Fermi sea \( \lambda_F \) (non-interacting particles)
Fermi sea  $\lambda_F$
Fermi sea $\lambda_F$
Fermi sea $\lambda_F$
Fermi sea $\lambda_F$
Question:
What is the most favorable arrangement of these two spheres?

Answer: The energy of the system does not depend on $r$ as long as $r > 2a$. NB Assuming that a liquid drop model for the fermions is accurate!

This is a very strange answer! Isn’t it? Something is amiss here.

Fermi sea $\lambda_F$
Let us try to think of this situation now in quantum mechanical terms.

The dark blue region is really full of de Broglie’s waves, which is in the absence of homogeneities are simple plane waves.

When inhomogeneities are present, there are a lot of scattered waves.

Also, there are some almost stationary waves, which reflect back and forth from the two tips of the empty spheres.

As in the case of a musical instrument, in the absence of damping, the stable “musical notes” correspond to stationary modes.

Problems: 1) There is a large number of such modes.
2) The tip-to-tip modes cannot be absolutely stable, as the reflected wave disperses in the rest of the space.
Fermionic Casimir effect for normal and superfluid systems

A force from nothing onto nothing

Aurel Bulgac (Seattle)
Piotr Magierski (Warsaw)
Andreas Wirzba (Juelich)
H.B.G. Casimir (1948): two parallel uncharged metallic plates attract each other in vacuum

\[ \frac{F^{\parallel}(L)}{A} = -\frac{\hbar c}{L^4} \frac{\pi^2}{240} \approx -1.3 \times 10^{-7} \frac{1}{L^4} N \mu m^4/cm^2 \]

\[ \varepsilon^{\parallel}(L) = -\frac{\hbar c}{L^3} \frac{\pi^2}{720} A \]

Origin: zero-point fluctuations of e.m. field modified by the addition of the two plates relative to free case

⇒ change in the energy of the vacuum: \[ \sum \hbar \omega_k |_{\text{plates}(L)} - \sum \hbar \omega_k |_{\text{free}} \]

Casimir effect: *Mesoscopic manifestation of quantum fluctuations of the vacuum*

- Related to van der Waals forces (however, the latter always attractive!)
- Experimental confirmation in the last decade (for the sphere-plate system!)
  - S. Lamoureux, *Phys. Rev. Lett.* 78 (1997);
Casimir effect and vacuum energy in QFT:

- Infinite zero-point energy can be subtracted (discarded) by suitable redefinition of energy-origin (e.g. normal ordering)
- However, energy-origin can be re-defined only once (and only in homogeneous space (no b.c.'s) and without gravity)

\[ \varepsilon_C = \lim_{V \to \infty} \frac{\varepsilon_C}{V} \quad (V : \text{quantization box}) \]

\[ = \lim_{\Lambda \to \infty} \lim_{V \to \infty} (-1)^{2S} \left( \sum_{k,\nu_{\text{deg}}} \frac{1}{2} \hbar \omega_k |_{V,\Lambda,C} - \sum_{k,\nu_{\text{deg}}} \frac{1}{2} \hbar \omega_k |_{V,\Lambda,0} \right) \]

Constraints by

\[ \begin{cases} \text{gravity} & (\text{e.g. non-flat metric}) \\ \text{external fields} & (\text{e.g. anomalies}) \\ \text{internal fields} & (\text{e.g. QCD-vacuum}) \\ \text{geom.-dep. b.c.'s} & (2 \text{ plates, 2 half-spheres, } \ldots) \end{cases} \]
Gravity serious: “cosmological constant problem!”

The critical mass density \( \rho_{\text{crit.}} = \frac{3H_0^2}{8\pi G_N} \approx h_0^2 \times 10.5 \text{ GeV m}^{-3} = h_0^2 \times 4.8 \times 10^{-47} \text{ GeV}^4 \)

compared with vacuum energy density \( \rho_V = \frac{\Lambda_{\text{cosm.}}}{8\pi G_N} + \varepsilon \)

where \( \varepsilon = \frac{\varepsilon_0}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar \omega_k e^{-\hbar \omega_k/\Lambda_{\text{UV}}} \approx \frac{3\hbar c}{2\pi^2} \left( \frac{\Lambda_{\text{UV}}}{\hbar c} \right)^4 \)

<table>
<thead>
<tr>
<th>Theory</th>
<th>( \Lambda_{\text{UV}} )</th>
<th>Scale [GeV]</th>
<th>( \varepsilon = \frac{1}{2} \int \hbar \omega )</th>
<th>( \varepsilon / \rho_{\text{crit.}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ToE</td>
<td>( M_P )</td>
<td>1.2 \times 10^{19}</td>
<td>10^{76} GeV^4</td>
<td>10^{123}</td>
</tr>
<tr>
<td>GUT</td>
<td>String</td>
<td>5 \times 10^{17}</td>
<td>10^{72} GeV^4</td>
<td>10^{120}</td>
</tr>
<tr>
<td>SU(5)SUSY</td>
<td>2 \times 10^{16}</td>
<td>10^{65} GeV^4</td>
<td>10^{112}</td>
<td></td>
</tr>
<tr>
<td>SU(5)\text{min.}</td>
<td>10^{15}</td>
<td>10^{60} GeV^4</td>
<td>10^{107}</td>
<td></td>
</tr>
<tr>
<td>SUSY break</td>
<td>( \Lambda_{\text{had.}} )</td>
<td>10^{3}</td>
<td>10^{12} GeV^4</td>
<td>10^{59}</td>
</tr>
<tr>
<td>EW</td>
<td>Higgs vacuum</td>
<td>2.5 \times 10^{2}</td>
<td>10^{9} GeV^4</td>
<td>10^{56}</td>
</tr>
<tr>
<td></td>
<td>Z, W</td>
<td>10^{2}</td>
<td>10^{8} GeV^4</td>
<td>10^{55}</td>
</tr>
<tr>
<td>QCD</td>
<td>( \Lambda_{QCD} )</td>
<td>0.2</td>
<td>10^{-3} GeV^4</td>
<td>10^{44}</td>
</tr>
<tr>
<td>EM</td>
<td>\hbar \omega_{\text{phot.}}</td>
<td>1 eV</td>
<td>10^{-36} GeV^4</td>
<td>10^{11}</td>
</tr>
<tr>
<td>?</td>
<td>10^{-3} eV</td>
<td>\sim 40K</td>
<td>10^{-46} GeV^4</td>
<td>1</td>
</tr>
</tbody>
</table>
Original Casimir effect:

\[ \text{1) "box" geometry} \]
\[ \text{2) EM b.c.'s: } \mathbf{n} \cdot \mathbf{B} = 0 \quad \& \quad \mathbf{n} \wedge \mathbf{E} = 0. \]
\[ \Rightarrow 2 \text{ decoupled scalar fields:} \]

\[ \phi^D(x, y, z, t)|_{z=0} = \phi^D(x, y, z, t)|_{z=L} = 0 \quad \text{(Dirichlet b.c.'s)} \]
\[ \phi^D_{nk\perp}(x, y, z, t) = \sin(k_z z) e^{i(k_x x + k_y y)} e^{-i\omega_{n,k_x,k_y} t} \]

\[ k_z = \left( \frac{\pi}{L} \right) n, \quad n = 1, 2, 3, \ldots ; \quad \omega_{n,k_x,k_y} = c \sqrt{k_x^2 + k_y^2 + \left( \frac{n\pi}{L} \right)^2} \]

\[ \frac{\partial}{\partial z} \phi^N(x, y, z, t)|_{z=0} = \frac{\partial}{\partial z} \phi^N(x, y, z, t)|_{z=L} = 0 \quad \text{(Neumann b.c.'s)} \]
\[ \phi^N_{nk\perp}(x, y, z, t) = \cos(k_z z) e^{i(k_x x + k_y y)} e^{-i\omega_{n,k_x,k_y} t} \]

\[ k_z = \left( \frac{\pi}{L} \right) n, \quad n = 0, 1, 2, 3, \ldots . \]

\[ E_C^{\text{EM}}(L) = \lim_{\Lambda \to \infty} A \int \int \frac{dk_x dk_y}{(2\pi)^2} \left( \sum_{n=1}^{\infty} \frac{1}{2} \hbar \omega_{n,k_x,k_y} + \sum_{n=0}^{\infty} \frac{1}{2} \hbar \omega_{n,k_x,k_y} \right) e^{-\frac{n\omega_{n,k_x,k_y}}{\Lambda}} \]
\[ = \lim_{\Lambda \to \infty} \hbar c A L \left[ \frac{3(\Lambda/\hbar c)^4}{\pi^2} + (-1+1) \left( \frac{\Lambda/\hbar c}{4\pi L} \right)^3 - \frac{\pi^2}{720 L^4} + O \left( (\hbar c)^2/\Lambda^2 L^6 \right) \right] \]
Experiments: (<12 until 1998)

- **Early efforts**: relied on cantilever or spring balances
  - J.T.G. Overbeek & M.J. Sparnaay (1954); B.V. Deriagin & I.I. Abrikosova (1957): (glass or quartz l). M.J. Sparnaay (1958): first exp. with two metal plates; sensitivity: $(0.1 - 1) \times 10^{-8}$ N and 17 mV. Outcome: exponent $n$ in the $1/L^n$ Casimir force determined as $4 \pm 1$.

- **Resurgence**: (1996 $\rightarrow$): sphere-plate systems under proximity force theorem: $F_{0}^{\text{ol}}(L) = 2\pi R \frac{h \gamma_{20} L^{3}}{L^{3}}$
    - mech. hysteresis eliminated; piezoelectrical transducers; $\Delta F = 10^{-11}$ N, 10% accuracy.
    - atomic force microscope: $32 - 1000$ nm, $\Delta F = 10^{-12}$ N and 1-2 % accuracy.
    - capacitance bridge; seesaw device; Microelectromech. Systems (MEMS); $L_{min} \approx 70$ nm.
  - G. Bressi et al., PRL 88 (2002): two $1.2 \times 1.2$ mm$^2$ plates, 0.5–3 μm, 15 % accuracy.

- **Ideal case**:
  - perfect conductors without impurities at $T = 0$.

- **Reality**:
  - finite temperature (only important for $L > 1 \mu m$); imperfect conductors (realistic metals, Lifshitz theory); surface roughness (geometry averaging, $\langle r^2 \rangle_{s,r.}^{1/2} \approx 3$ nm)
Atomic Force Microscope:

Geometry dependence

- Before 1968: Casimir effect $\leftrightarrow$ van der Waals forces
  \[
  \begin{align*}
  &\{ \text{2 parallel plates} \\
  &\quad \text{plate & sphere} \\
  &\quad \text{sphere & sphere} \} \\
  &\text{attractive}
  \end{align*}
  \]

- Casimir’s idea of stabilizing the electron (1953):
  Then:
  \[
  \begin{align*}
  \mathcal{E}_{em} &= \frac{e^2}{2a} \\
  \mathcal{E}_C &= -C \frac{\hbar c}{2a^3} 4\pi a^2
  \end{align*}
  \]
  $\Rightarrow$ stabilization if $C = \frac{e^2}{4\pi \hbar c}$ (independently of the radius $a$).

- However, T. Boyer (1968): force is repulsive!
  (Similarly, R. Balian & C. Bloch (1972-74): Casimir calculations in cavities depend on (bag-)geometry)

\[\therefore \text{“Generalization” of concept of Casimir energy}\]
“Generalization” of concept of Casimir energy:

1) geometry dependence

**Casimir energy** \( \equiv \) vacuum energy from the *geometry-dependent* part of the *density of states* (d.o.s)

\( \leftrightarrow \) *shell* correction energy in Nucl. Phys.

**d.o.s.:** \[ \rho(E) \equiv \sum_{E_k} \delta(E - E_k) = \rho_0(E) + \rho_{\text{bulk}}(E) + \delta \rho_C(E, \text{geom.-dep.}) \]

**N.o.s.:** \[ \mathcal{N}(E) \equiv \sum_{E_k} \Theta(E - E_k) = \int_0^E dE' \rho(E') \]

**Casimir Energy:** \[ \mathcal{E}_C \equiv \int dE E \delta \rho_C(E, \text{geom.-dep.}) = -\int dE \mathcal{N}_C(E, \text{geom.-dep.}) \]
Further “Generalization” of concept of Casimir energy:

2) matter fields

- **Here**: space not “filled” with fluctuating EM modes, but with gas of non-interacting (non-relativistic) fermions.

- **Similarity**: ∃ mode sums $\sum \hbar \omega_k$ with constant degeneracy factor, (because of Pauli’s exclusion principle).

- **Difference**: ∃ of second scale: fermi energy = chemical potential $\mu$ (at $T \approx 0$) in addition to geometric size and distance scale(s).

- **Concrete**: Matter fields (fermions) in the space between voids build up a quantum pressure on the voids

∃ effective interaction between empty regions of space in the background of non-interacting fermions
Applications

In the inner crust of a neutron star:

- nuclei start to loose neutrons due to increasing pressure and density: below saturation density \( \Rightarrow \) chain of phases between 0.03 \( \text{fm}^{-3} \) and 0.1 \( \text{fm}^{-3} \):
  - nuclei \( \rightarrow \) rods \( \rightarrow \) slabs \( \rightarrow \) tubes \( \rightarrow \) bubbles \( \rightarrow \) uniform matter
- Liquid drop model: *meat-ball, spaghetti, lasagne* vacua from interplay between Coulomb and surface energies with phase differences of a few keV/fm\(^3\).
- However, neglected shell correction energy of the same order!

Inside a neutron/quark star:

- Quark matter droplets immersed in hadronic matter at \( \rho \gg \rho_{nm} \)

In the lab:

- *Buckyballs* immersed in an electron gas (in liquid mercury)
- or Buckyballs immersed in liquid \(^3\text{He}\).
- (Bosonic) cavities in dilute atomic Fermi condensates.
Let me now go back to my starting theme and consider spherical inhomogeneities in an otherwise featureless (normal) Fermi sea.
- Casimir energy for fermions between two impenetrable (parallel) planes at distance $L$:
  \[ E_C = \mu F(k_F L) \quad \text{with} \quad \mu = \hbar^2 k_F^2 / 2m \]  
as natural scale.

- For more complicated geometries → involved computations!

- However, case of immersed non-overlapping spherical voids still relatively simple:
  
  Krein (1953) trace-formula: level density var. \( \delta \rho(E) \leftrightarrow \frac{d}{dE} \) of phase shift \( \frac{1}{2i} \ln \det S_n(E) \)

  \[
  \delta \rho(E) = \bar{\rho}(E) - \bar{\rho}_0(E) = \frac{1}{2\pi i} \frac{d}{dE} \text{tr} \ln S_n(E)
  \]
  of the $n$-sphere S-matrix

Extract geometry-dependent Casimir fluctuations from multiple-scattering part

⇒ Calculation mapped onto a quantum mechanical billiard problem:

hyperbolic point-particle scattering off $n$ spheres or $n$ disks

References:
B. Eckhardt, J. Phys. A20 (1987);  
P. Gaspard & S. Rice, J. Chem. Phys. 90 (1989);  
P. Cvitanović & B. Eckhardt, Phys. Rev. Lett. 63 (1989);  
Digression 1: E. Beth & G.E. Uhlenbeck (1937)

predecessor of Krein (1962) formula for **spherical** potentials:

Idea: **spherical boxes**: \[ \lim_{R \to \infty} \left( \frac{R}{k} \right) = - \frac{R}{k} \]

Asymptotically:
\[ u_{k\ell}(r) \sim \sin \left( kr - \frac{1}{2} \ell \pi + \eta(k) \right) \]
\[ u^{(0)}_{k\ell}(r) \sim \sin \left( kr - \frac{1}{2} \ell \pi \right) \]

and Dirichlet b.c.'s:
\[ u_{k\ell}(R) = u^{(0)}_{k\ell}(R) = 0. \]

**EV conditions ( (2\ell+1)-fold degenerate )**:
\[ k_{\ell,n} R - \frac{1}{2} \ell \pi + \eta(k_{\ell,n}) = \pi n, \quad n=0,1,2,\ldots \quad \text{(with potential)} \]
\[ k^{(0)}_{\ell,n} R - \frac{1}{2} \ell \pi = \pi n, \quad n=0,1,2,\ldots \quad \text{(without potential)} \]

A change of \( n \) by one unit, for \( \ell \) fixed, leads to
\[ \Delta k_{\ell} \left( R + \frac{\partial}{\partial k} \eta(k) \right) = \pi = \Delta k^{(0)}_{\ell} R, \]

such that the conditions \( \tilde{\rho}_{\ell}(k) \Delta k_{\ell} = \tilde{\rho}_{\ell}^{(0)}(k) \Delta k^{(0)}_{\ell} = 2\ell + 1 \quad \text{(note the averaging !)} \)

induce the formula
\[ \tilde{\rho}_{\ell}(k) - \tilde{\rho}_{\ell}^{(0)}(k) = \frac{2\ell+1}{\pi} \frac{\partial}{\partial k} \eta(k) \]
Digression 2: Semiclassical interpretation of Krein formula

Determinant of scattering matrix is semiclassically a sum over periodic orbits (+Weyl terms)

Consider the difference of the densities of states of two bounded reference systems:

- The container-induced periodic orbits cancel!
- However, $\exists$ further spurious periodic orbits whose lengths grow with increasing $R$.
- Removal of long orbits by exponential damping or averaging:

$$\lim_{\epsilon \to 0^+} \lim_{R \to \infty} \left\{ \rho^{(n)}(k + i\epsilon; R) - \rho^{(0)}(k + i\epsilon; R) \right\}$$

$$= \frac{1}{2\pi} \text{Im} \frac{d}{dk} \ln \det S^{(n)}(k) \bigg|_{k \text{ real}}$$

Note the order of the limits!
Digression 3: Casimir energy in fermionic background

Chem. potential \( \mu = \hbar^2 k_F^2 / 2m \) or Fermi momentum \( k_F \) natural UV-cutoff (\( \Lambda_{UV} = \mu \) or \( k_{UV} = k_F \))

Casimir energy at fixed total number of fermions \( \mathcal{N} \):

\[ \mathcal{N} = \int_0^\mu \rho(E) \, dE = \int_0^{\mu_0} \left[ \rho_0(E) + \rho_W(E) \right] \, dE \]

\[ \Rightarrow \quad \int_0^\mu \rho(E) \, dE = - \int_0^{\mu_0} \left[ \rho(E) - \rho_0(E) - \rho_W(E) \right] \, dE = - \frac{d}{dE} \mathcal{N}_C(E) \]

\[ \mathcal{E}_C|_{\mathcal{N}} = \int_0^\mu E \rho(E) \, dE - \int_0^{\mu_0} E \left[ \rho_0(E) + \rho_W(E) \right] \, dE \]

\[ = \int_0^{\mu_0} E \rho_C(E) \, dE + \mu_0 \int_0^\mu \rho(E) \, dE + \mathcal{O}(V^{-1}) \]

\[ = \int_0^{\mu_0} (E - \mu_0) \rho_C(E) \, dE = - \int_0^{\mu_0} \mathcal{N}_C(E) \, dE. \]

Grandcanonical Casimir energy at fixed \( \mu = \mu_0 \):

\[ \widetilde{\mathcal{E}}_C|_{\mu} - \mu \mathcal{N}_C(\mu) = \mathcal{E}_C|_{\mathcal{N}} + \mathcal{O}(V^{-1}) \]
1. "infinite" container: $\rho(E) = \rho_0(E)$  (fermionic background)

2. $n$ bubbles (of radii $a_i$) "punched out" at "infinite" separation:

$$\rho(E) = \rho_0(E) + \sum_{i=1}^{n} \rho_{W}(E, a_i)$$  (note the excluded volume !)

3. geometry-dependent arrangement of $n$ bubbles:

$$\rho(E) = \rho_0(E) + \sum_{i=1}^{n} \rho_{W}(E, a_i) + \delta \rho_C(E, \{a_i\}, \{r_{ij}\})$$

4. Krein formula (note the averaging):

$$\delta \rho(E) = \bar{\rho}(E) - \bar{\rho}_0(E) = \frac{1}{2\pi i} \frac{d}{dE} \ln \det S_n(E)$$

5. Multiple scattering matrix

$$\delta \rho_C(E, \{a_i\}, \{r_{ij}\}) = -\frac{1}{\pi} \text{Im} \left( \frac{d}{dE} \ln \det M(E) \right)$$

6. Casimir energy:

$$\mathcal{E}_C = \int_0^\mu dE (E - \mu) \delta \bar{\rho}_C = -\int_0^\mu dE \mathcal{N}_C$$

(see A.W., Phys. Rep. 309 (1999))
Multi-scattering matrix for \(n\) spheres of radii \(a_j\) and distances \(r_{jj'} (j, j' = 1, 2, \ldots n)\):

\[
M_{lm,l'm'}^{jj'} = \delta^{jj'} \delta_{ll'} \delta_{mm'} + (1 - \delta^{jj'}) i^{2m + l' - l} \sqrt{4\pi(2l + 1)(2l' + 1)} \left(\frac{a_j}{a_{j'}}\right)^2 \frac{j_l(ka_j)}{h_l^{(1)}(ka_j')} \times \sum_{l=0}^{\infty} \sum_{\tilde{m}=-l'}^{l'} \sqrt{2\tilde{l} + 1} i^{l} \left(\begin{array}{ccc}
\tilde{l} & l' & l \\
0 & 0 & 0
\end{array}\right) \left(\begin{array}{ccc}
\tilde{m} & \tilde{m} & \tilde{m} \\
-m & \tilde{m} & -m
\end{array}\right) D_{l',\tilde{m}}^{(j,j')} h_l^{(1)}(kr_{jj'}) Y_{\tilde{l}}^{m-\tilde{m}}(\hat{r}_{jj'})
\]

In the limit of small scatterers:

\[
M^{jj'}(E) \approx \delta^{jj'} - (1 - \delta^{jj'}) f_j(E) \frac{\exp(ikr_{jj'})}{r_{jj'}} (\text{+ p-wave})
\]

Two spheres of radius \(a\) at distance \(r\)

\[
\nu_{deg} \frac{a^2}{\pi r^2} \sin[2(r - a)k] + \mathcal{O}((ka)^3),
\]

in the small-scatterer limit:

\[
\mathcal{N}_{C}^{oo}(E) = -\frac{1}{\pi} \text{Im} \ln \det M^{oo}(E) \approx \nu_{deg} \frac{a^2}{4\pi r(r - 2a)} \sin[2(r - 2a)k] \quad (\text{Gutzwiller's trace formula})
\]

whereas the semiclassical result (for the single two-bounce periodic orbit with no repetitions) reads:

\[
\mathcal{N}_{C,sci}^{oo}(E) = \nu_{deg} \frac{a^2}{4\pi r(r - 2a)} \sin[2(r - 2a)k] \quad (\text{Gutzwiller's trace formula})
\]
Two-sphere case:

N.o.S \( \mathcal{N}_{C}^{\infty}(E) = -\frac{1}{\pi} \text{Im} \ln \det M^{\infty}(E) \approx \nu_{\text{deg}} \frac{a^2}{4\pi r(r-2a)} \sin[2(r-2a)k] \)

\[ \mathcal{E}_{C}^{\infty} = -\int_{0}^{\mu(k_F)} dE \mathcal{N}_{C}^{\infty}(E) \approx -\nu_{\text{deg}} \mu \frac{a^2}{2\pi r(r-2a)} j_1[2(r-2a)k_F], \quad (k_F a > 1) \]

(of longer range than molecular van der Waals forces!)

Sphere-plate geometry:

\[ \mathcal{E}_{C}^{0} \approx -\nu_{\text{deg}} \frac{a}{2\pi(r-a)} j_1[2(r-a)k_F], \quad (k_F a > 1) \]
Three and four spheres:

- *periodic orbit summation*: ∃ of genuine three and *more-body* interactions
- However, *2-bounce orbit* dominates in equilateral three- and four sphere systems
  (max. correction due to 3-bounce orbit is ∼ 10% at r ≈ 2.5α)

- *Billiard analogy*: difficult to make *long* shots, especially with many bounces
  – the slightest error ruins the shot.
Dominance of two-body interactions:

the Casimir energy satisfies the rule $E_3 \approx 3E_2$ for 3 identical spheres (equilateral triangle)

and $E_4 \approx 6E_2 \approx 2E_3$ for 4 identical spheres (symmetric tetrahedron) if $k_F a \gg 1$.

($k_F a \leq 1$: corrections up to 10% and 25% for 3 & 4 spheres)
Let us now consider the role of fermionic superfluidity
What happens at the boundary of a normal and superfluid regions?

\[
\begin{pmatrix}
\hbar - \lambda & \Delta \\
\Delta^+ & -\left(\hbar^* - \lambda\right)
\end{pmatrix}
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix}
= E_k
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix}
\]

outside    inside

reflected particle
incident particle
Cooper pair

Andreev reflection
Fermionic Casimir Effect of two superfluid grains in a normal Fermi gas

*Specular reflection* replaced by *Andreev reflection*

⇒ particle retro-reflected as a hole

\[ \frac{k_p^2}{2} + \frac{k_h^2}{2} = k_F^2 \] (particle/hole states symmetric)

\[ k_p \sin \theta_p = k_h \sin \theta_h \] (modified Snell’s law),

⇒ disks/spheres have focusing effect

stable orbits if \( \frac{R}{2a} \leq \frac{k_p}{k_p - k_h} \) satisfied

(here \( R = 6a \) and \( k_p / k_h = 1.5 \))
Apply

1. Bogoliubov-de Gennes equations with pairing field \( \Delta_{1,2}(\vec{r}) = \Delta e^{\pm i \phi_\Delta} / 2 \) at circular grain 1 and 2,
2. mod. Krein formula \( \delta N(E) = -\frac{1}{\pi} \ln \det M(E) \) with \( M(E) \) multi-scattering matrix in p-h space,
3. asymptotic approximation valid for large separations \( R \gg a \)

\[
\delta N(E) \approx \frac{4|T_{ph}(a, \mu, \Delta)| \cos \phi_\Delta}{\pi^2 \sqrt{k_h k_p R}} \sin[(k_p - k_h)R + \phi_{ph}]
- \frac{2|T_{pp}(a, \mu, \Delta)|}{\pi^2 k_p R} \cos(2k_p R + \phi_{pp}) + \frac{2|T_{hh}(a, \mu, \Delta)|}{\pi^2 k_h R} \cos(2k_h R - \phi_{hh})
\]

\( T_{ph} = \left( \sum_m t_{ph}^m / 2 \right)^2 \), \( T_{pp} = \left( \sum_m (-1)^m t_{pp}^m / 2 \right)^2 \) etc. structure functions (only dep. on single scatterer)

\[ k_p^2 / 2m - \mu \equiv E \leq \Delta \text{ dominant} \]

(\( \mu = 200/a^2 \),
\gap \Delta = 50/a^2,
rel. gap phase \( \phi_\Delta = 0 \),
\( R = 6a \))
Quantum Corrections to the GS Energy of Inhomogeneous NM

The Casimir energy for various phases. The lattice constants are:
L = 23, 25 and 28 fm respectively.
u — anti-filling factor
\( \tilde{\rho}_0 \) — average density

A. Bulgac and P. Magierski
Nucl. Phys. 683, 695 (2001)
Nucl. Phys, 703, 892 (2002) (E)
The Casimir energy for the displacement of a single void in the lattice

A. Bulgac and P. Magierski
Nucl. Phys. 683, 695 (2001)
Nucl. Phys. 703, 892 (2002) (E)
Deformation of the rod-like phase lattice

\[ \alpha \beta \sin \gamma = 1 \]

volume conservation

A. Bulgac and P. Magierski
Nucl. Phys. 683, 695 (2001)
Nucl. Phys, 703, 892 (2002) (E)
FIG. 2. The total energy density (1) of npe matter as a function of the proton quadrupole moment $Q_p = Q_{20}$ (middle subfigures). The integrated proton and neutron densities (see text for definition) corresponding to nuclear configurations indicated by arrows are shown in the lower and upper subfigures.
"rippled" slabs

nuclei

FIG. 3. The same as in the Fig. 2 but for different densities.

FIG. 4. The energy density difference $\Delta E/V$ between nuclear phases as a function of the total density. Solid curve denotes the difference between the spherical and rodlike phase. Dotted curve denotes the difference between the spherical and slablike phase. Smaller subfigures show the energy density of npe matter as a function of the proton quadrupole moment for four different densities. Parameter $d$ denotes the length of the cubic box.
Various contributions to the energy density as a function of the proton quadrupole moment.

Let us change gears and consider the scalar Casimir effect in quantum field theory between a sphere and a plane.
Note: asymptotically \( L/a \gg 1 \) s-wave scattering dominates:

\[
E_s(L) \sim -\frac{\pi^3 \hbar c a}{1440 L^2} \frac{90}{\pi^4} \frac{2}{1 + a/L} \frac{1 + a/2L}{2} = -\frac{\pi^3 \hbar c a}{1440 L^2} \times 1.847 \ldots
\]

whereas

\[
E_{p-wave}(L) \sim -\frac{\pi^3 \hbar c a^3}{1440 L^4} \frac{90}{\pi^4} \frac{1}{1 + a/L} \frac{1 + a/2L}{2}
\]

Scalar Casimir effect of a Dirichlet sphere and a plate

Antonello Scardicchio, R.L. Jaffe, NPB 704, 552 (2005)
Conclusions

A qualitatively new form of Casimir energy

In normal Fermi systems the Casimir energy could be either attractive or repulsive, depending on the relative arrangement of the scatterers

Semiclassical arguments show that one can disentangle two-body (typically dominant) and many-body interactions

In superfluid Fermi systems Andreev reflection leads to a very strong enhancement of the Fermionic Casimir interaction, which seems to be predominantly repulsive

Expected to show up in a variety of systems, neutron stars, various condensed matter systems, cosmology etc.

The methods developed by us proved extremely useful in calculating the familiar Casimir interaction in a more transparent fashion and easy manner