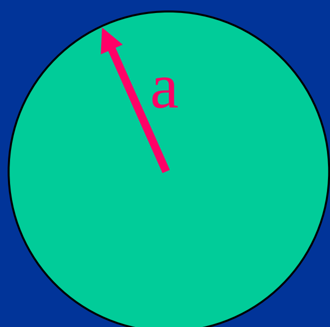
Fermi sea λ_F

(non-interacting particles)



Fermi sea λ_F

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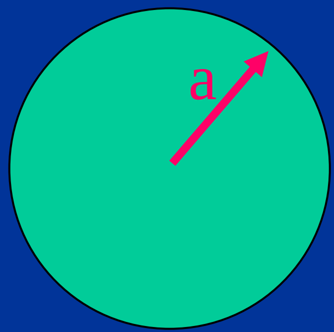

Fermi sea λ_F

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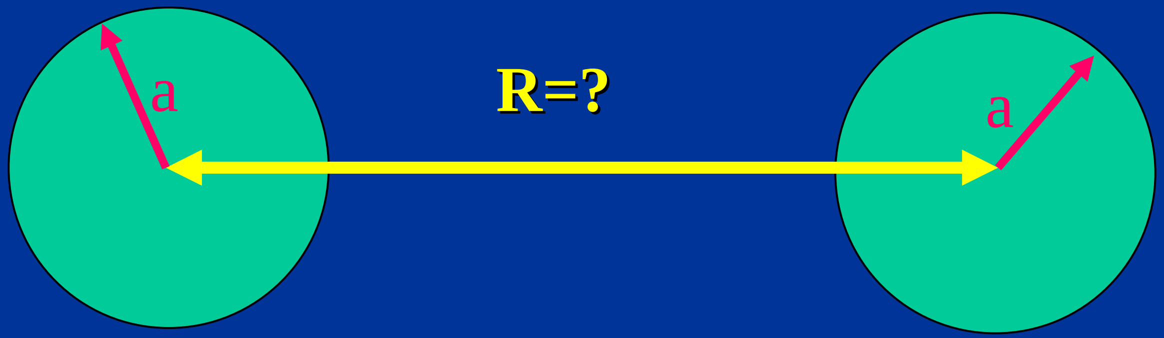
Question:

What is the most favorable arrangement of these two spheres?



Answer: The energy of the system does not depend on r as long as r > 2a.
NB Assuming that a liquid drop model for the fermions is accurate!

This is a very strange answer! Isn't it? Something is amiss here.

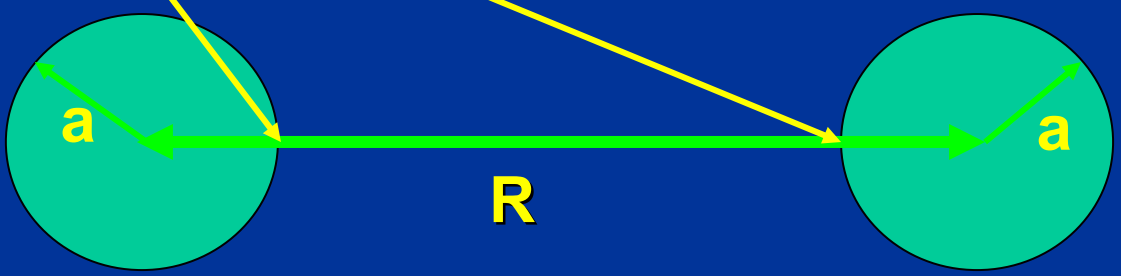
Fermi sea λ_F

Let us try to think of this situation now in quantum mechanical terms.

The dark blue region is really full of de Broglie's waves, which is in the absence of homogeneities are simple plane waves.

When inhomogeneities are present, there are a lot of scattered waves.

Also, there are some almost stationary waves, which reflect back and forth from the two tips of the empty spheres.



As in the case of a musical instrument, in the absence of damping, the stable "musical notes" correspond to stationary modes.

Problems: 1) There is a large number of such modes.
2) The tip-to-tip modes cannot be absolutely stable, as the reflected wave disperses in the rest of the space.

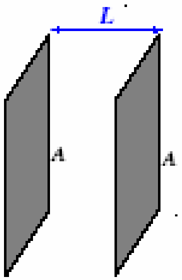
Fermionic Casimir effect for normal and superfluid systems

A force from nothing onto nothing

AW

Aurel Bulgac (Seattle)
Piotr Magierski (Warsaw)
Andreas Wirzba (Juelich)

- H.B.G. Casimir (1948): two parallel uncharged metallic plates attract each other in vacuum



$$\frac{F^{||}(L)}{A} = -\frac{\hbar c}{L^4}\frac{\pi^2}{240} \approx -1.3\times 10^{-7}\frac{1}{L^4}\mathrm{N}\frac{\mu\mathrm{m}^4}{\mathrm{cm}^2}$$

$$\mathcal{E}^{||}(L) = -\frac{\hbar c}{L^3}\frac{\pi^2}{720}A$$

- Origin: zero-point fluctuations of e.m. field modified by the addition of the two plates relative to free case

  $\Rightarrow$ *change* in the energy of the vacuum: $\displaystyle\sum\!\!\!\!\!\!\int \hbar\omega_k|_{\mathrm{plates}(L)} - \sum\!\!\!\!\!\!\int \hbar\omega_k|_{\mathrm{free}}$

Casimir effect: *Mesoscopic manifestation of quantum fluctuations of the vacuum*

- Related to van der Waals forces (however, the latter always attractive !)
- experimental confirmation in the last decade (for the sphere-plate system!)
  - S. Lamoureaux, *Phys. Rev. Lett.* **78** (1997);
  - U. Mohideen & A. Roy, *Phys. Rev. Lett.* **81** (1998);
  - 9-Feb-2001 issue of *New York Times* about Casimir effect in *MicroElectroMechanical Systems*; etc.

Many of the subsequesnt slides are borrowed from my collaborator's talk A. Wirzba

Casimir effect and vacuum energy in QFT:

- Infinite zero-point energy can be subtracted (discarded) by suitable redefinition of energy-origin (e.g. normal ordering)
- However, energy-origin can be re-defined only *once* (and only in homogeneous space (no b.c.'s) and without gravity)

$\therefore$ Casmir effect $=$ difference of (properly regularized) eigen-mode sums of *constrained* QFT
$-$ eigen-mode sums of *free* QFT

$$\varepsilon_C = \lim_{V\to\infty} \frac{\mathcal{E}_C}{V} \qquad (V : \text{quantization box})$$

$$= \lim_{\Lambda\to\infty} \lim_{V\to\infty} \underbrace{(-1)^{2S}}_{\text{statistic}} \left(\sum\!\!\!\!\!\!\int_{\vec{k},\nu_{\rm deg}} \frac{1}{2}\hbar\omega_k|_{V,\Lambda,C} - \sum\!\!\!\!\!\!\int_{\vec{k},\nu_{\rm deg}} \frac{1}{2}\hbar\omega_k|_{V,\Lambda,\emptyset} \right)$$

Constraints by:

gravity	(e.g. non-flat metric)
external fields	(e.g. anomalies)
internal fields	(e.g. QCD-vacuum)
geom.-dep. b.c.'s	(2 plates, 2 half-spheres, ...)

Gravity serious: "cosmological constant problem!"

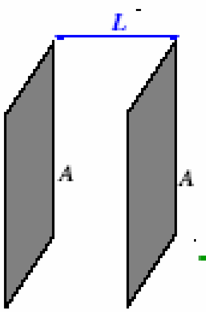
The *critical mass density* $\rho_{\rm crit.} = \frac{3H_0^2}{8\pi G_N} \approx h_0^2 \times 10.5\,{\rm GeV\,m^{-3}} = h_0^2 \times 4.8 \times 10^{-47}{\rm GeV}^4$

compared with *vacuum energy density* $\rho_V = \frac{\Lambda_{cosm.}}{8\pi G_N} + \varepsilon$

where $\varepsilon = \frac{\mathcal{E}_0}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2}\hbar\omega_k e^{-\hbar\omega_k/\Lambda_{UV}} \approx \frac{3\hbar c}{2\pi^2}\left(\frac{\Lambda_{UV}}{\hbar c}\right)^4$

Theory	Λ_{UV}	Scale [GeV]	$\varepsilon =$ "$\frac{1}{2}\int \hbar\omega$"	$\varepsilon/\rho_{\rm crit.}$
ToE	M_P	1.2×10^{19}	10^{76} GeV^4	10^{123}
	String	5×10^{17}	10^{72} GeV^4	10^{120}
GUT	SU(5)$_{\rm SUSY}$	2×10^{16}	10^{65} GeV^4	10^{112}
	SU(5)$_{\rm min.}$	10^{15}	10^{60} GeV^4	10^{107}
	SUSY$_{\rm break}$	10^3	10^{12} GeV^4	10^{59}
EW	Higgs vacuum	2.5×10^2	10^9 GeV^4	10^{56}
	Z, W	10^2	10^8 GeV^4	10^{55}
QCD	$\Lambda_{had.}$	1	1GeV^4	10^{47}
	Λ_{QCD}	0.2	10^{-3} GeV^4	10^{44}
EM	$\hbar\omega_{\rm phot.}$	1 eV	10^{-36} GeV^4	10^{11}
?	10^{-3} eV	$\sim 40K$	10^{-46} GeV^4	1

Original Casimir effect:



1) "box" geometry
2) EM b.c.'s : $\mathbf{n}\cdot\mathbf{B}=0$ & $\mathbf{n}\wedge\mathbf{E}=0.$ $\Rightarrow$ 2 decoupled scalar fields:

TE: $\phi^{\mathrm{D}}(x,y,z,t)|_{z=0}=\phi^{\mathrm{D}}(x,y,z,t)|_{z=L}=0$ (Dirichlet b.c.'s)

$$\phi^{\mathrm{D}}_{nk_\perp}(x,y,z,t)=\sin(k_z z)\,e^{i(k_x x+k_y y)}e^{-i\omega_{n,k_x,k_y}t}$$

$$\boxed{k_z=\left(\frac{\pi}{L}\right)n\,,\;\; n=1,2,3,\cdots;\;\;\omega_{n,k_x,k_y}=c\sqrt{k_x^2+k_y^2+\left(\frac{n\pi}{L}\right)^2}}$$

TM: $\frac{\partial}{\partial z}\phi^{\mathrm{N}}(x,y,z,t)|_{z=0}=\frac{\partial}{\partial z}\phi^{\mathrm{N}}(x,y,z,t)|_{z=L}=0$ (Neumann b.c.'s)

$$\phi^{\mathrm{N}}_{nk_\perp}(x,y,z,t)=\cos(k_z z)\,e^{i(k_x x+k_y y)}e^{-i\omega_{n,k_x,k_y}t}$$

$$\boxed{k_z=\left(\frac{\pi}{L}\right)n\,,\;\; n=0,1,2,3,\cdots.}$$

$$\begin{aligned}\mathcal{E}_C^{||\mathrm{EM}}(L) &= \lim_{\Lambda\to\infty}A\iint\frac{dk_xdk_y}{(2\pi)^2}\left(\sum_{n=1}^{\infty}\frac{1}{2}\hbar\omega_{n,k_x,k_y}+\sum_{n=0}^{\infty}\frac{1}{2}\hbar\omega_{n,k_x,k_y}\right)e^{-\frac{\hbar\omega_{n,k_x,k_y}}{\Lambda}}\\ &= \lim_{\Lambda\to\infty}\hbar cAL\left[\frac{3(\Lambda/\hbar c)^4}{\pi^2}+(-1+1)\frac{(\Lambda/\hbar c)^3}{4\pi L}-\frac{\pi^2}{720L^4}+\mathcal{O}\left((\hbar c)^2/\Lambda^2L^6\right)\right]\end{aligned}$$

Experiments: (<12 until 1998)

- *Early efforts:* relied on cantilever or spring balances

  J.T.G. Overbeek & M.J. Sparnaay (1954); B.V. Deriagin & I.I. Abrikosova (1957): (glass or quartz !).

  M.J. Sparnaay (1958): first exp. with two *metal* plates; sensitivity: (0.1 – 1) $\times 10^{-8}\,\mathrm{N}$ and 17 mV.

  Outcome: exponent n in the $1/L^n$ Casimir -force determined as 4 ± 1.

  P.H.G.M. van Blokland & J.T.G. Overbeek (1978): sphere-plate system, $\sim 50\%$ accuracy.

- *Resurgence: (1996 $\rightarrow$):* sphere-plate systems under proximity force theorem: $F^{\circ|}(L) = 2\pi R \frac{-\hbar c \pi^2}{720 L^3}$

  S.K. Lamoreaux, PRL 78 (1997) & PRL 81 (1998): torsion pendulum (60 cm !)

  mech. hysteresis eliminated; piezzoelectrical transducers; $\Delta F = 10^{-11}\,\mathrm{N}$, 10% accuracy.

  U. Mohideen & A. Roy, PRL 81 (1998) and A. Roy et al., PRD 60 (1999):

  atomic force microscope; 32 – 1000 nm, $\Delta F = 10^{-12}\,\mathrm{N}$ and 1-2 % accuracy.

  H.B. Chan, F. Capasso et al., Science 291 (2001): (N.Y. Times, 9-Feb.-2001)

  capacitance bridge; seesaw device; Microelectromech. Systems (MEMS);$L_{min} \approx 70\,\mathrm{nm}$.

  Th. Ederth, PRA 62 (2000): two coated (with hydrocarbon layers) crossed cylinders, 20-100 nm.

  G. Bressi et al., PRL 88 (2002): two 1.2$\times$1.2 mm^2 plates, 0.5– 3 μm, 15 % accuracy.

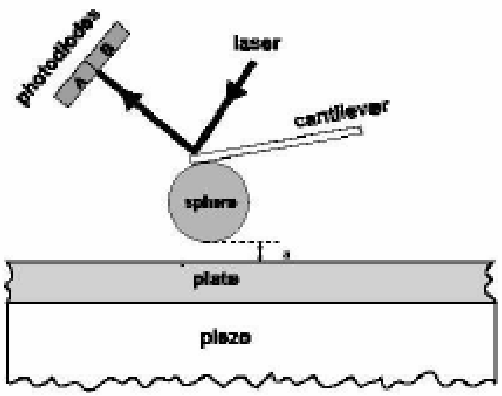
- Ideal case:

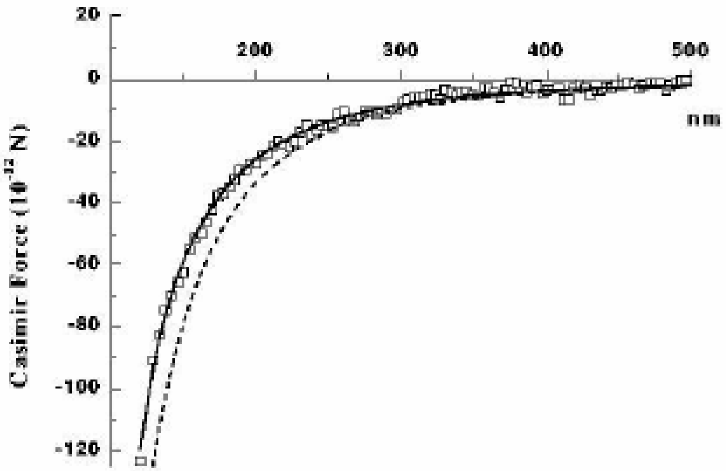
  perfect conductors without impurities at $T = 0$.

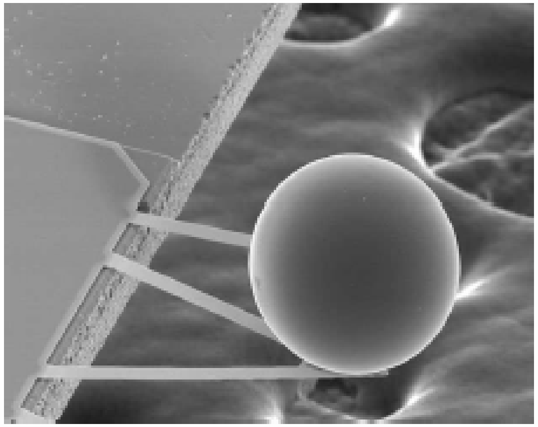
- Reality:

  finite temperature (only important for $L > 1\,\mu\mathrm{m}$); imperfect conductors (realistic metals, Lifshitz theory); surface roughness (geometry averaging, $\langle r^2 \rangle^{1/2}_{\mathrm{s.r.}} \approx$ 3 nm)

Atomic Force Microscope:



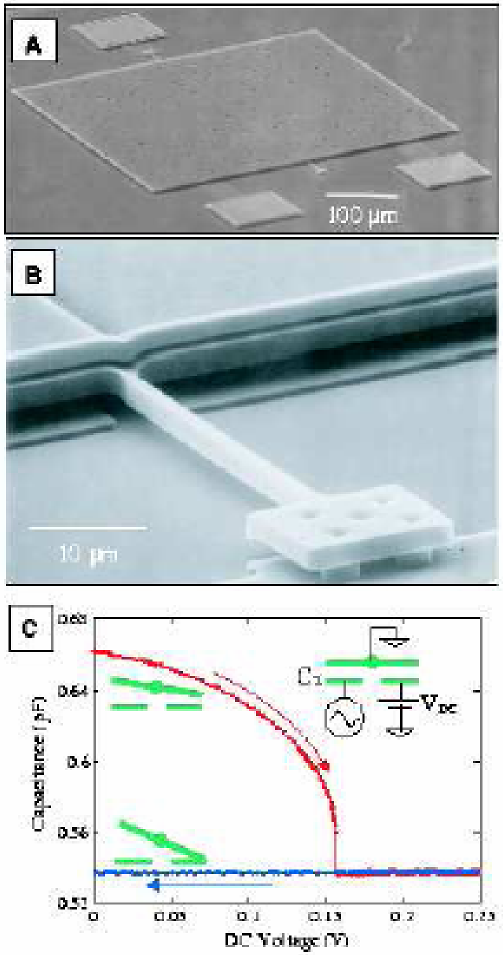


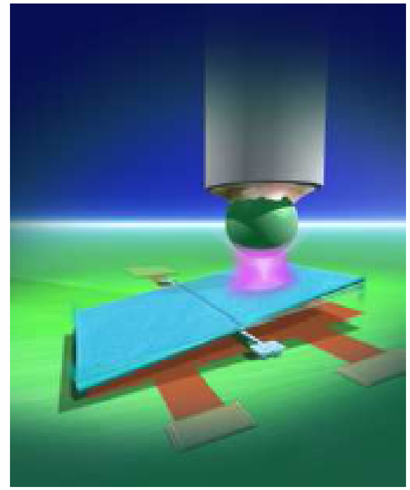


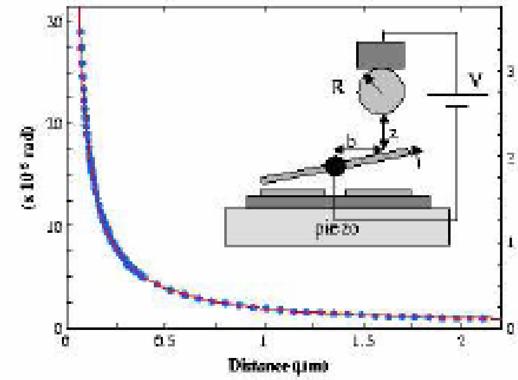
M. Bordag, U. Mohideen, V.M. Mostepanenko, Phys. Rep. 353 (2001)
and U. Mohideen and A. Roy, Phys. Rev. Lett. 81 (1998).

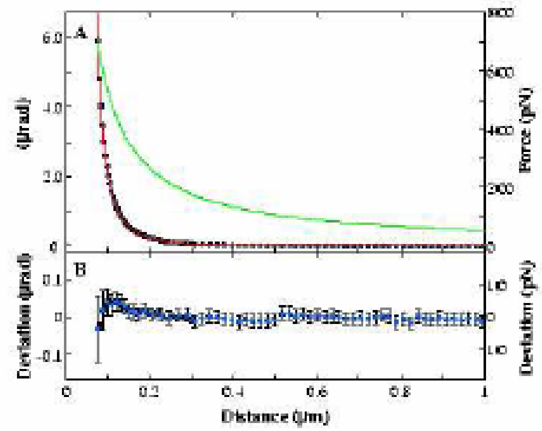
Seesaw MEMS:

see H.B. Chan, V.A. Aksyuk, R.N. Kleiman, D.J. Bishop, F. Capasso, Science 291 (2001); PRL 87 (2001) and Phys. Rev. Focus (Nov. 2, 2001).







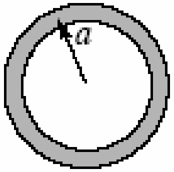


Geometry dependence

- Before 1968: Casimir effect $\leftrightarrow$ van der Waals forces

$$\left\{ \begin{array}{c} \text{2 parallel plates} \\ \text{plate \& sphere} \\ \text{sphere \& sphere} \end{array} \right\} \qquad \text{attractive}$$

- Casimir's idea of stabilizing the electron (1953):



electron as spherical shell of radius a with uniform charge distribution.

Then:

$$\begin{aligned} \mathcal{E}_{em} &= \frac{e^2}{2a} \\ \mathcal{E}_C &\stackrel{?}{=} -C\,\frac{\hbar c}{2a^3}\,4\pi a^2 \end{aligned}$$

$\Rightarrow$ stabilization if $C = \frac{e^2}{4\pi\hbar c}$ (indepently of the radius a).

- However, T. Boyer (1968): force is repulsive!
  (Similarly, R. Balian & C. Bloch (1972-74): Casimir calculations in cavities depend on (bag-)geometry)

$\therefore$ "Generalization" of concept of Casimir energy

"Generalization" of concept of Casimir energy:

1) *geometry dependence*

Casimir energy $\equiv$ vacuum energy from the *geometry-dependent* part of the density of states (d.o.s)

(↔ *shell* correction energy in Nucl. Phys.)

d.o.s.: $$\rho(E) \equiv \sum_{E_k} \delta(E - E_k) = \rho_0(E) + \rho_{\text{bulk}}(E) + \delta\rho_C(E, \textit{geom.-dep.})$$

N.o.s.: $$\mathcal{N}(E) \equiv \sum_{E_k} \Theta(E - E_k) = \int_0^E dE'\, \rho(E')$$

Casimir Energy: $$\mathcal{E}_C \equiv \int dE\, E\, \delta\rho_C(E, \textit{geom.-dep.}) = -\int dE\, \mathcal{N}_C(E, \textit{geom.-dep.})$$

Further "Generalization" of concept of Casimir energy:

2) *matter fields*

- *Here* : space not "filled" with *fluctuating* EM modes, but with gas of *non-interacting* (non-relativistic) fermions.
- *Similarity* : $\exists$ mode sums $\sum\!\!\!\!\!\!\int \hbar\omega_k$ with *constant* degeneracy factor, (because of Pauli's exclusion principle).
- *Difference* : $\exists$ of second scale: fermi energy = *chemical potential* μ (at $T \approx 0$) in addition to geometric size and distance scale(s).
- *Concrete* : Matter fields (fermions) in the space between voids build up a quantum pressure on the voids

$\exists$ effective *interaction* between empty regions of space in the background of *non-interacting* fermions

Applications

In the inner crust of a neutron star:

- nuclei start to loose neutrons due to increasing pressure and density:
  below saturation density $\Rightarrow$ chain of phases between 0.03 fm^{-3} and 0.1 fm^{-3}:

  nuclei $\longrightarrow$ rods $\longrightarrow$ slabs $\longrightarrow$ tubes $\longrightarrow$ bubbles $\longrightarrow$ uniform matter

  - Liquid drop model: *meat-ball*, *spaghetti*, *lasagne* vacua from interplay between Coulomb and surface energies with phase differences of a few keV/fm^3.
    - However, neglected shell correction energy of the same order !
      (see A. Bulgac & P. Magierski, *Nucl. Phys.* **A 683** (2001) 695.)

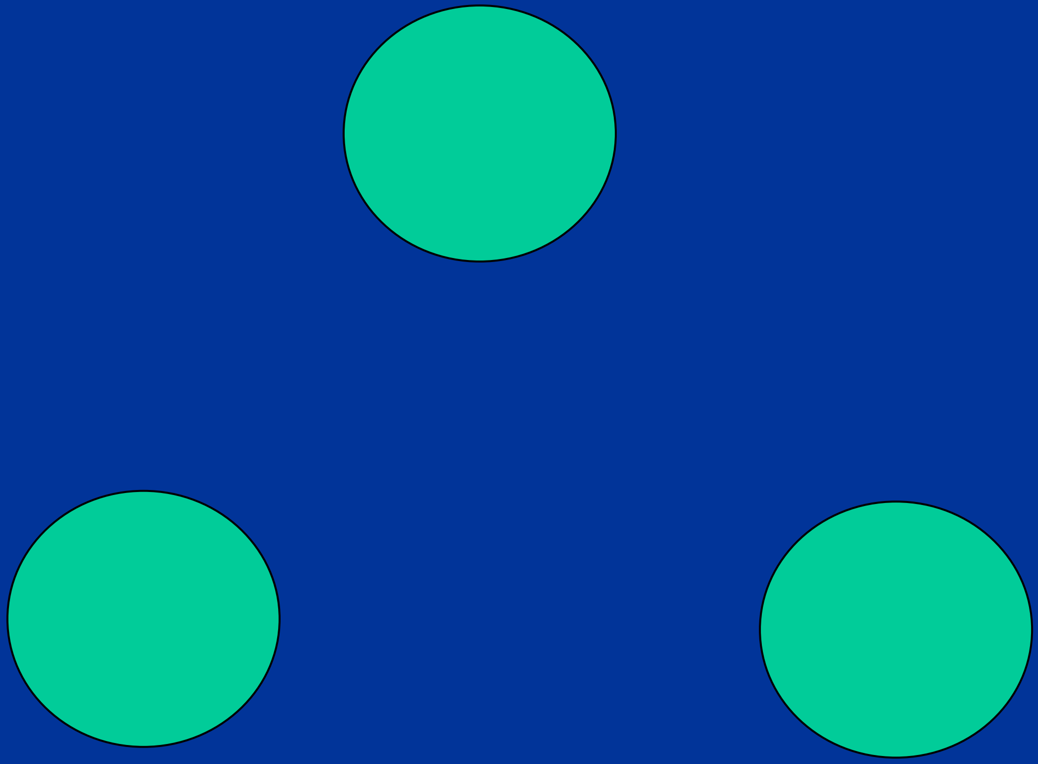
Inside a neutron/quark star:

- Quark matter droplets immersed in hadronic matter at $\rho \gg \rho_{nm}$

In the lab:

- Buckyballs immersed in an electron gas (in liquid mercury)
- or Buckyballs immersed in liquid ^{3}He.
- (Bosonic) cavities in dilute atomic Fermi condensates.

Let me now go back to my starting theme and consider spherical inhomogeneities in an otherwise featureless (normal) Fermi sea



Calculation

- Casimir energy for fermions between two impenetrable (parallel) planes at distance L:

$$\mathcal{E}_C = \mu\, F(k_F L) \quad \text{with} \quad \mu = \hbar^2 k_F^2/2m \text{ as natural scale .}$$

- For more complicated geometries $\longrightarrow$ involved computations !

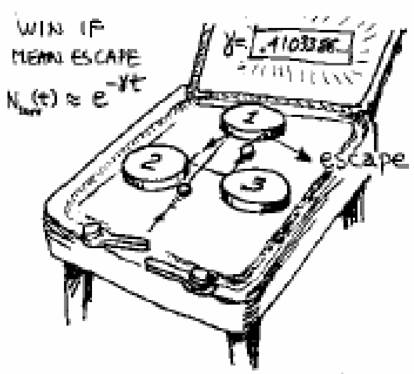
- However, case of immersed non-overlapping *spherical* voids still relatively simple:

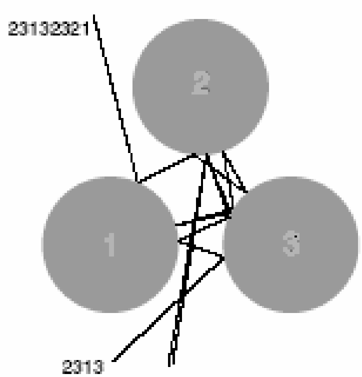
Krein (1953) trace-formula : level density var. $\delta\rho(E) \leftrightarrow \frac{d}{dE}$ of phase shift $\frac{1}{2i}\ln\det S_n(E)$

$$\boxed{\delta\rho(E) = \bar\rho(E) - \bar\rho_0(E) = \frac{1}{2\pi i}\,\frac{d}{dE}\mathrm{tr}\ln S_n(E)}$$ of the n-sphere S-matrix

Extract *geometry-dependent* Casimir fluctuations from *multiple*-scattering part

$\Rightarrow$ Calculation mapped onto a quantum mechanical *billiard* problem:
hyperbolic point-particle scattering off *n* spheres or *n* disks



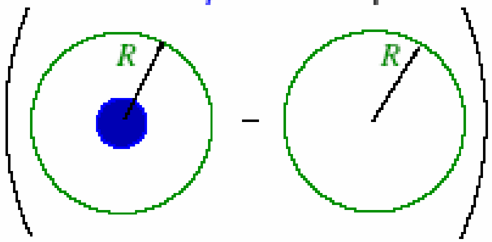


References:
B. Eckhardt, *J. Phys.* **A20** (1987);
P. Gaspard & S. Rice, *J. Chem. Phys.* **90** (1989);
P. Cvitanović & B. Eckhardt, *Phys. Rev. Lett.* **63** (1989);
M. Henseler, A. Wirzba & T. Guhr, *Ann. Phys.* **258** (1997).

Digression 1: *E.Beth & G.E. Uhlenbeck (1937)*

predecessor of *Krein (1962)* formula for *spherical* potentials:

Idea: *spherical boxes:* $\lim_{R\to\infty}$



Asymptotically:

$$u_{k\ell}(r) \sim \sin\left(kr - \tfrac{1}{2}\,\ell\pi + \eta_\ell(k)\right)$$
$$u^{(0)}_{k\ell}(r) \sim \sin\left(kr - \tfrac{1}{2}\ell\pi\right)$$

and Dirichlet b.c.'s: $u_{k\ell}(R) = u^{(0)}_{k\ell}(R) = 0.$

$\Rightarrow$ EV conditions ($(2\ell$+1)-fold degenerate) :

$$k_{\ell,n}R - \tfrac{1}{2}\ell\pi + \eta_\ell(k_{\ell,n}) = \pi n, \quad {\scriptstyle n=0,1,2,\cdots} \quad \text{(with potential)}$$
$$k^{(0)}_{\ell,n}R - \tfrac{1}{2}\ell\pi = \pi n, \quad {\scriptstyle n=0,1,2,\cdots} \quad \text{(without potential)}$$

A change of n by one unit, for ℓ fixed, leads to

$$\Delta k_\ell\left(R + \frac{\partial}{\partial k}\eta_\ell(k)\right) = \pi = \Delta k^{(0)}_\ell\, R\,,$$

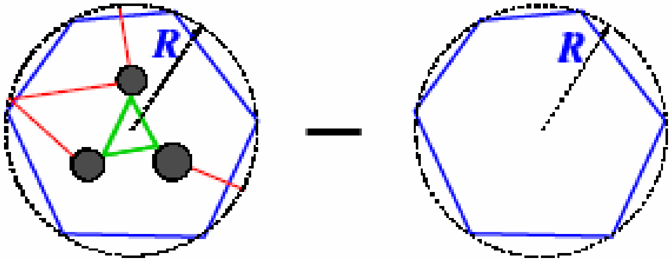
such that the conditions $\bar\rho_\ell(k)\Delta k_\ell = \bar\rho^{(0)}_\ell(k)\Delta k^{(0)}_\ell = 2\ell + 1$ (note the averaging !)

induce the formula $\boxed{\bar\rho_\ell(k) - \bar\rho^{(0)}_\ell(k) = \frac{2\ell+1}{\pi}\,\frac{\partial}{\partial k}\eta_\ell(k)}$

Digression 2: *Semiclassical interpretation of Krein formula*

Determinant of scattering matrix is semiclassically a sum over periodic orbits (+Weyl terms)

Consider the difference of the densities of states of two *bounded reference systems* :



- The *container-induced* periodic orbits cancel!
- However, ∃ further *spurious periodic orbits* whose lengths grow with increasing R.
- Removal of long orbits by exponential damping or averaging:

$$\lim_{\epsilon\to 0_+} \lim_{R\to\infty} \left\{ \rho^{(n)}(k+i\epsilon;R) - \rho^{(0)}(k+i\epsilon;R) \right\}$$

$$= \frac{1}{2\pi} \mathrm{Im} \frac{\mathrm{d}}{\mathrm{d}k} \ln \det \mathbf{S}^{(n)}(k) \Big|_{k\,\mathrm{real}}$$

Note the order of the limits!

Digression 3: Casimir energy in fermionic background

chem. potential $\mu = \hbar^2 k_F^2/2m$ or Fermi momentum k_F *natural UV-cutoff* ($\Lambda_{UV} = \mu$ or $k_{UV} = k_F$)

Casimir energy at *fixed* total number of fermions $\mathcal{N}$:

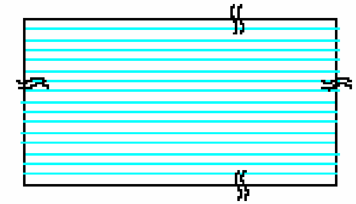
$\Rightarrow$ chem. potentials μ & μ_0 at *finite* and *infinite* separation of the scatterers differ:

$$\begin{aligned}
\mathcal{N} &= \int_0^{\mu} \rho(E)\,dE = \int_0^{\mu_0} [\rho_0(E) + \rho_W(E)]\,dE \\
\Rightarrow \quad & \underline{\int_{\mu_0}^{\mu} \rho(E)\,dE} = -\int_0^{\mu_0} \underbrace{[\rho(E) - \rho_0(E) - \rho_W(E)]}_{\rho_C(E) = \frac{d}{dE}\mathcal{N}_C(E)}\,dE \\
\mathcal{E}_C|_{\mathcal{N}} &= \int_0^{\mu} E\,\rho(E)\,dE - \int_0^{\mu_0} E[\rho_0(E) + \rho_W(E)]\,dE \\
&= \int_0^{\mu_0} E\,\rho_C(E)\,dE + \mu_0 \underline{\int_{\mu_0}^{\mu} \rho(E)\,dE} + \mathcal{O}(V^{-1}) \\
&= \int_0^{\mu_0} (E - \mu_0)\,\rho_C(E)\,dE = -\int_0^{\mu_0} \mathcal{N}_C(E)dE\,.
\end{aligned}$$

Grandcanonical Casimir energy at *fixed* $\mu = \mu_0$:

$$\widetilde{\mathcal{E}_C}|_{\mu} - \mu\mathcal{N}_C(\mu) = \mathcal{E}_C|_{\mathcal{N}} + \mathcal{O}(V^{-1})$$

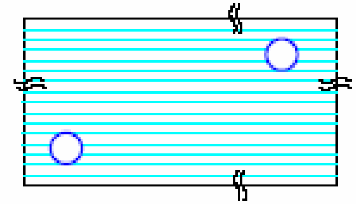
1. *"infinite" container:* $\rho(E) = \rho_0(E)$ (fermionic background)



2. n bubbles (of radii a_i) "punched out" at "infinite" separation:

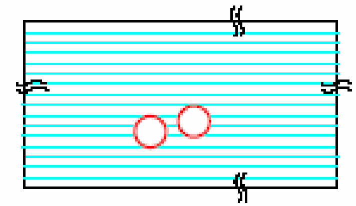
$$\rho(E) = \rho_0(E) + \sum_{i=1}^{n} \underbrace{\rho_W(E, a_i)}_{\textit{Weyl term}}$$

(note the excluded volume !)



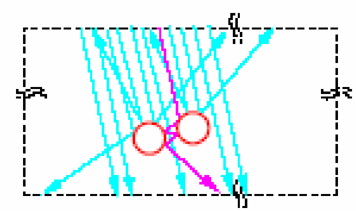
3. *geometry*-dependent arrangement of n bubbles:

$$\rho(E) = \rho_0(E) + \sum_{i=1}^{n} \rho_W(E, a_i) + \delta\rho_C(E, \{a_i\}, \{\mathbf{r}_{ij}\})$$



4. Krein formula (note the averaging):

$$\delta\rho(E) \quad = \quad \bar\rho(E) - \bar\rho_0(E) = \frac{1}{2\pi i}\frac{d}{dE}\overbrace{\ln\det S_n(E)}^{2i\eta_n(E)}$$



5. *Multiple scattering matrix*

$$\overbrace{\det S_n(E)} = \prod_i \det S_1(E, a_i)\,\frac{\det M^\dagger(k^*)}{\det M(k)}$$

$$\hookrightarrow \quad \delta\bar\rho_C(E, \{a_i\}, \{\mathbf{r}_{ij}\}) = -\frac{1}{\pi}\mathrm{Im}\left(\frac{d}{dE}\ln\det M(E)\right)$$

(see A.W., *Phys. Rep.* **309** (1999))

6. Casimir energy:

$$\therefore\ \mathcal{E}_C = \int_0^\mu dE\,(E-\mu)\,\delta\bar\rho_C = -\int_0^\mu dE\,\overline{N}_C$$

Multi-scattering matrix for n spheres of radii a_j and distances $r_{jj'}$ $(j, j'=1, 2, \ldots n)$

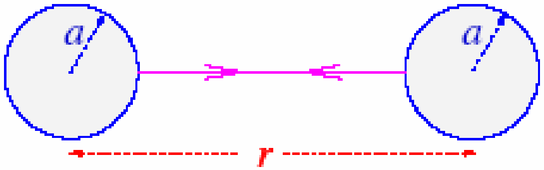
$$M^{jj'}_{lm,l'm'} = \delta^{jj'}\delta_{ll'}\delta_{mm'} + (1-\delta^{jj'})\, i^{2m+l'-l}\sqrt{4\pi(2l+1)(2l'+1)}\left(\frac{a_j}{a_{j'}}\right)^2 \frac{j_l(ka_j)}{h^{(1)}_{l'}(ka_{j'})}$$
$$\times \sum_{\tilde{l}=0}^{\infty}\sum_{\tilde{m}=-l'}^{l'} \sqrt{2\tilde{l}+1}\, i^{\tilde{l}} \begin{pmatrix} \tilde{l} & l' & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{l} & l' & l \\ m-\tilde{m} & \tilde{m} & -m \end{pmatrix} D^{l'}_{m',\tilde{m}}(j,j')\, h^{(1)}_{\tilde{l}}(kr_{jj'})\, Y^{m-\tilde{m}}_{\tilde{l}}(\hat{r}^{(j)}_{jj'})$$

M. Henseler, A. Wirzba & T. Guhr, *Ann. Phys.* **258** (1997) 286.

In the limit of small scatterers:

$$M^{jj'}(E) \approx \delta^{jj'} - (1-\delta^{jj'})\, \underbrace{f_j(E)}_{s\text{-wave}} \frac{\exp(ikr_{jj'})}{r_{jj'}} \quad (+\ p\text{-wave})$$

Two spheres of radius a at distance r



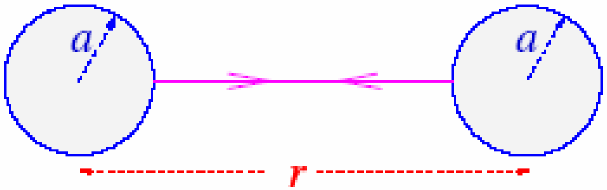
in the *small-scatterer* limit:

$$\mathcal{N}^{oo}_C(E) = -\frac{1}{\pi}\mathrm{Im}\ln\det M^{oo}(E) \approx \nu_{\rm deg}\frac{a^2}{\pi r^2}\sin[2(r-a)k] + \mathcal{O}\left((ka)^3\right),$$

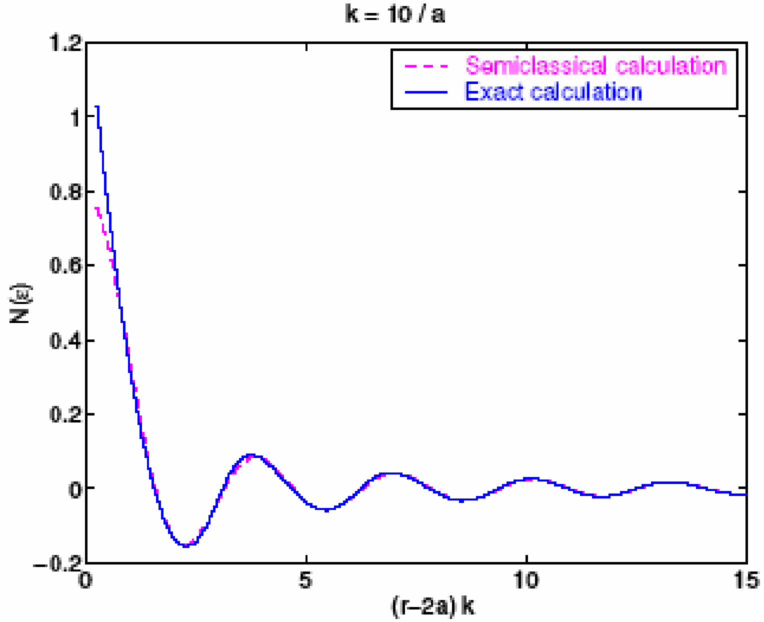
whereas the *semiclassical* result (for the single *two-bounce periodic orbit* with no repetions) reads:

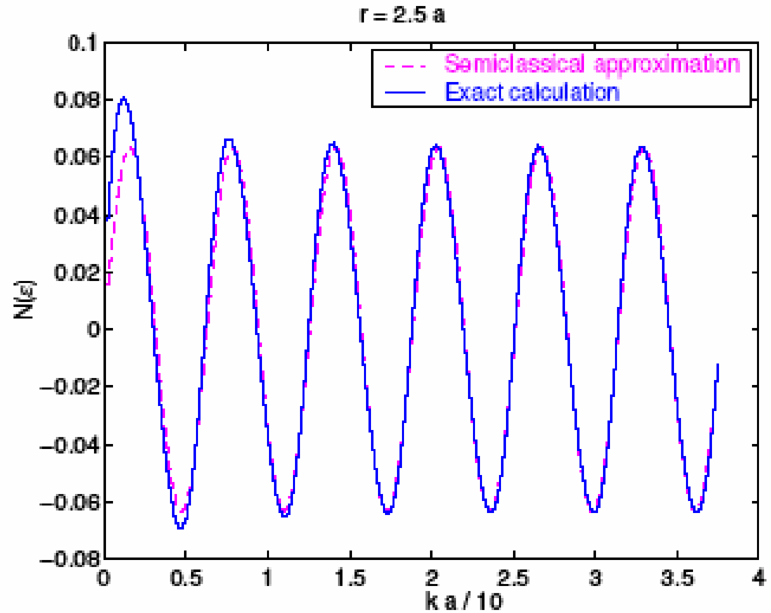
$$\mathcal{N}^{oo}_{C,scl}(E) = \nu_{\rm deg}\frac{a^2}{4\pi r(r-2a)}\sin[\underbrace{2(r-2a)k}_{S_{po}(k)/\hbar}] \qquad \text{(Gutzwiller's \underline{trace} formula)}$$

Two-sphere case:



N.o.S $\mathcal{N}_C^{oo}(E) = -\frac{1}{\pi}\mathrm{Im}\ln\det M^{oo}(E) \approx \nu_{\deg}\frac{a^2}{4\pi r(r-2a)}\sin[2(r-2a)k]$





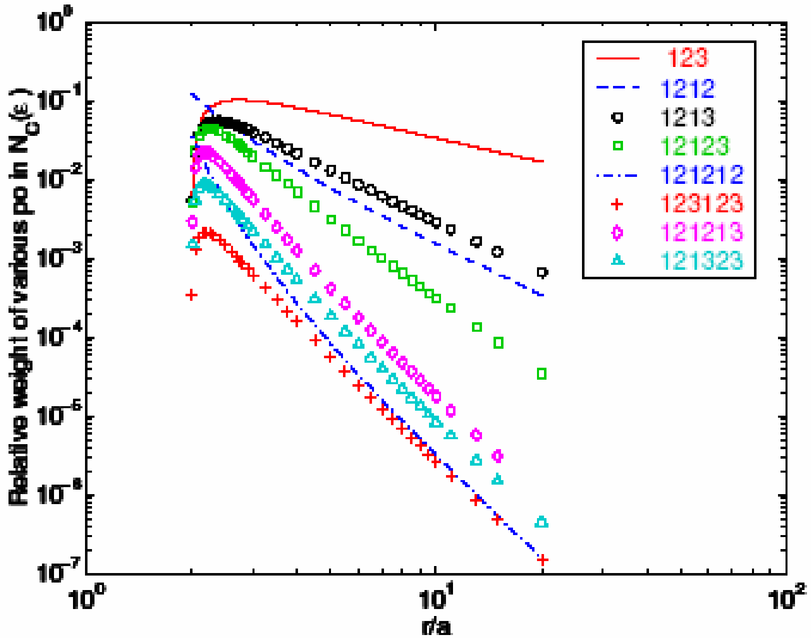
$$\mathcal{E}_C^{oo} = -\int_0^{\mu(k_F)} dE\, \mathcal{N}_C^{oo}(E) \approx -\nu_{\deg}\,\mu\frac{a^2}{2\pi r(r-2a)} j_1[2(r-2a)k_F], \qquad (k_F a > 1)$$

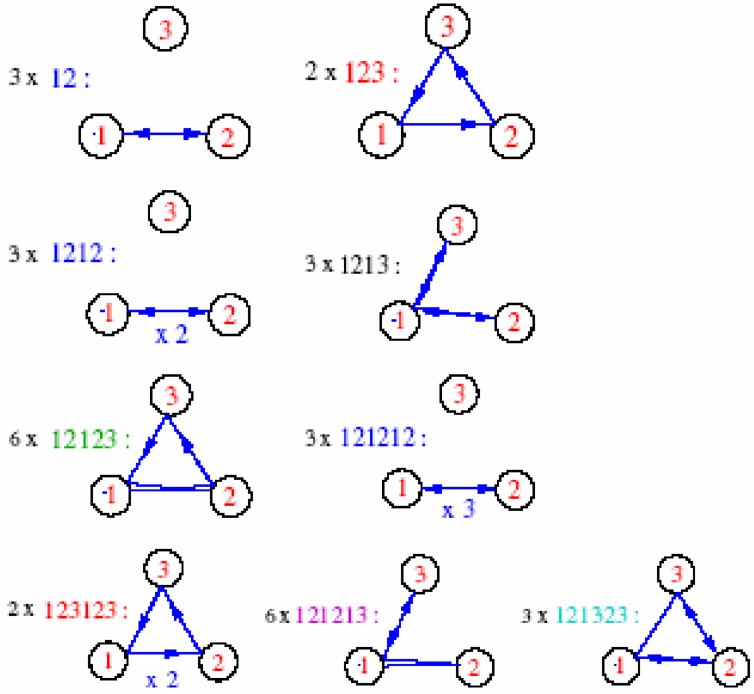
(of longer range than molecular van der Waals forces !)

Sphere-plate geometry: $\mathcal{E}_C^{o|} \approx -\nu_{\deg}\,\mu\frac{a}{2\pi(r-a)} j_1[2(r-a)k_F], \qquad (k_F a > 1)$

Three and four spheres:

- *periodic orbit* summation: ∃ of genuine *three* and *more-body* interactions
- However, *2-bounce orbit* dominates in equilateral three- and four sphere systems
  (max. correction due to 3-bounce orbit is $\sim 10\,\%$ at $r \approx 2.5a$)





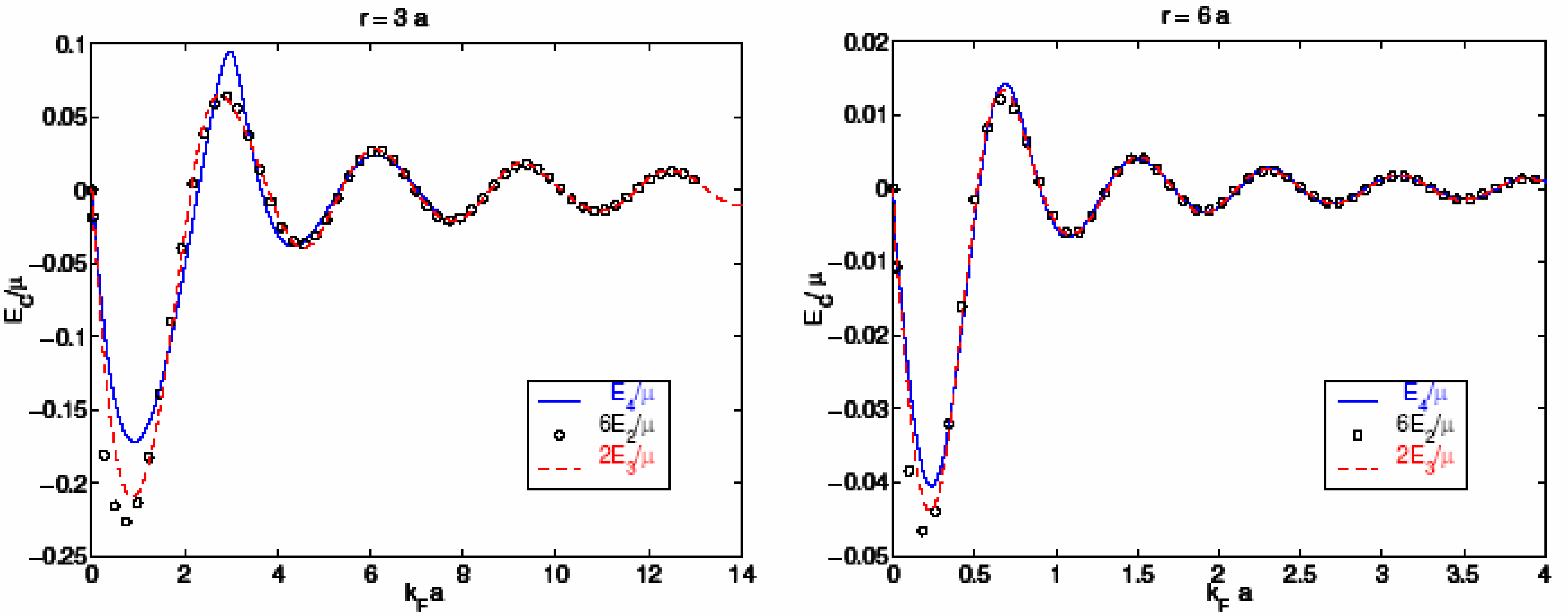
- *Billiard analogy* : difficult to make *long* shots, especially with many bounces
  – the slightest error ruins the shot.

Dominance of two-body interactions:

the Casimir energy satisfies the rule $\boxed{E_3 \approx 3E_2}$ for 3 identical spheres (*equilateral triangle*)

and $\boxed{E_4 \approx 6E_2 \approx 2E_3}$ for 4 identical spheres (*symmetric tetrahedron*) if $k_F a \gg 1$.

($k_F a \leq 1$: corrections up to 10 % and 25 % for 3 & 4 spheres)

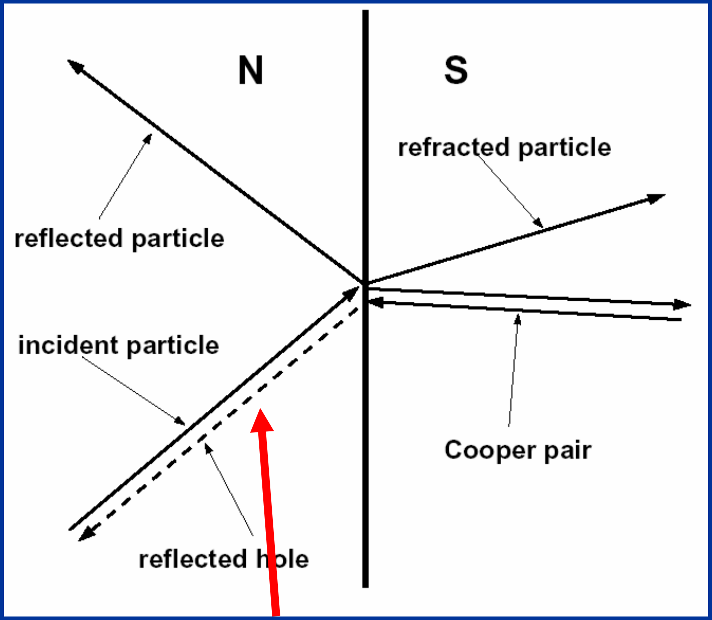


Let us now consider the role of fermionic superfluidity

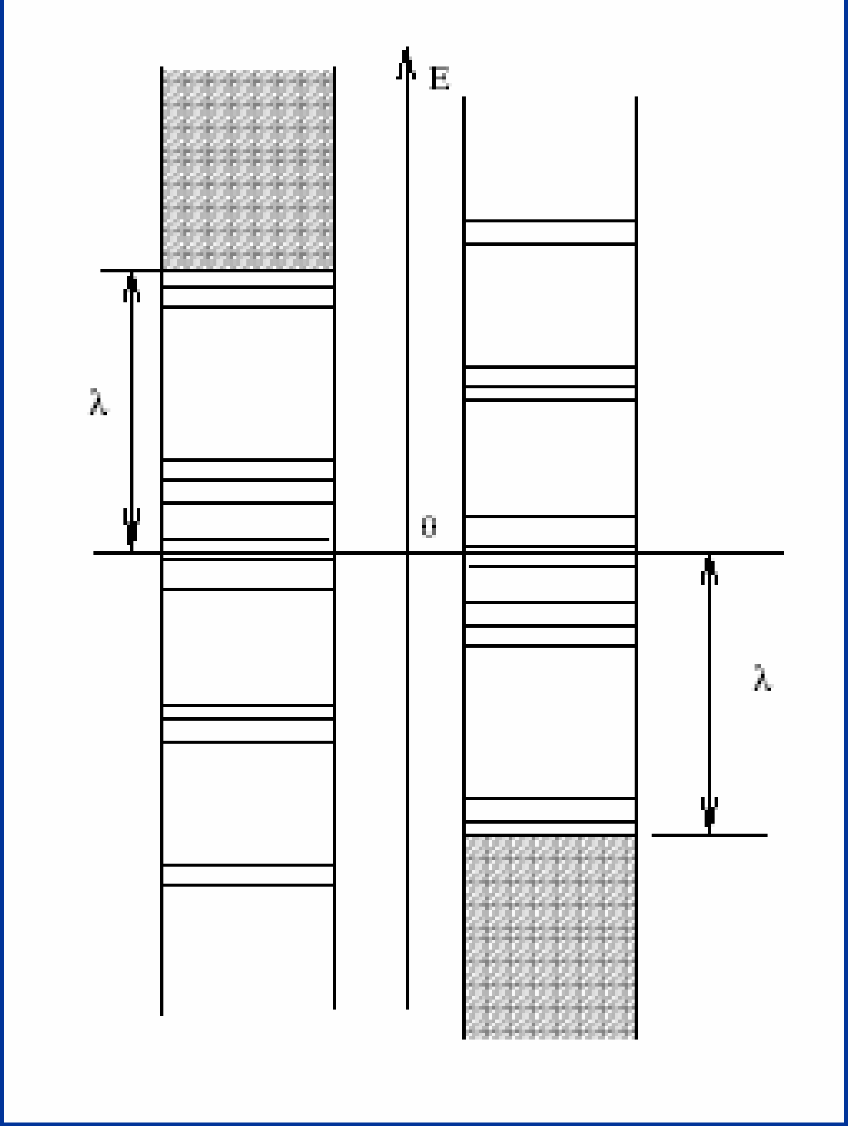
What happens at the boundary of a normal and superfluid regions?

$$\begin{pmatrix} h-\lambda & \Delta \\ \Delta^{+} & -(h^{*}-\lambda) \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

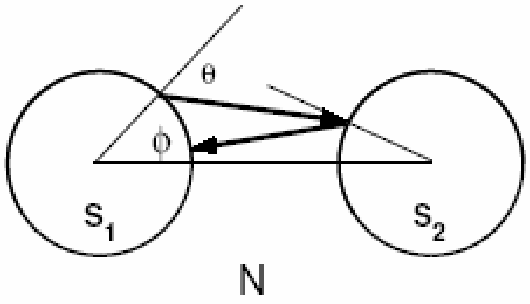
outside inside



Andreev reflection



Fermionic Casimir Effect of two superfluid grains in a normal Fermi gas



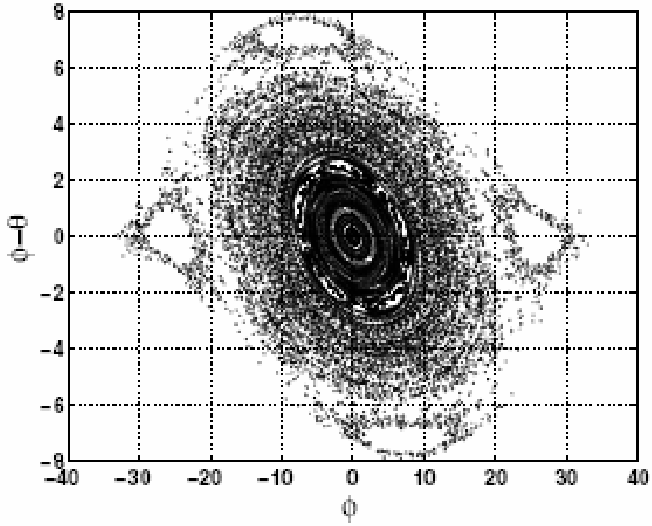
Specular reflection replaced by *Andreev reflection*

$\Rightarrow$ particle retro-reflected as a hole

$k_p^2/2 + k_h^2/2 = k_F^2$ (particle/hole states symmetric)

$k_p \sin\theta_p = k_h \sin\theta_h$ (modified Snell's law),

$\Rightarrow$ disks/spheres have focusing effect



stable orbits if $\frac{R}{2a} \leq \frac{k_p}{k_p - k_h}$ satisfied

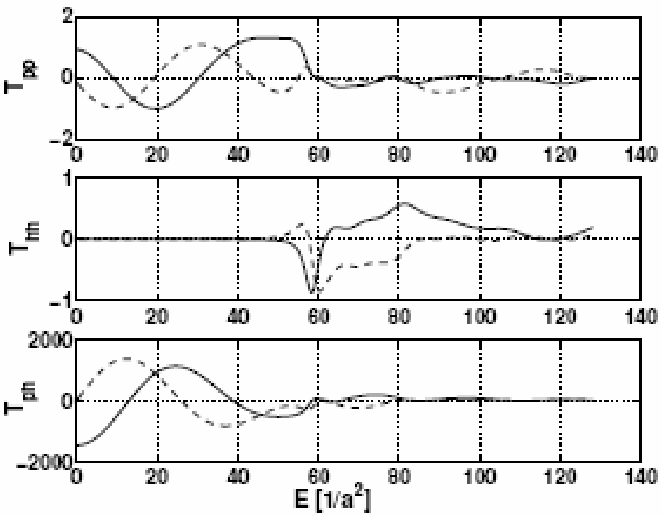
(here $R = 6a$ and $k_p/k_h = 1.5$)

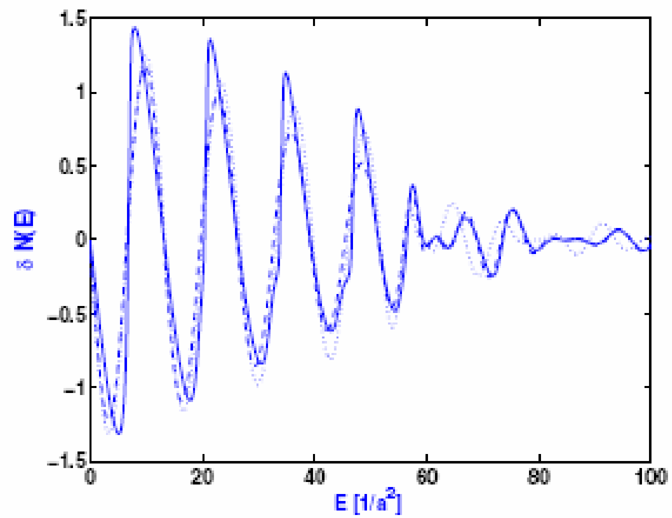
Apply

1. Bogoliubov-de Gennes equations with pairing field $\Delta_{1,2}(\vec{r}) = \Delta e^{\pm i\phi_\Delta/2}$ at circular grain 1 and 2,
2. mod. Krein formula $\delta N(E) = -\frac{1}{\pi}\ln\det M(E)$ with $M(E)$ multi-scattering matrix in p-h space,
3. asymptotic approximation valid for large separations $R \gg a$

$$\delta N(E) \approx \frac{4|T_{ph}(a,\mu,\Delta)|\cos\phi_\Delta}{\pi^2\sqrt{k_h k_p}R}\sin[(k_p - k_h)R + \phi_{ph}] - \frac{2|T_{pp}(a,\mu,\Delta)|}{\pi^2 k_p R}\cos(2k_p R + \phi_{pp}) + \frac{2|T_{hh}(a,\mu,\Delta)|}{\pi^2 k_h R}\cos(2k_h R - \phi_{hh})$$

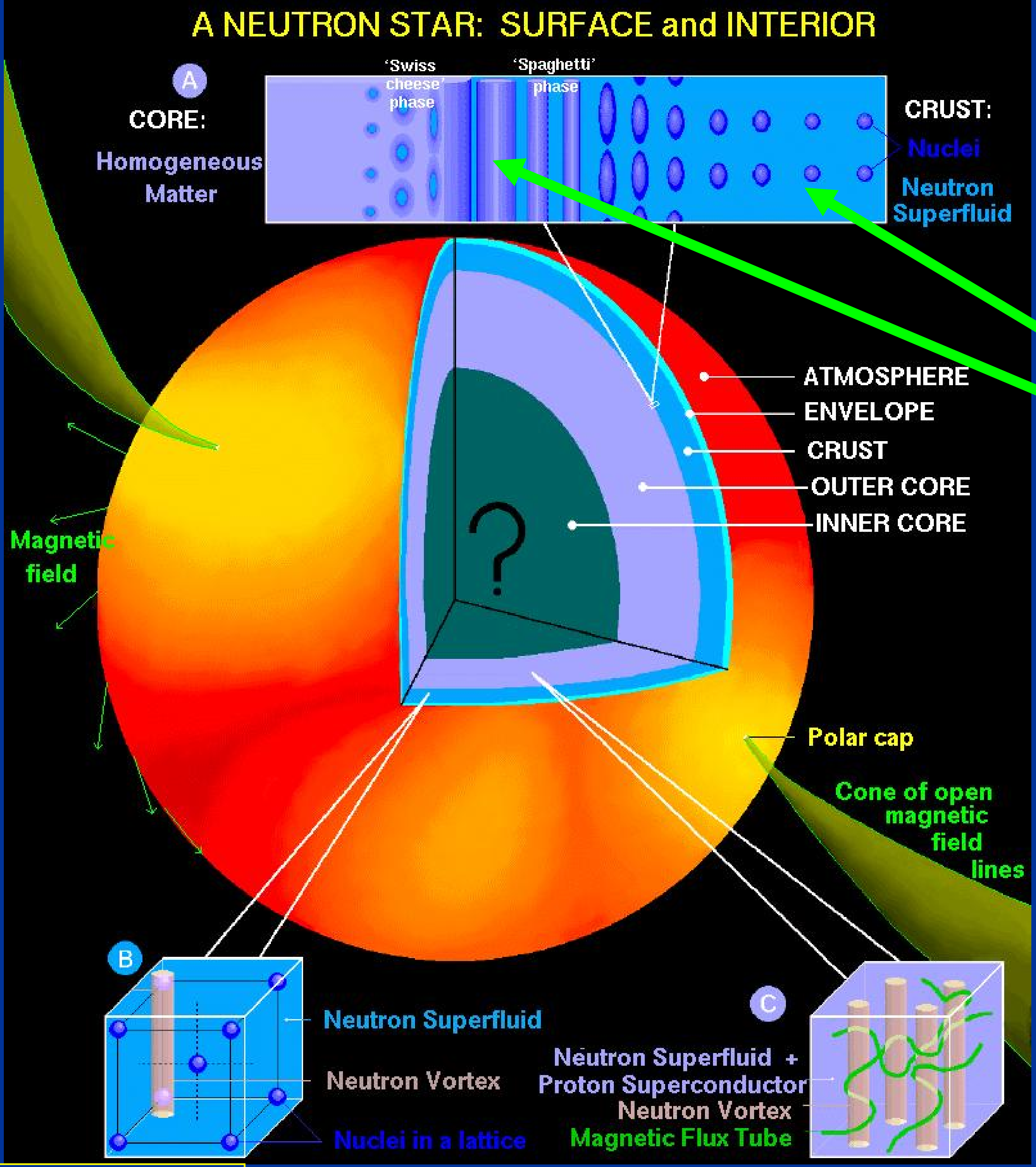
$T_{ph} = \left(\sum_m t_{ph}^m/2\right)^2$, $T_{pp} = \left(\sum_m (-1)^m t_{pp}^m/2\right)^2$ etc. structure functions (only dep. on single scatterer)





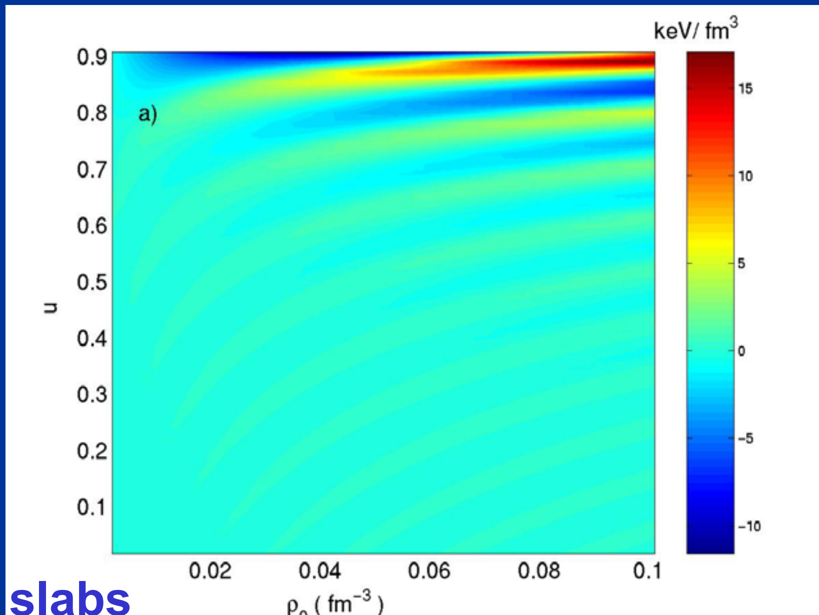
$k_p^2/2m - \mu \equiv E \leq \Delta$ *dominant*

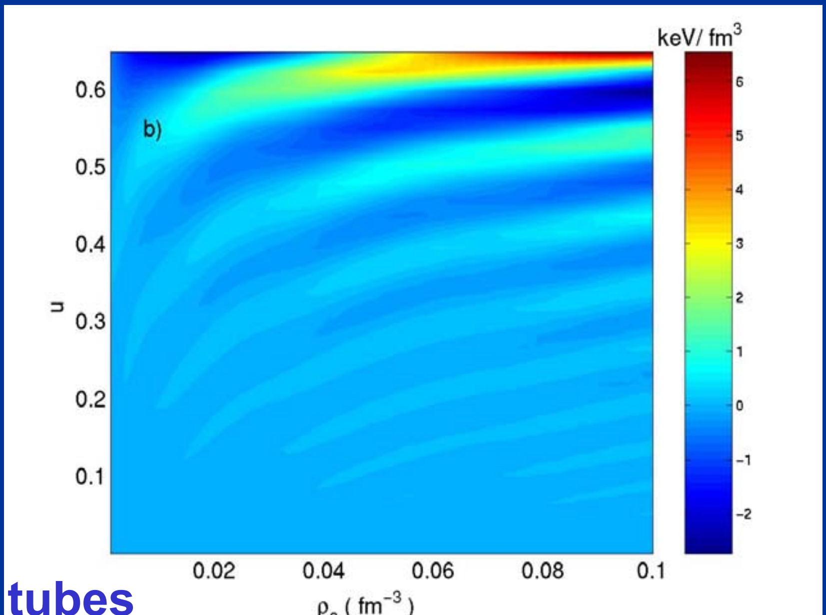
($\mu = 200/a^2$,
gap $\Delta = 50/a^2$,
rel. gap phase ϕ_Δ = 0,
$R = 6a$)

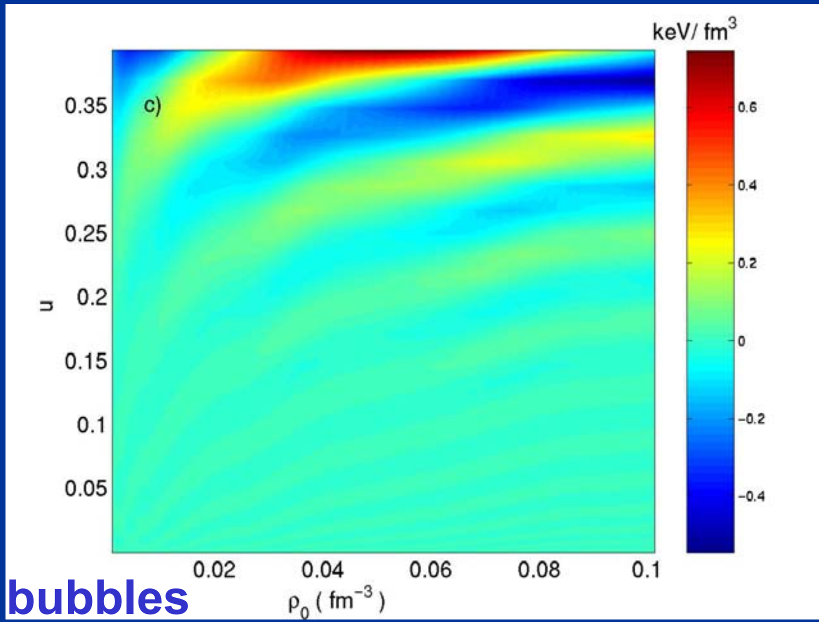


Author Danny Page

Quantum Corrections to the GS Energy of Inhomogeneous NM



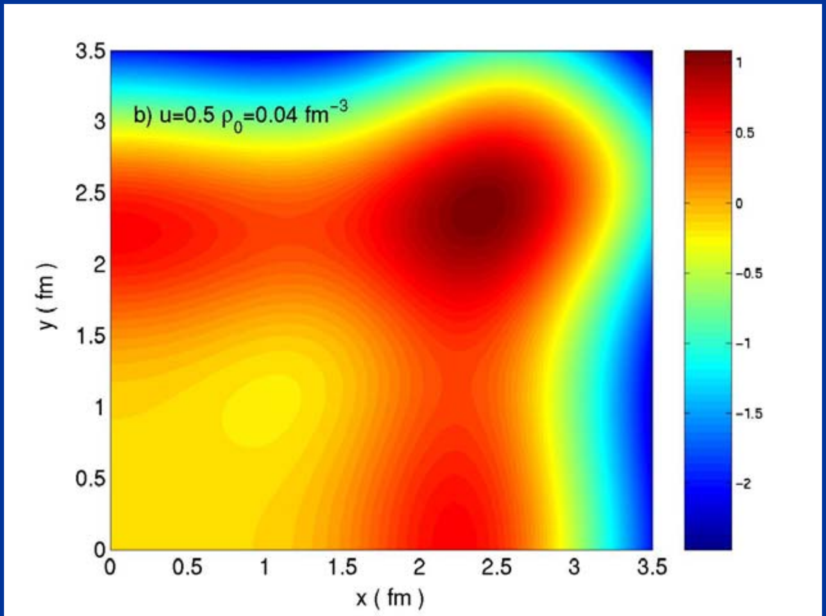




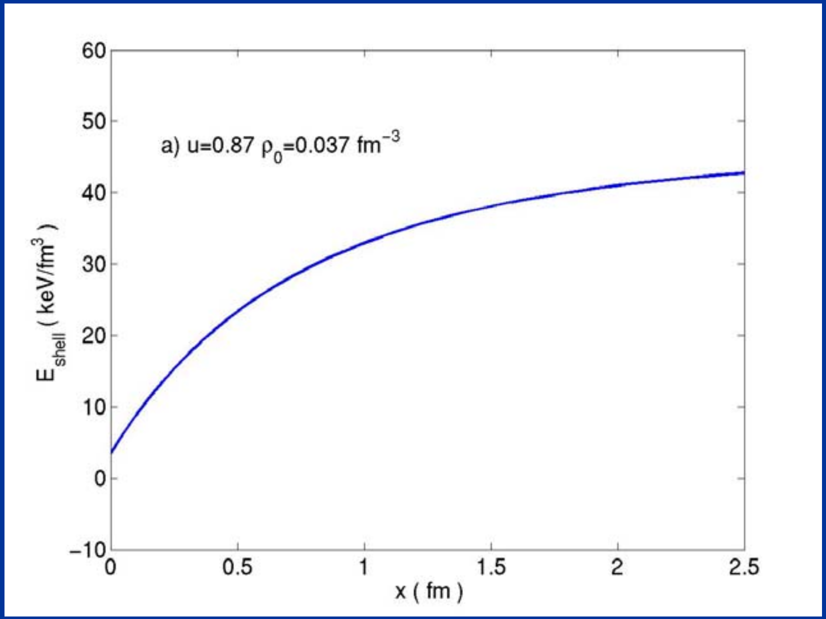
The Casimir energy for various phases.
The lattice constants are:
L = 23, 25 and 28 fm respectively.
u – anti-filling factor
ρ_0 – average density

A. Bulgac and P. Magierski
Nucl. Phys. 683, 695 (2001)
Nucl. Phys, 703, 892 (2002) (E)

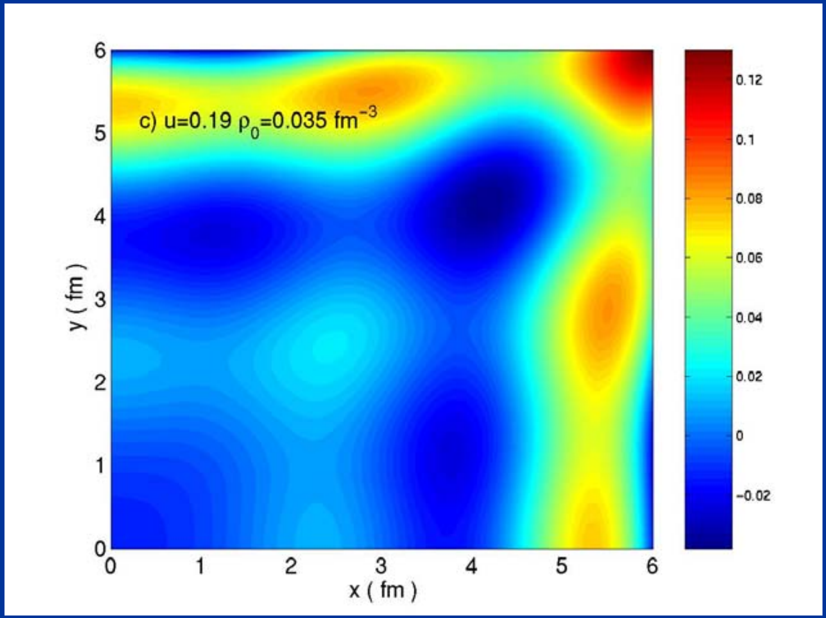
The Casimir energy for the displacement of a single void in the lattice



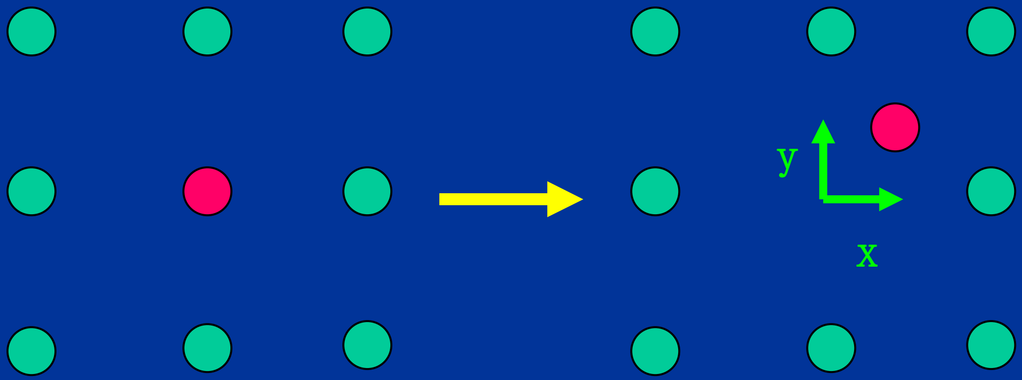
Rod phase



Slab phase

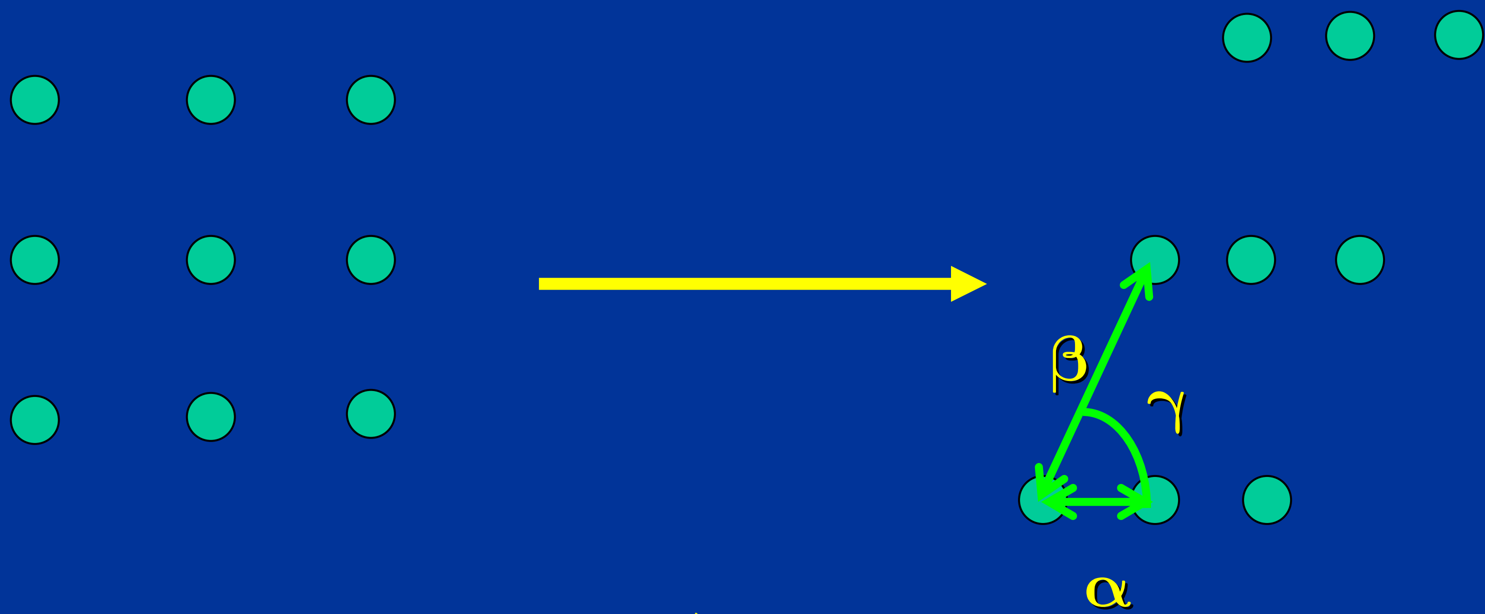


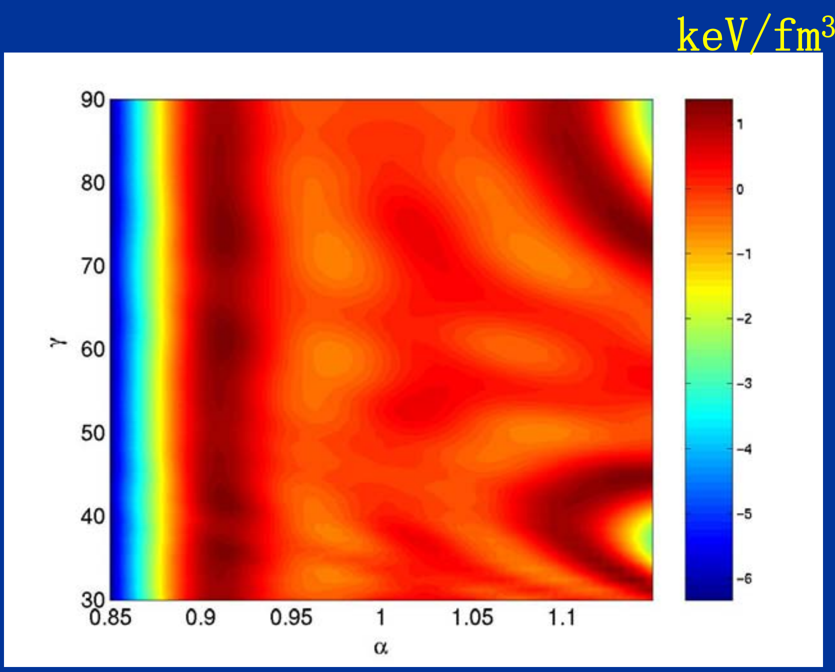
Bubble phase



A. Bulgac and P. Magierski
Nucl. Phys. 683, 695 (2001)
Nucl. Phys. 703, 892 (2002) (E)

Deformation of the rod-like phase lattice





$$\alpha\beta\sin\gamma = 1$$

volume conservation

A. Bulgac and P. Magierski
Nucl. Phys. 683, 695 (2001)
Nucl. Phys, 703, 892 (2002) (E)

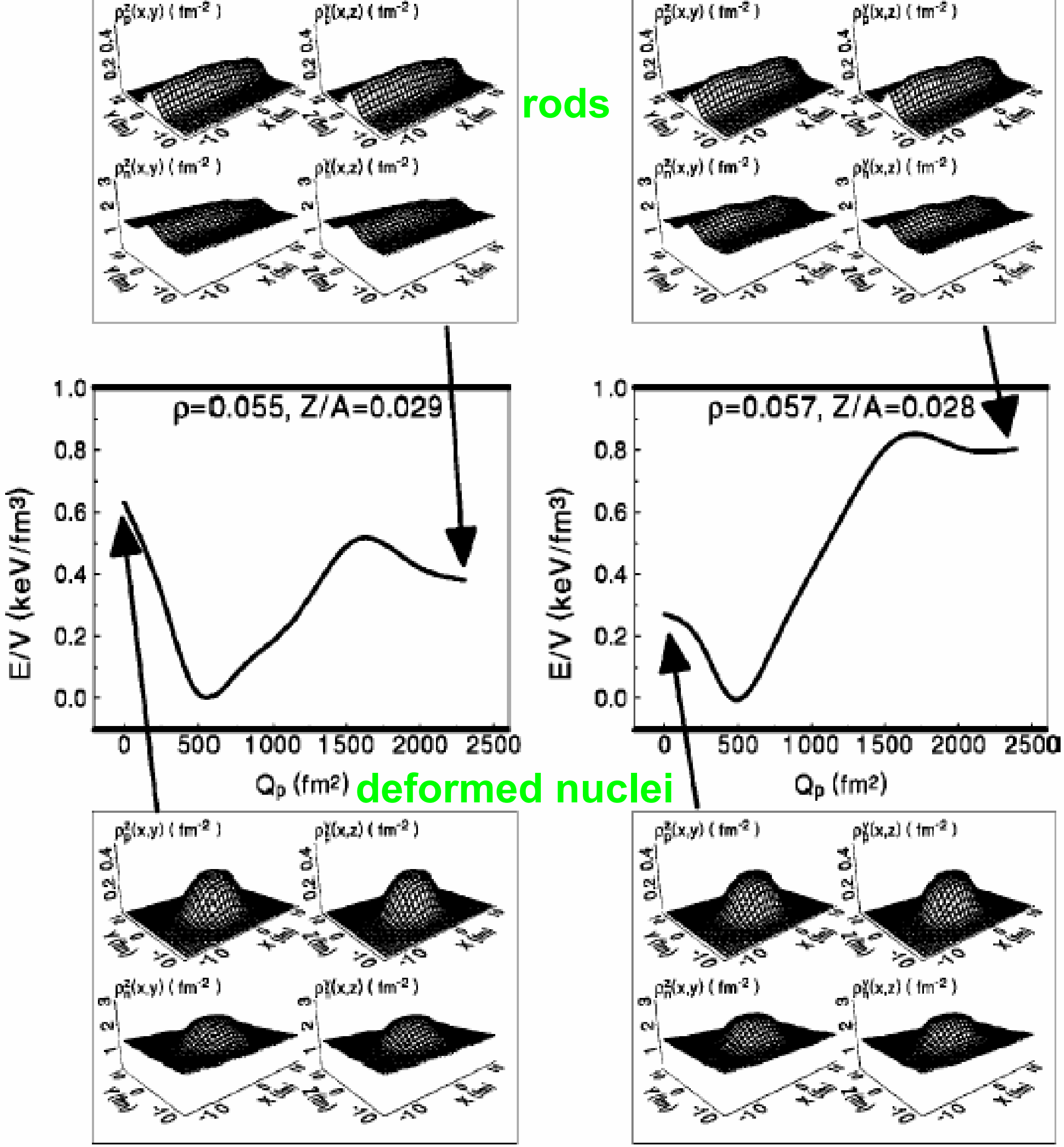


FIG. 2. The total energy density (1) of *npe* matter as a function of the proton quadrupole moment $Q_p = Q_{20}^p$ (middle subfigures). The integrated proton and neutron densities (see text for definition) corresponding to nuclear configurations indicated by arrows are shown in the lower and upper subfigures.

Skyrme HF with SLy4, Magierski and Heenen, Phys. Rev. C 65, 045804 (2002)

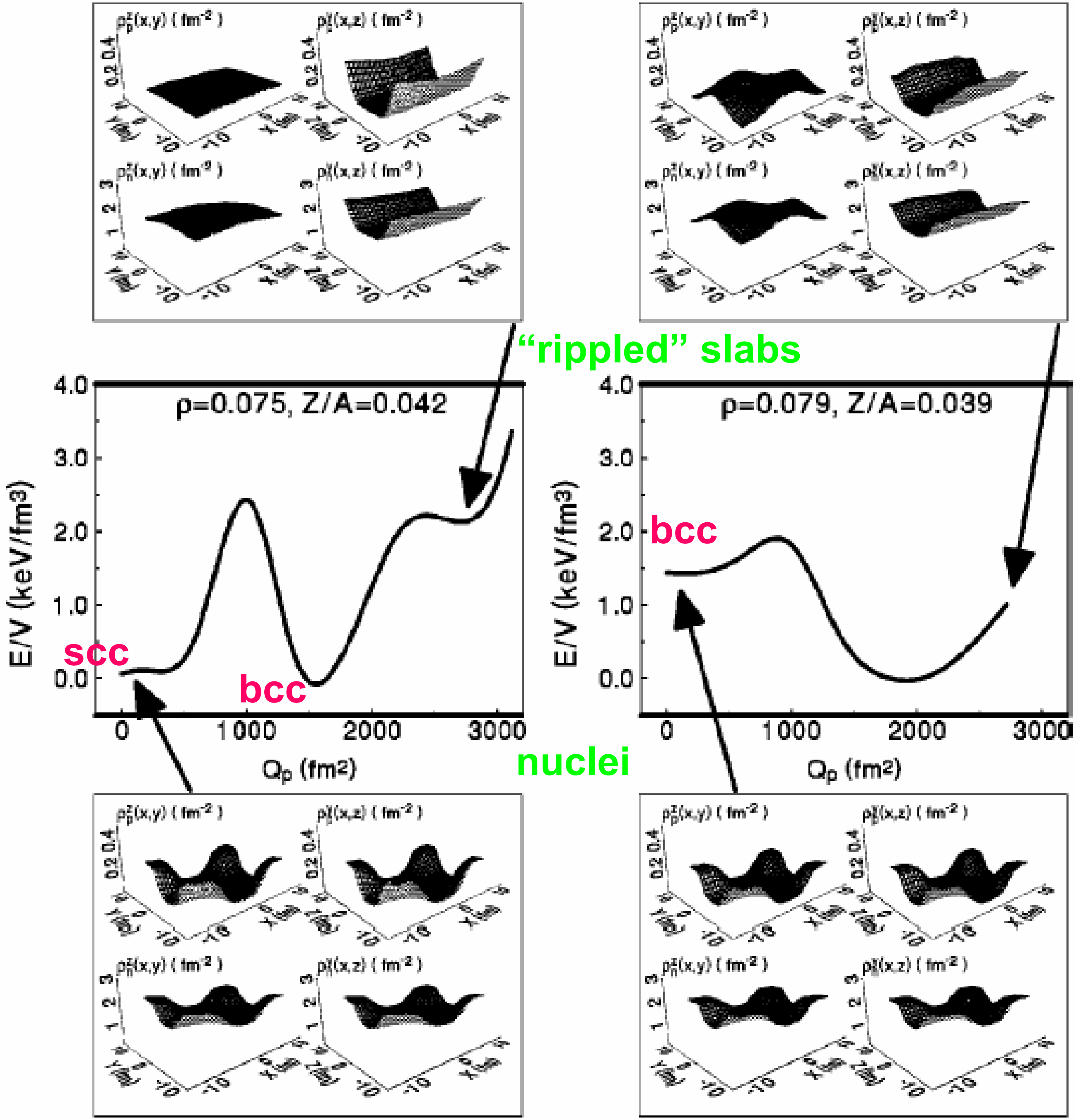


FIG. 3. The same as in the Fig. 2 but for different densities.

Skyrme HF with SLy4, Magierski and Heenen, Phys. Rev. C 65, 045804 (2002)

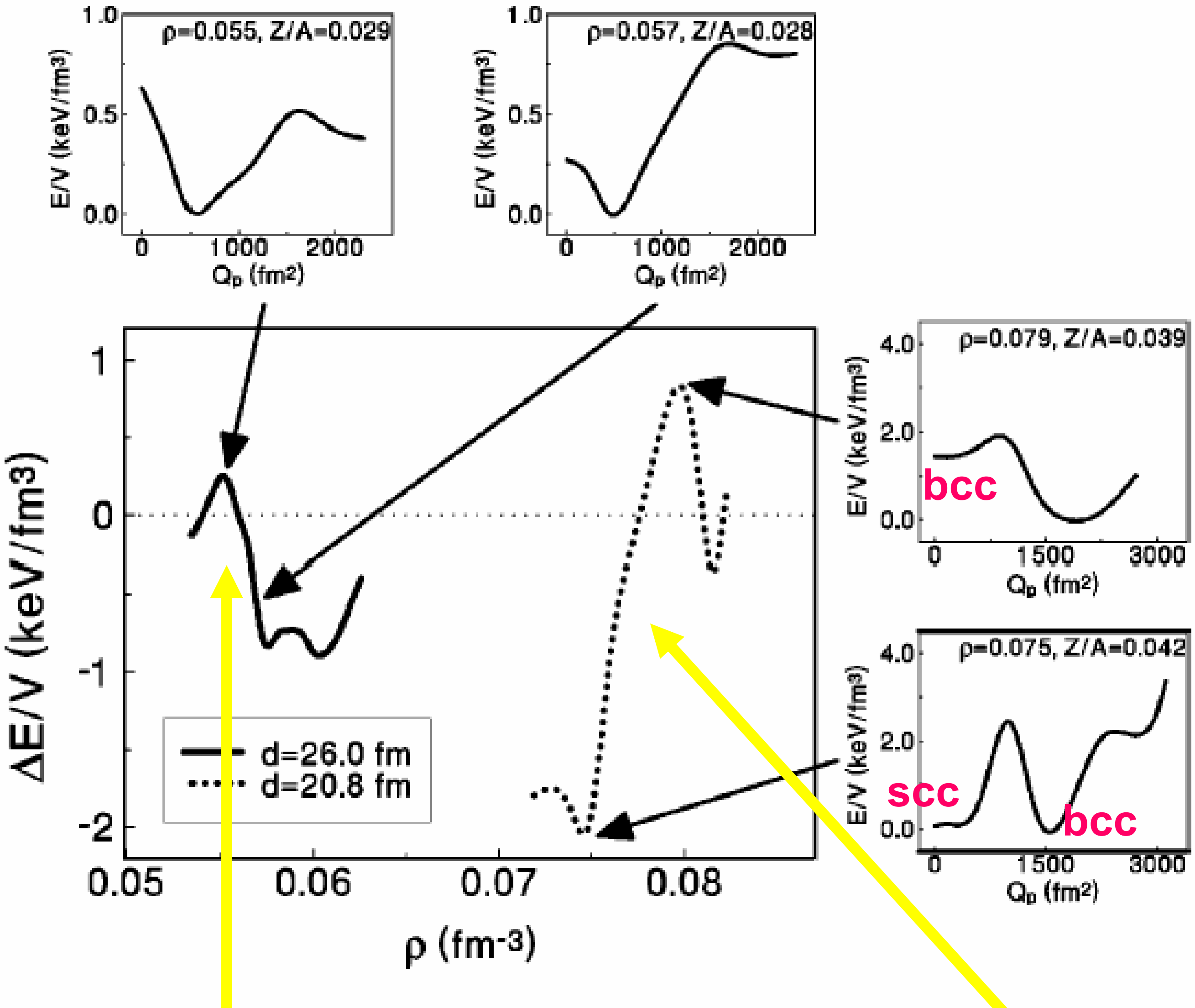
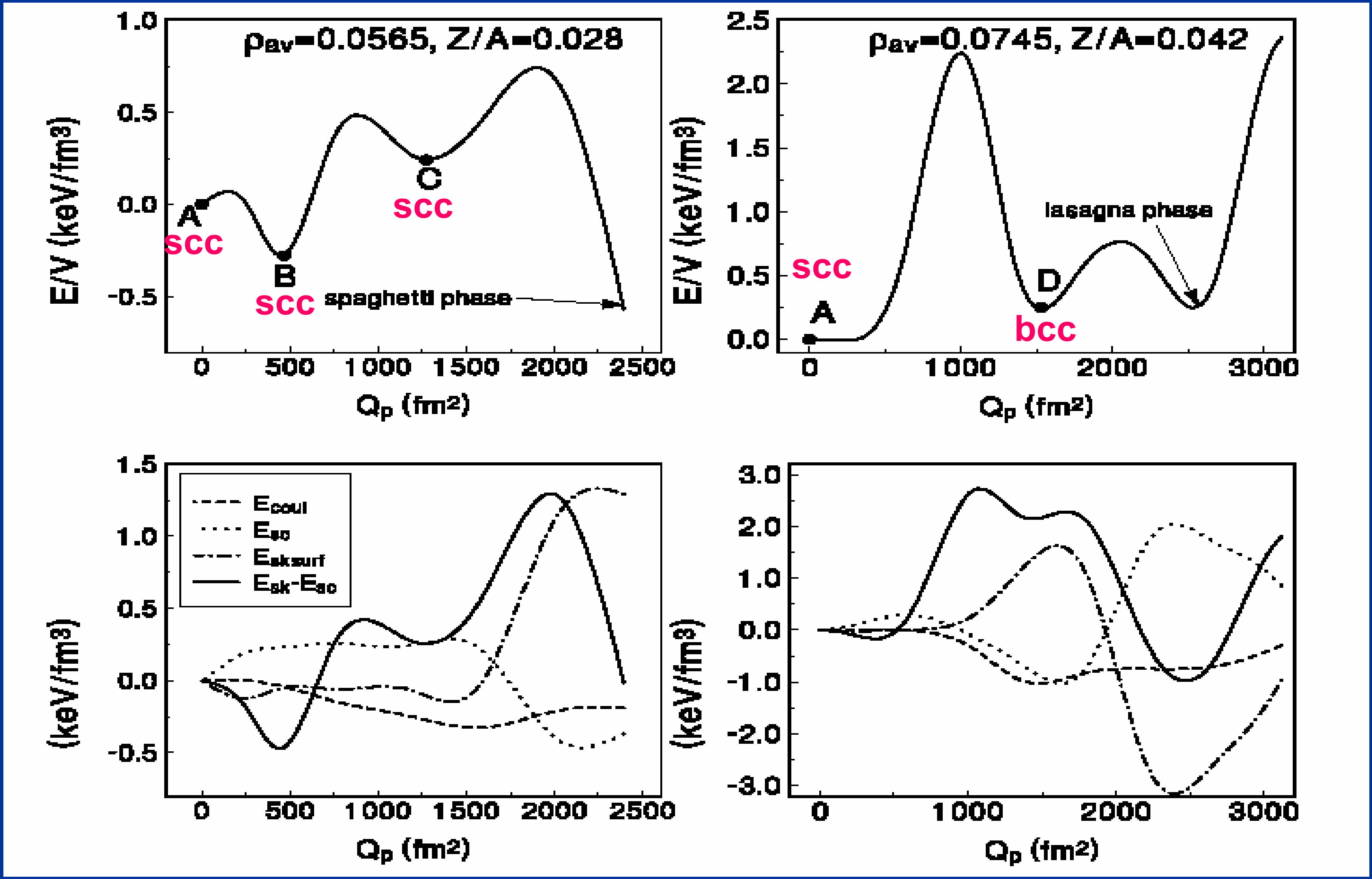


FIG. 4. The energy density difference $\Delta E/V$ between nuclear phases as a function of the total density. Solid curve denotes the difference between the spherical and rodlike phase. Dotted curve denotes the difference between the spherical and slablike phase. Smaller subfigures show the energy density of *npe* matter as a function of the proton quadrupole moment for four different densities. Parameter d denotes the length of the cubic box.

ΔE between spherical and rod-like phases

ΔE between spherical and slab-like phases

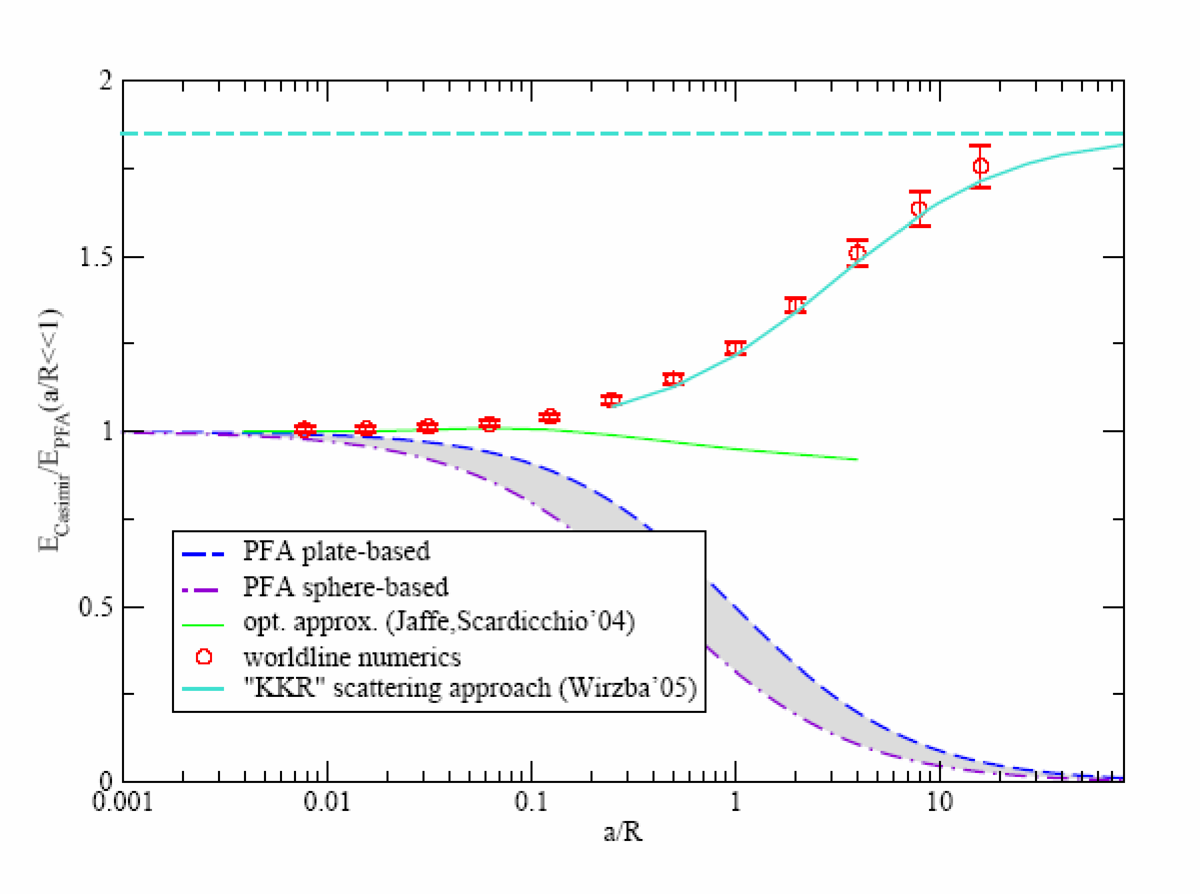
Skyrme HF with SLy4, Magierski and Heenen, Phys. Rev. C 65, 045804 (2002)



Various contributions to the energy density as a function
of the proton quadrupole moment.

Magierski, Bulgac and Heenen, Nucl.Phys. A719, 217c (2003)

Let us change gears and consider the scalar Casimir effect in quantum field theory between a sphere and a plane



Note: asymptotically ($L/a \gg 1$) s-wave scattering dominates:

$$\mathcal{E}(L) \sim -\frac{\pi^3 \hbar c\, a}{1440 L^2} \frac{90}{\pi^4} \frac{2}{(1+a/L)(1+a/2L)} \to -\frac{\pi^3 \hbar c\, a}{1440 L^2} \frac{90}{\pi^4} \times 2 = -\frac{\pi^3 \hbar c\, a}{1440 L^2} \times 1.847\cdots$$

whereas

$$\mathcal{E}_{\rm p-wave}(L) \sim -\frac{\pi^3 \hbar c\, a^3}{1440 L^4} \frac{90}{\pi^4} \frac{1}{(1+a/L)(1+a/2L)^3}$$

Scalar Casmir effect of a Dirichlet sphere and a plate

Holger Gies, K. Langfeld & L. Moyaerts, JHEP **0306**, 018 (2003)
R.L. Jaffe, Antonello Scardicchio, PRL **92**, 070402 (2004)
Antonello Scardicchio, R.L. Jaffe, NPB **704**, 552 (2005)

Conclusions

A qualitatively new form of Casimir energy

In normal Fermi systems the Casimir energy could be either attractive or repulsive, depending on the relative arrangement of the scatterers

Semiclassical arguments show that one can disentangle two-body (typically dominant) and many-body interactions

In superfluid Fermi systems Andreev reflection leads to a very strong enhancement of the Fermionic Casimir interaction, which seems to be predominantly repulsive

Expected to show up in a variety of systems, neutron stars, various condensed matter systems, cosmology etc.

The methods developed by us proved extremely useful in calculating the familiar Casimir interaction in a more transparent fashion and easy manner