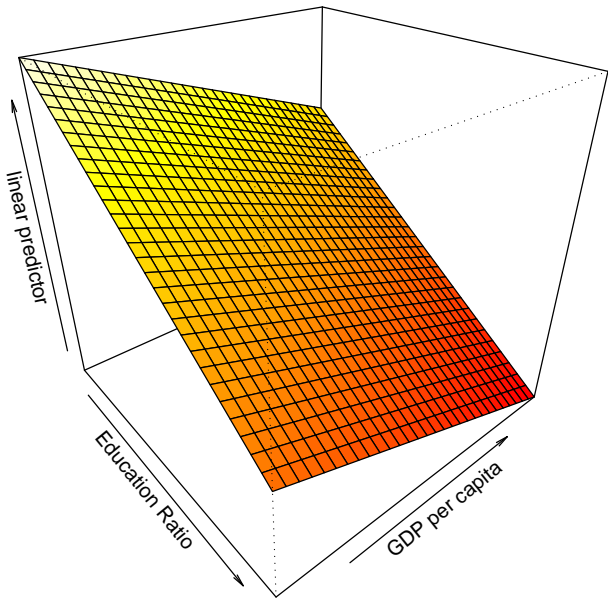


Examining which cases are big outliers may suggest additional variables to include as covariates

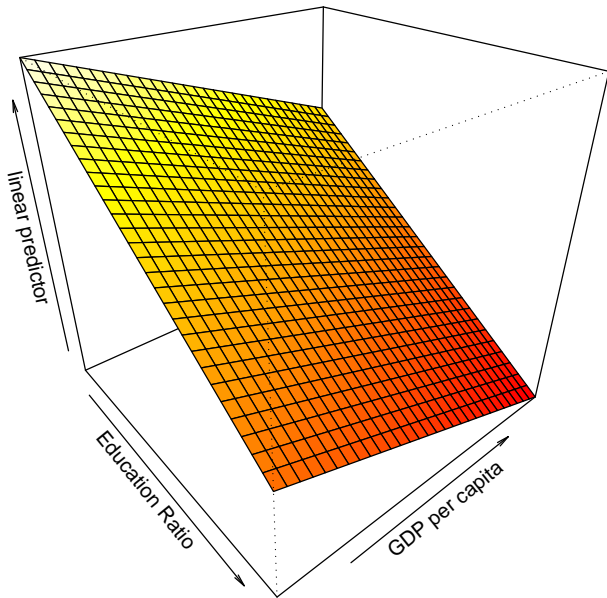


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Think of what the missing cases have in common

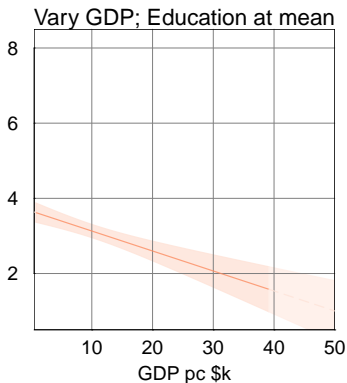
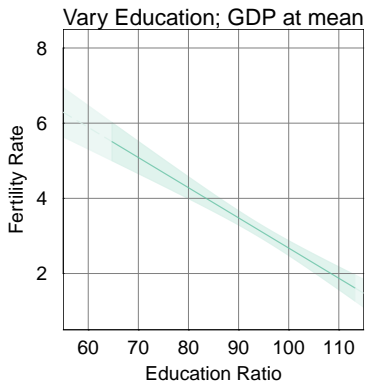


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Visualizing the modelled relationship between many variables is tricky

We can do it with a 3D plot for 2 covariates, but not for 3 or more

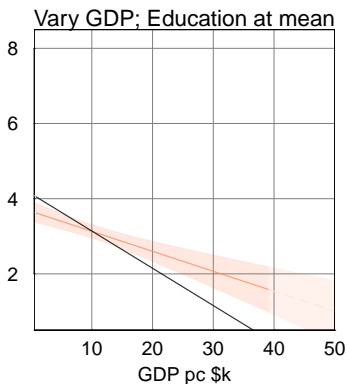
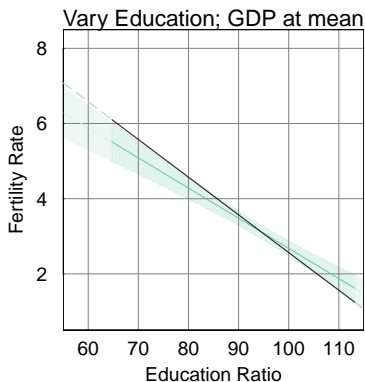


An alternative that works for any number of covariates:

Plot out the model predictions as a function of each covariate,

while holding the other covariates fixed, e.g., at their means

Then predict what Fertility rate should happen on average if the country had average GDP but variable Education (or vice versa)



Let's compare the multiple regression estimates (in color) with the bivariate regression results (in black)

How are they different? Are the bivariate results affected by omitted variable bias?

## Regression models including Agricultural Labor

Variable	Model			
	1	2	3	4
Intercept	11.15 (2.64)	2.76 (0.18)	1.83 (0.15)	8.95 (2.79)
Education Ratio	-0.09 (0.03)			-0.06 (0.03)
GDP per capita (\$k)		-0.04 (0.01)		-0.03 (0.01)
Agriculture Labor			0.02 (0.01)	0.004 (0.008)
<i>N</i>	72	72	72	72
$R^2$	0.13	0.17	0.14	0.26
RMSE	0.94	0.92	0.93	0.88

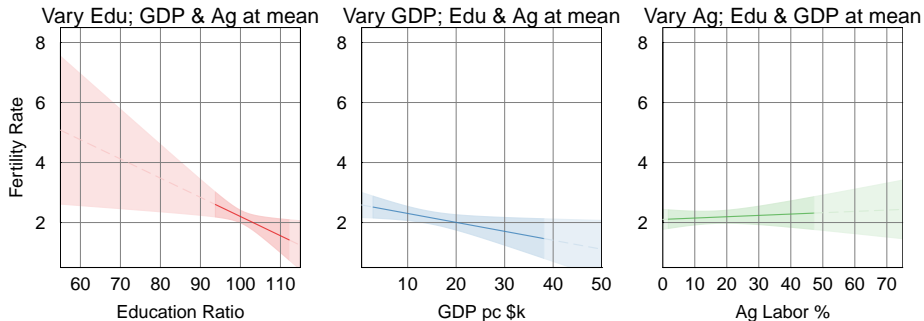
Standard errors in parentheses

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Edu Ratio	-0.09 [-0.14, -0.04]			-0.06 [-0.12, -0.01]
GDP pc		-0.04 [-0.06, -0.02]		-0.03 [-0.06, -0.003]
Ag Labor			0.02 [0.01, 0.03]	0.004 [-0.01, 0.02]
<i>N</i>	72	72	72	72
<i>R</i> <sup>2</sup>	0.13	0.17	0.14	0.26
RMSE	0.94	0.92	0.93	0.88

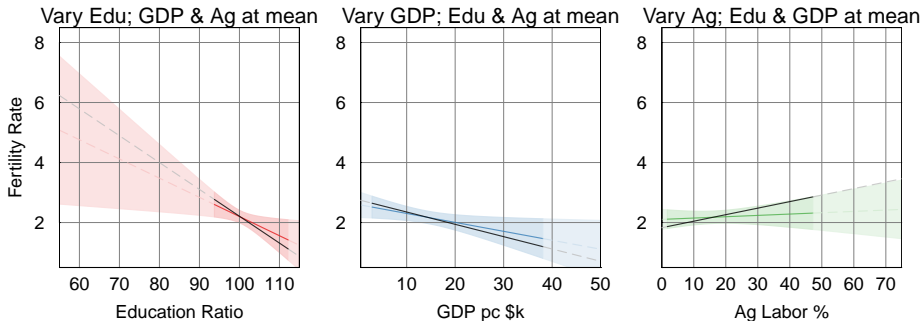
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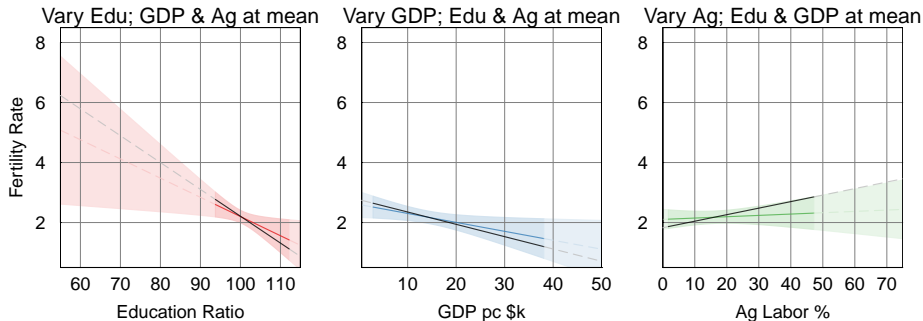


How do we interpret these plots?

The dashed lines indicate extrapolation:  
no observed data have these values for the covariates



The black lines show the bivariate results. Was there omitted variable bias?



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**YES.** The apparent effect of Ag Labor was a mirage: just the omitted effect of GDP per capita. If we control for GDP, we see Ag Labor has no effect.

## Warning!

Linear regression is powerful, but easy to misuse

We mentioned one assumption last time: That the error term is Normally distributed

To this we now add two additional assumptions

**Correct specification** The model contains all the covariates that produce  $Y$ . If any omitted cause of  $Y$  is correlated with the included  $X$ 's, then  $\hat{\beta}$  can no longer be trusted.

**No endogeneity of  $Y$**  None of the included  $X$ 's are caused by  $Y$

More on these assumptions next time