Quantum Lévy Processes and Fractional Kinetics

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Exotic stochastic processes are shown to emerge in the quantum evolution of complex systems. Using influence function techniques, we consider the dynamics of a system coupled to a chaotic subsystem described through random matrix theory. We find that the reduced density matrix can display dynamics given by Lévy stable laws. The classical limit of these dynamics can be related to fractional kinetic equations. In particular we derive a fractional extension of Kramers equation.

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Whether one studies deterministic Hamiltonian or dissipative systems, one finds that transport in chaotic systems often resembles some type of stochastic process. The dynamics of such systems leads to a rich spectrum of behaviors, from enhanced diffusion such as tracer diffusion in flows and turbulent diffusion in the atmosphere, to dispersive diffusion [1]. Much effort has been spent in recent years to understand such classical stochastic processes in chaotic systems, leading to the development of approaches ranging from fractional kinetic equations [2–4], Lévy flights [5] to random walks in random environments [5,6] and stochastic webs [7]. One of the common features to all of these is the use of Lévy stable laws [8]. It was shown by Lévy [9], in studies of extension of the central limit theorem, that a continuous class of non-gaussian processes satisfy the same fundamental equation that gives rise to the theory of gaussian processes, namely the Chapman-Kolmogorov equation for the conditional probability $P(q,q',t)$:

$$P(q-q';t) = \int dq" \, P(q-q", t-t")P(q"-q', t"). \quad (1)$$

(Translational invariance assumed for simplicity). The standard solution, $P(q-q',t) \propto \exp(-(q-q')^2/4Dt)$, gives rise to the gaussian processes and the usual form of the Fokker-Planck equation. The general solutions of Lévy provide a way to generalize Brownian motion.

The non-gaussian processes which satisfy (1) are called Lévy stable laws, and have the form:

$$P(q,t) = \mathcal{L}_\alpha^A(q) = \frac{1}{2\pi}\int \exp\left\{ikq - A|k|^\alpha\right\} dk \qquad (2)$$

where $0 < \alpha \le 2$, with $\alpha = 2$ corresponding to Gaussian processes. The Lévy stable laws satisfy the scaling relation:

$$\mathcal{L}_\alpha^A(q) = A^{-1/\alpha}\mathcal{L}_\alpha^1(qA^{-1/\alpha}) \qquad (3)$$

where for $A=1$ we drop the superscript: $\mathcal{L}_\alpha^1(x) = \mathcal{L}_\alpha(x)$. For $\alpha < 2$, these distributions are characterized by infinite second moments. Never-the-less, these non-gaussian processes can be related to anomalous transport in a variety of physical systems [6]. On the other hand, chaotic systems are known to undergo anomalous diffusion and transport. We have recently shown that turbulent diffusion can arise in complex quantum systems. Here we find that the general form of such quantum chaotic backgrounds can give rise to quantum evolution characterized by Lévy stable laws. Further, we can now connect, in the semi-classical limit, such processes to fractional kinetic theory, which was initially introduced as a phenomenological approach to classical anomalous diffusion.

We would like to study the problem of a particle coupled to a chaotic environment, quantum mechanically. It has been realized in recent years that the quantum counterpart of chaos is random matrix theory. For systems with time-reversal symmetry, the random matrices are real-symmetric. In this letter we will examine the class of quantum dynamic processes which can be realized through the interaction of a particle with a random matrix background. In contrast to the Caldeira-Leggett approach [10], we assume from the outset that the background is chaotic, and not necessarily thermal. We denote the coordinates of the background by (x,p) and that of the test particle by (X,P). The Hamiltonian for the background plus interaction is taken to have the following form:

$$H_b = h_0(x,p) + h_1(X,x,p). \qquad (4)$$

In the basis of (many-body) eigenstates of h_0, $h_0 \mid n\rangle = \varepsilon_n \mid n\rangle$ $(n = 1,...,N)$, we define the matrix of H_b as

$$[H_b]_{ij} = \varepsilon_i\delta_{ij} + [h_1(X)]_{ij}. \qquad (5)$$

The average level density is $\rho(\varepsilon) = \rho_0\exp(\beta\varepsilon)$. For a background with constant average level density, $\beta = 0$, while for a general many body system, $\beta > 0$. The chaotic properties of the interaction of the background with the particle are incorporated into the correlation function (second cumulant):

$$\langle [h_1(X)]_{ij}[h_1(Y)]_{kl}\rangle = \mathcal{G}_{ij}(X,Y)\Delta_{ijkl}. \qquad (6)$$

Here $\Delta_{ijkl} = [\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$, and all other cumulants vanish. In our analysis, the integration over the chaotic part, given by $h_1(X)$, is defined through a gaussian measure for parametric random matrices, which can be defined as

$$P[h_1(X)]dh_1 \propto \mathcal{D}h_{1,ij}(X) \exp\left\{-\frac{1}{2}\int dX dY \right. \\ \left. \mathrm{Tr}\left[h_1(X)\mathcal{G}^{-1}(X,Y))h_1(Y)\right]\right\}. \tag{7}$$

In that case, if we assume a translationally invariant measure, $\mathcal{G}^{-1}(X,Y) = \mathcal{G}^{-1}(X-Y)$, then the only non-vanishing cumulant is the second moment given in Eq. (6). The character of the interaction of the background with the test particle is incorporated into the correlation function $\mathcal{G}(X-Y)$. We use the form [12,13]:

$$\mathcal{G}_{ij}(X) = \frac{\Gamma^{\downarrow}}{2\pi\sqrt{\rho(\varepsilon_i)\rho(\varepsilon_j)}} \exp\left[-\frac{(\varepsilon_i - \varepsilon_j)^2}{2\kappa_0^2}\right] G\left(\frac{X}{X_0}\right). \tag{8}$$

This describes a parametric, banded, random matrix where the strength of matrix elements decreases with increasing level density. Here $G(x) = G(-x) = G^*(x) \leq 1$, $G(0) = 1$, and the spreading width $\Gamma^{\downarrow}$, κ_0 (linked with the effective band width $N_0 \approx \kappa_0\rho(\varepsilon)$) and correlation length X_0 are characteristic of the background.

In order for the measure (7) to be positive definite, it has been shown that G must be a positive definite function, and consequently, can decorrelate no faster than Gaussian [11]:

$$G(X) \simeq 1 - |X|^{\alpha} + \cdots, \qquad \alpha \in [0,2]. \tag{9}$$

For $\alpha > 2$, the argument in the exponent can change sign, and the measure (7) becomes unbounded [11]. As the position X of the slow particle changes, the energy levels E_n of $[h_1(X)]_{ij}$ change. Using the above measure, the average fluctuations are

$$\langle(\delta E_n(X))^2\rangle = \frac{4N}{\pi^2}\delta X^{\alpha} = D_{\alpha}\delta X^{\alpha}. \tag{10}$$

The energy-spacing fluctuations have a behavior which is similar to a Lévy process, except on short distance scales, characterized by the diffusion constant D_{α}. The character of these fluctuations in the eigenvalues E_n, indicated by α, will be seen to be related Lévy stable laws which describe the time evolution of the density matrix for a particle evolving in this chaotic bath.

To develop the dynamical evolution of a free particle evolving in the presence of a chaotic background, we take the Hamiltonian of the form:

$$H_{ij}(X,P) = \delta_{ij}(\frac{P^2}{2M} + U(X)) + H_{b,ij}(X). \tag{11}$$

The correlated, random-matrix bath can be integrated out in an influence functional formalism [12]. This has recently been done up to $o(\beta^2)$ [13]. For our purposes, the $o(\beta)$ action is sufficient. In this case the effective equation for the density matrix of the test particle has the form:

$$i\hbar\partial_t\rho(X,Y,t) = \left\{\frac{P_X^2}{2M} - \frac{P_Y^2}{2M} + U(X) - U(Y) \right. \\ -\frac{\beta\Gamma^{\downarrow}\hbar}{4X_0M}G'\left(\frac{X-Y}{X_0}\right)(P_X - P_Y) \\ \left. +i\Gamma^{\downarrow}\left[G\left(\frac{X-Y}{X_0}\right) - 1\right]\right\}\rho(X,Y,t) \tag{12}$$

where $G'(X)$ above represents $-\alpha|X|^{\alpha-1}$. Consider first a test-particle interacting with a background with constant average level density ($U(X) = 0$ and $\beta = 0$). This evolution equation can be solved by passing to the coordinates $r = (X+X')/2$, $s = X - X'$. In these variables, the density matrix has the form:

$$\rho(r,s,t) = \int dr' \int \frac{dk}{2\pi\hbar}\rho_0(r', s - \frac{kt}{m}) \exp\left[\frac{ik(r-r')}{\hbar} \right. \\ \left. + \frac{\Gamma^{\downarrow}M}{\hbar k}\int_{s-kt/M}^{s} ds'[G(s'/X_0) - 1]\right] \tag{13}$$

An initial wavepacket, $\psi_0(X) = \exp[-X^2/4\sigma^2]/[2\pi\sigma^2]^{1/4}$, provides an initial density matrix $\rho_0(X,X') = (1/\sqrt{2\pi\sigma^2})\exp[-(4r^2+s^2)/8\sigma^2]$.

For the diffusive dynamics of the test particle, we are interested in the diagonal component of the density matrix $\rho(X,X,t) = \rho(r, s=0, t)$:

$$\rho(r,0,t) = \iint \frac{dr'dk}{2\pi\hbar}\rho_0\left(r', -kt/M\right)\exp\left[\frac{ik(r-r')}{\hbar} \right. \\ \left. - \int_{-kt/M}^{0} ds' \frac{M\Gamma^{\downarrow}}{k\hbar}\left|\frac{s'}{X_0}\right|^{\alpha}\right] \tag{14}$$

$$= \int \frac{dk}{2\pi\hbar}\exp\left[-k^2\left[\frac{\sigma^2}{2\hbar^2} + \frac{t^2}{8M\sigma^2}\right] \right. \\ \left. - \frac{\Gamma^{\downarrow}t^{\alpha+1}}{(\alpha+1)\hbar(MX_0)^{\alpha}}|k|^{\alpha} + ik\frac{r}{\hbar}\right]. \tag{15}$$

$\rho(X,X,t)$ is nothing more than the spatial probability distribution $P(X,t)$ for the process. We can now express it in terms of a convolution of Lévy stable laws:

$$\rho(X,X,t) = \int dX' \mathcal{L}_{\alpha}^{a(t)}(X')\mathcal{L}_2^{b(t)}(X-X') \tag{16}$$

where

$$a(t) = \frac{\Gamma^{\downarrow}}{(\alpha+1)\hbar}\left(\frac{\hbar}{MX_0}\right)^{\alpha} t^{\alpha+1} \tag{17}$$

$$b(t) = \frac{\sigma^2}{2} + \frac{\hbar^2}{8M^2\sigma^2}t^2 \tag{18}$$

As both functions in the integrand of Eq. (16) are positive definite, the spatial probability $P(X,t)$ is also positive definite. Notice that the restriction of $0 < \alpha \leq 2$, which came from the short–distance statistical correlations and the requirement of a positive definite statistical measure, is also the necessary requirement on the Lévy stable law to keep the resulting time evolution positive definite. Hence the character of the short distance fluctuations is directly responsible for the long–time behavior of the quantum system.

Consider now the short–time and long–time behavior of the dynamics. For $1 < \alpha < 2$, in the limit of long times, we expect the $t^{\alpha+1}$ term to dominate over t^2 in (15), so that the density asymptotically approaches a Lévy stable law:

$$\rho(X,X,t) \longrightarrow a(t)^{-1/\alpha}\mathcal{L}_\alpha\left(a(t)^{-1/\alpha}X\right), \tag{19}$$

while for very short times, the Gaussian process is the dominant behavior:

$$\rho(X,X,t) \longrightarrow \frac{\sqrt{2}}{\sigma}\,\mathcal{L}_2\left(\frac{\sqrt{2}}{\sigma}X\right) \tag{20}$$

So while the general solution is convolution, one can see that the quantum dynamics can exhibit a crossover from Gaussian diffusion to a Lévy stable law. The scaling properties of the stable laws do not have any contradiction with quantum mechanics, the solution is only approximate. In the exact convolution, the Gaussian $\mathcal{L}_2$ regulates the large k, or short distance, behavior of the distribution in the long time limit. Specifically, the $|k|^\alpha$ term dominates in the long time limit only for momenta $k < k_c$ where $k_c = (a(t)/b(t))^{1/(2-\alpha)}$. For the special case of $\alpha = 2$, the result is Gaussian, but the dynamics can be anomalous. For a non-zero coupling to the background, one can have turbulent-like diffusion, where the quantum expectation value behaves as $\langle X^2\rangle \sim t^3$ [14]. When the level density of the background is not constant, $\beta > 0$, it has been recently found that one can recover Brownian diffusion [13]. For general α, and $\beta > 0$, however, the results are not yet known.

For the range $0 < \alpha < 1$, the long–time behavior is essentially gaussian. At short–times, the dynamics is influenced by $\mathcal{L}_\alpha$, and there is a cross-over from short time stable law dynamics to normal Gaussian expansion of the wavepacket. Again the shortest length scales are regulated by the initial wavepacket.

Efforts to understand unusual stochastic behaviors of dynamical systems has led to the development of extensions of the Fokker-Planck (FP) equation [2–4]. These are phenomenological fractional kinetic equations (restricted to one dimension) in which certain derivatives are replaced by derivatives of 'fractional' order [16]. Such approaches have also found applications in a wide range of problems from turbulence to diffusion in porous or viscoelastic media [15]. We can now explore the type of stochastic process which emerges in the classical limit of our quantum Lévy process, and the connection to multi-dimensional fractional kinetic theory.

A typical type of phenomenological fractional FP equation has the form

$$\begin{aligned}\frac{\partial^\beta P(Q,t)}{\partial t^\beta} &= \frac{\partial^\mu}{\partial(-Q)^\mu}\left(A(Q)P(Q,t)\right) \\ &\quad+\frac{1}{2}\frac{\partial^{2\nu}}{\partial(-Q)^{2\nu}}\left(B(Q)P(Q,t)\right).\end{aligned} \tag{21}$$

where $\mu = \nu = 1$ in Ref. [4], $\mu = \nu$ in Ref. [3] and $\beta = \nu = 1$ in Ref. [2]. Here the symbol $\partial^\mu/\partial x^\mu$ and so forth represent the Riemann-Liouville fractional derivative [16], except for Ref. [2], where it represents the Fourier transform of $-k^\mu$. This equation, while formally constructed, is phenomenological. It is defined to reproduce anomalous diffusion through scaling formulas such as $Q^2 \sim t^\gamma$, where γ is a function of β, μ, ν. A few points should be made here. Generally, the coefficients A and B are defined as limits whose existence is postulated but not known. Further, either the form of the fractional derivatives is taken to provide this scaling law, or power law noise is chosen to obtain them. Such dynamics can then be related to Lévy processes [1]. Finally, the extension of these equations to phase space becomes tenuous, since it is not clear how to include momentum. Not only is it unclear if one should take fractional derivatives with respect to coordinates, momenta or both, but the existence of the corresponding coefficients A, B,... is unknown. Through our transport equation, we can provide a microscopic interpretation of these coefficients as well as a systematic manner to construct a fractional kinetic equation in phase space whose quantum limit results in Lévy processes.

To obtain a classical transport equation, we construct the Wigner transform $f(Q,P,t)$ of the density matrix $\rho(X,Y,t)$ as

$$f(Q,P,t) = \frac{1}{2\pi\hbar}\int dR \exp\left\{-\frac{iPR}{\hbar}\right\}\rho\left(Q+\frac{R}{2}, Q-\frac{R}{2}, t\right). \tag{22}$$

Applying this to our evolution equation, taking the leading order terms in $\hbar$, we find

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{2i\pi\hbar^2}\int dR \exp\left(-\frac{iPR}{\hbar}\right)\left\{-\frac{\hbar^2}{2M}\partial_Q\partial_R\right. \\ &\quad+U\left(Q+\frac{R}{2}\right) - U\left(Q-\frac{R}{2}\right) - i\Gamma^{\downarrow}\left|\frac{R}{X_0}\right|^\alpha \\ &\quad\left.+i\gamma\hbar X_0\alpha\left|\frac{R}{X_0}\right|^{\alpha-1}\partial_R\right\}\rho\left(Q+\frac{R}{2}, Q-\frac{R}{2}, t\right).\end{aligned} \tag{23}$$

This leads naturally to the Reisz fractional integro-differential operator. This operator, applied to a function $f(P)$, is defined as [16]

$$(-\Delta_P)^{\alpha/2}f = \mathcal{F}^{-1}|X|^\alpha\mathcal{F}f, \tag{24}$$

where Δ_P is the Laplacian (in our case with respect to the momentum P), and $\mathcal{F}$ represents a Fourier transform. (This operator is distinct from that proposed in [2] which did not have the absolute value, and from [3] which uses the Riemann–Liouville form of this operator. The Reisz operator is defined as a fractional integral for Reα <0 and as a fractional derivative for Reα >0 through analytic continuation.) It is convenient to define the operator $D_P^\alpha = (-i/\hbar)^\alpha(-\Delta_P)^{\alpha/2}$, since $D_P^1[Pf] = \partial(Pf)/\partial P$ and $D_P^2[f] = \partial^2 f/\partial P^2$. Then the classical limit of our quantum Lévy process gives rise to a fractional extension of Kramers equation:

$$\frac{\partial f(Q,P,t)}{\partial t} + \frac{P}{M}\frac{\partial f(Q,P,t)}{\partial Q} - \frac{\partial U(Q)}{\partial Q}\frac{\partial f(Q,P,t)}{\partial P} \quad (25)$$

$$= \gamma_\alpha \left\{ D_P^{\alpha-1}[Pf(Q,P,t)] - \frac{2TM}{\alpha\hbar^2}(i\hbar)^\alpha D_P^\alpha[f(Q,P,t)] \right\},$$

where $T = 1/\beta$ is the temperature, the velocity is $V = \dot{Q} = P/M$, and the generalized friction coefficient is given by:

$$\gamma_\alpha = \frac{\beta\Gamma^\downarrow\hbar\alpha}{2MX_0^\alpha}. \quad (26)$$

For $\alpha = 2$ we recover Kramers equation [17]. What we see is that it is not the coordinates which acquire the fractional character, as usually assumed, but the momenta. Because the coupling to the background is not momentum dependent, the correlation function $G(X)$ results only in fractional derivatives with respect to momenta. This can be traced back to the nature of the chaotic correlations in Eq. (12). Further, these processes, related to Lévy stable laws, do not require the introduction of fractional time derivatives. We note here that our transport theory has a consistent classical limit for all of these transport coefficients only when they remain finite as $\hbar \to 0$. This requires in turn that the parameters of our theory cannot remain constant as $\hbar \to 0$, if we are to recover a well defined classical transport. Finally, we observe that this approach provides finite coefficients D_{QQ}, D_{PP}, D_{QP} and so forth (eg A, B,...) for a fractional kinetic equation in phase space.

We have shown that the quantum evolution of a wavepacket in a chaotic environment can lead to reduced density matrices which behave as Lévy stable laws, and are regulated on short distances. The short distance energy fluctuations of the background, which are characterized by a parameter $\alpha \in (0,2]$, are found to be precisely related to the quantum time evolution with a Lévy stable law of the same character α. For $\alpha = 2$ one has gaussian processes which can display normal to turbulent-like diffusion or even Brownian diffusion ($\beta > 0$), while for $\alpha = 1$ one has the dynamics of the Dyson process. The general quantum evolution of a wavepacket displays a cross-over between Gaussian and Lévy dynamics. In passing to the classical limit of this behavior, we find that the dynamical evolution results in a fractional kinetic equation, which is a generalization of Kramers equation. For $\alpha = 2$ Kramers theory is recovered. This approach provides a means to develop fractional kinetic theory in more than one dimension, since the expansion coefficients are determined from the microscopic theory. It also provides the possibility to explore the connections between quantum and classical transport in chaotic systems, as well as the links between chaos, quantum statistical fluctuations, Lévy stable laws and classical fractional dynamics.

[1] M. Shlesinger, G. Zaslavsky and J. Klafter, *Nature* **363** (1993) 31.

[2] H. Fogedby, *Phys. Rev. Lett.* **73** (1994) 2517.

[3] G. Zaslavsky, *Physica* **D76** (1994) 110-122; *Chaos* **7** (1997) 753.

[4] G. Jumarie, *J. Math. Phys.* **33** (1992) 3536.

[5] M.Shlesinger, B. West and J.Klafter, *Phys. Rev. Lett.* **58** (1987) 1100; E.Montroll and B. West, in *Fluctuation Phenomena*, Eds. E. Montroll and J. Lebowitz (North-Holland, Amsterdam, 1979).

[6] J.P.Bouchaud, A.Comtet, A.Georges and P.Le Doussal, *Ann. Phys. (NY)* **201** (1990) 285; J.P.Bouchaud and A.Georges, *Phys. Rep.* **195** (1990) 127.

[7] G. Zaslavsky and B. Niyazov, *Phys. Rep.* **283** (1997) 73.

[8] See for example, *Lévy Flights and Related Topics in Physics*, Eds. M. Shlesinger, G. Zaslavsky and U. Frisch, (Springer-Verlag, Berlin, 1995).

[9] P. Lévy, *Calcul des Probabilities*, (Guthier-Villars, Paris, 1925).

[10] A.O. Caldeira and A.J. Leggett, Ann. Phys. **149**, 374–456 (1983); R.P. Feynman and F.L. Vernon, Ann. Phys. **24**, 118–173 (1963); P. Pechukas, Phys. Rev. **181**, 174–185 (1969).

[11] D. Kusnezov and C. Lewenkopf, *Phys. Rev.* **E53** (1996) 2283.

[12] A.Bulgac, G.DoDang and D.Kusnezov, *Phys. Rev.* **54**, 3468–3478, (1996); *Ann. Phys. (NY)* **242** (1995) 1.

[13] A.Bulgac, G.DoDang and D.Kusnezov, preprint (1998).

[14] D. Kusnezov, A. Bulgac and G. DoDang, *Phys. Lett.* **A234** (1997) 103.

[15] F. Mainardi, *Chaos, Solitons and Fractals*, **7** (1996) 17; R. Nigmatullin, Phys. Status Solidi, **B124** (1984) 389; E. Novikov, in *Lévy Flights and Related Topics in Physics*, Eds. M. Shlesinger, G. Zaslavsky and U. Frisch, (Springer-Verlag, Berlin, 1995), 35;

[16] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives*, (Gordon-Breach, Paris, 1993).

[17] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, (North–Holand, Amsterdam, 1990).