The Many Facets of Superfluidity in Dilute Fermi Systems

Aurel Bulgac

collaborators: Yongle Yu, P.F. Bedaque. A. C. Fonseca

Transparencies will be available shortly at
http://www.phys.washington.edu/~bulgac

One of my favorite times in the academic year occurs in early spring when I give my class of extremely bright graduate students, who have mastered quantum mechanics but are otherwise unsuspecting and innocent, a take-home exam in which they are asked to deduce superfluidity from first principles. There is no doubt a special place in hell being reserved for me at this very moment for this mean trick, for the task is impossible. Superfluidity, like the fractional quantum Hall effect, is an emergent phenomenon – a low-energy collective effect of huge numbers of particles that cannot be deduced from the microscopic equations of motion in a rigorous way and that disappears completely when the system is taken apart[A)]. There are prototypes for superfluids, of course, and students who memorize them have taken the first step down the long road to understanding the phenomenon, but these are all approximate and in the end not deductive at all, but fits to experiment. The students feel betrayed and hurt by this experience because they have been trained to think in reductionist terms and thus to believe that everything not amenable to such thinking is unimportant. But nature is much more heartless than I am, and those students who stay in physics long enough to seriously confront the experimental record eventually come to understand that the reductionist idea is wrong a great deal of the time, and perhaps always.

Robert B. Laughlin, Nobel Lecture, December 8, 1998

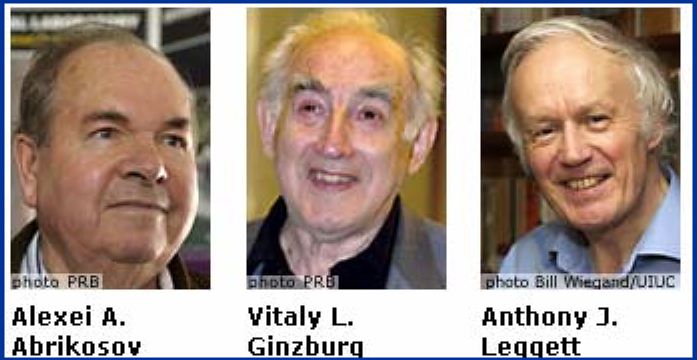
Superconductivity and superfluidity in Fermi systems

- ✓ Dilute atomic Fermi gases — $T_c > 10^{-12}$ eV
- Liquid ^{3}He — $T_c > 10^{-7}$ eV
- Metals, composite materials — $T_c > 10^{-3} – 10^{-2}$ eV
- ✓ Nuclei, neutron stars — $T_c > 10^5 – 10^6$ eV
- QCD color superconductivity — $T_c > 10^7 – 10^8$ eV

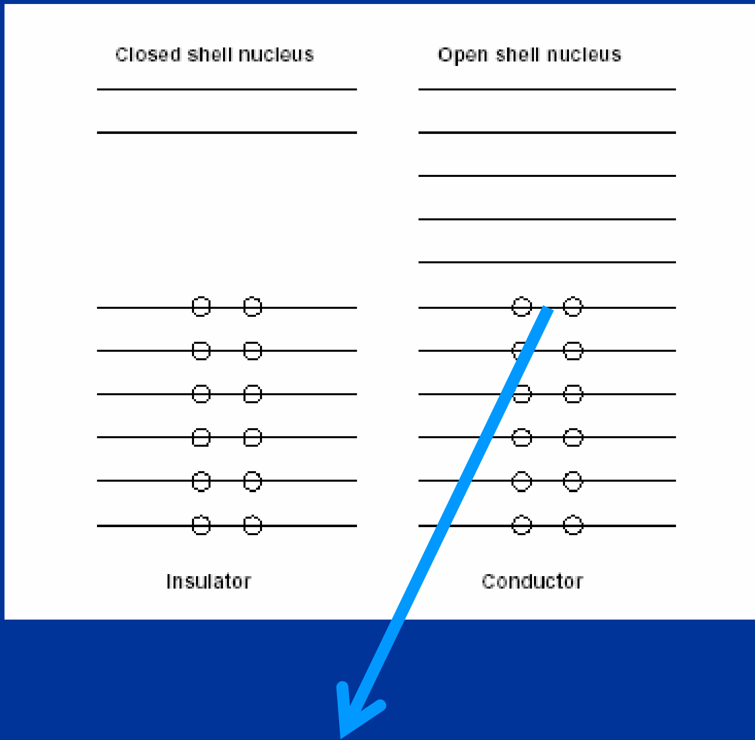
units (1 eV > 10^4 K)

Memorable years in the history of superfluidity and superconductivity of Fermi systems

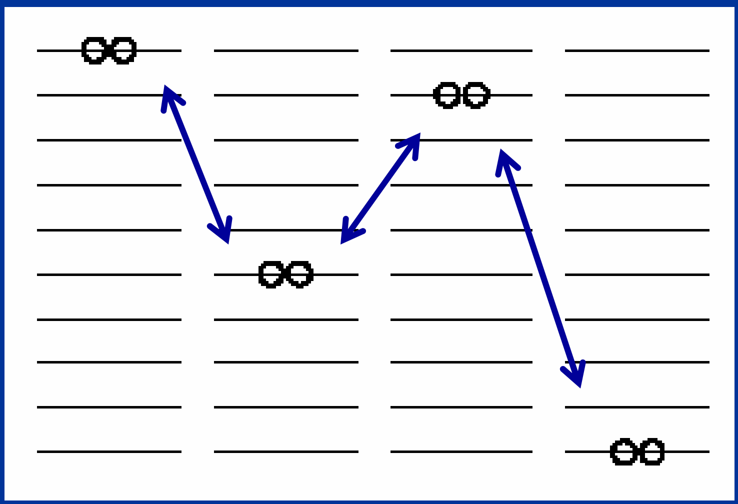
- 1913 Kamerlingh Onnes
- 1972 Bardeen, Cooper and Schrieffer
- 1973 Esaki, Giaever and Josephson
- 1987 Bednorz and Muller
- 1996 Lee, Osheroff and Richardson
- 2003 Abrikosov, Ginzburg and Leggett

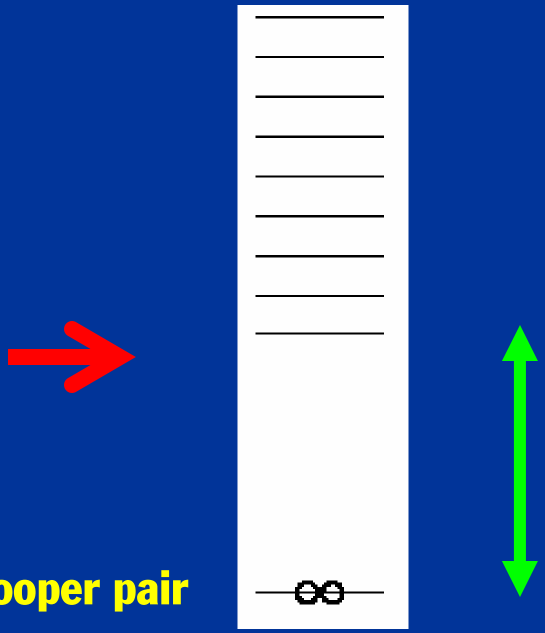


How pairing emerges?



Cooper's argument (1956)





Cooper pair

Gap 2D

In dilute Fermi systems only very few characteristics of the particle-particle interaction are relevant. Why?

- These systems are typically very cold

$$T \ll T_F = \frac{\varepsilon_F}{k_B}, \qquad \varepsilon_F = \frac{p_F^2}{2m}$$

- A dilute Fermi system is degenerate and the fastest particle has a momentum of the order of the Fermi momentum

$$p_F = (6\pi^2 n_\uparrow)^{1/3}\hbar = (6\pi^2 n_\downarrow)^{1/3}\hbar$$

$$n_\uparrow = n_\downarrow = \frac{n}{2}$$

- The wave functions are basically constant over the interaction volume of two particles and thus they cannot "see" any details, except the scattering length typically.

What is the scattering length?

$$k\ \mathrm{cotan}\ \delta_0(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + \ldots$$

$$\begin{cases} a > 0 & \text{a bound state exists} \\ a < 0 & \text{there is no bound state} \end{cases}$$

$$\psi(\vec{r}) = \exp(i\vec{k} \bullet \vec{r}) + \frac{f}{r}\exp(ikr) \underset{kr\to 0}{\approx} 1 - \frac{a}{r} + O(kr)$$

In the region outside the potential well

At very low energies the interaction of two particles can be approximated by the pseudo-potential

$$U(\vec{r}) = \frac{4\pi\hbar^2 a}{m}\delta(\vec{r}) = \begin{cases} > 0 \text{ (repulsive)} & \text{if} \quad a>0 \\ < 0 \text{ (attractive)} & \text{if} \quad a<0 \end{cases}$$

In dilute atomic systems experimenters can control nowadays almost anything:

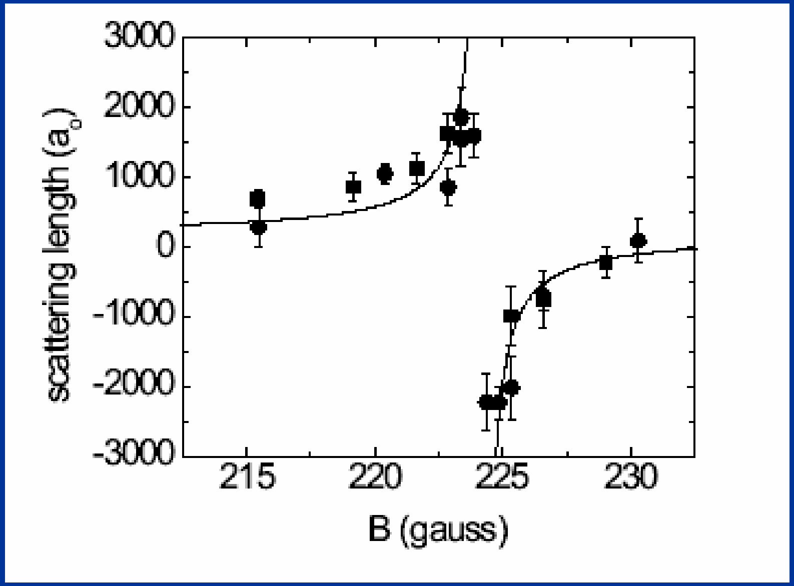
- The number of atoms in the trap
- The density of atoms
- Mixtures of various atoms
- The temperature of the atomic cloud
- <u>The strength of the atom-atom interaction</u>

Feshbach resonance

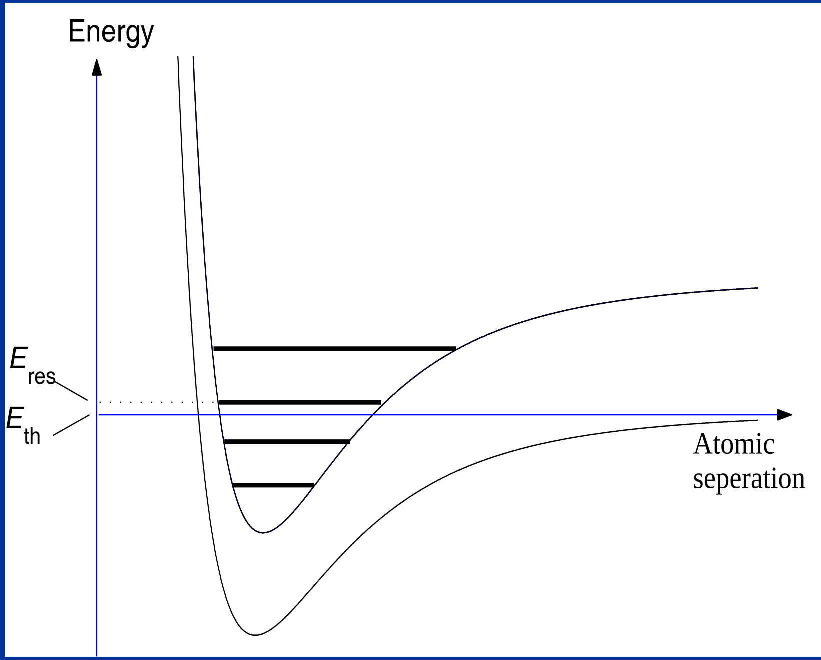
$$H = \frac{\vec{p}^{\,2}}{2\mu_r} + \sum_{i=1}^{2}(V_i^{hf} + V_i^{Z}) + V_0(\vec{r})P_0 + V_1(\vec{r})P_1 + V^d$$

$$V^{hf} = \frac{a_{hf}}{\hbar^2}\vec{S}^e \cdot \vec{S}^n, \quad V^Z = (\gamma_e S_z^e - \gamma_n S_z^n)B$$

Tiesinga, Verhaar, Stoof
Phys. Rev. A47, 4114 (1993)



Regal and Jin
Phys. Rev. Lett. 90, 230404 (2003)



BCS →BEC crossover

Leggett (1980), Nozieres and Schmitt-Rink (1985), Randeria *et al.* (1993)

If $a<0$ at $T=0$ a Fermi system is a BCS superfluid

$$\Delta \approx \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left(\frac{\pi}{2k_F a}\right), \quad \text{iff} \quad k_F\,|a| << 1 \text{ and } \xi = \frac{1}{k_F}\frac{\varepsilon_F}{\Delta} >> \frac{1}{k_F}$$

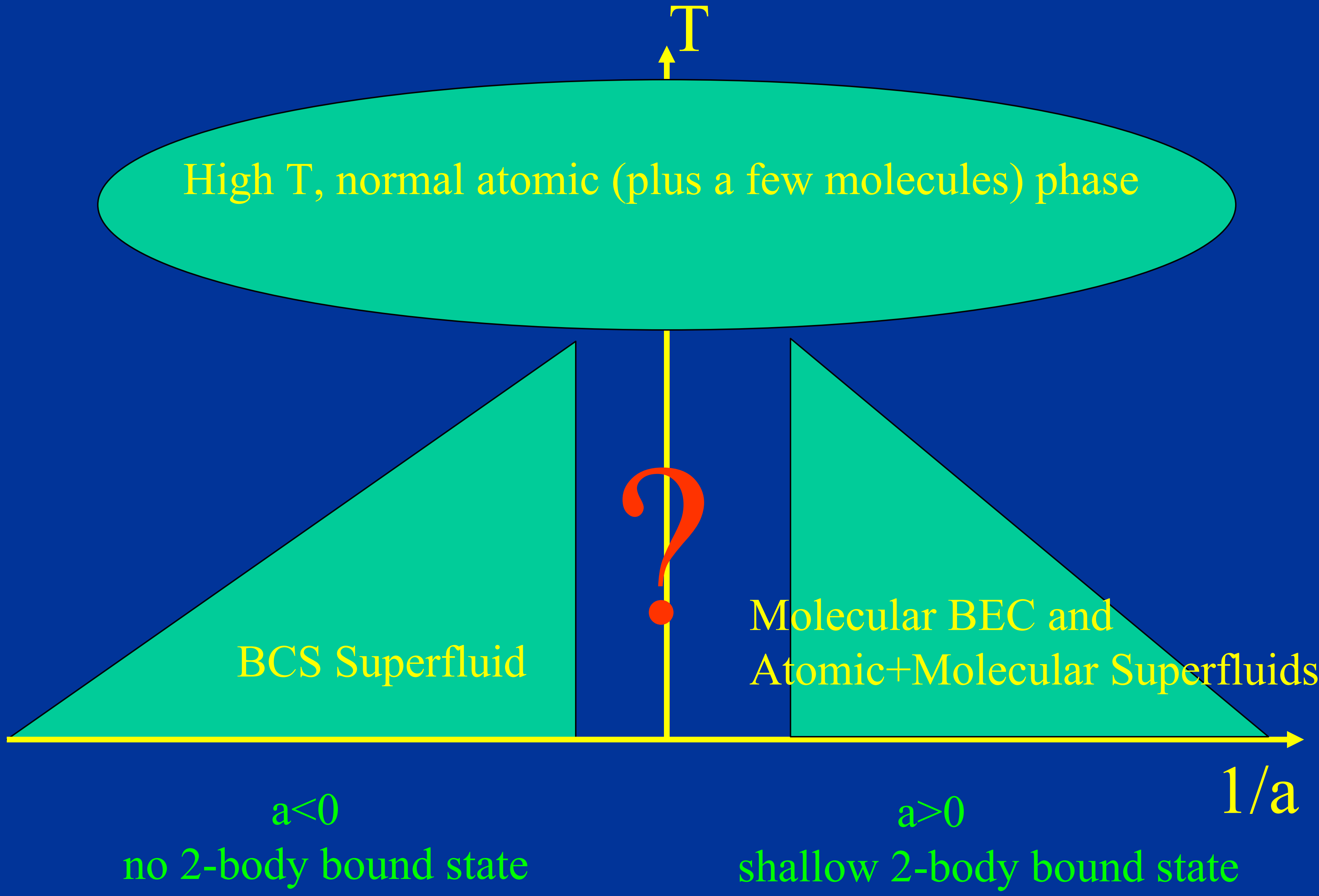
If $|a|=\infty$ and nr_0^3 á 1 a Fermi system is strongly coupled and its properties are universal. Carlson *et al*. PRL <u>91</u>, 050401 (2003)

$$\frac{E_{\text{normal}}}{N} \approx 0.54\,\frac{3}{5}\varepsilon_F, \qquad \frac{E_{\text{superfluid}}}{N} \approx 0.44\,\frac{3}{5}\varepsilon_F \quad \text{and } \xi = O(\lambda_F)$$

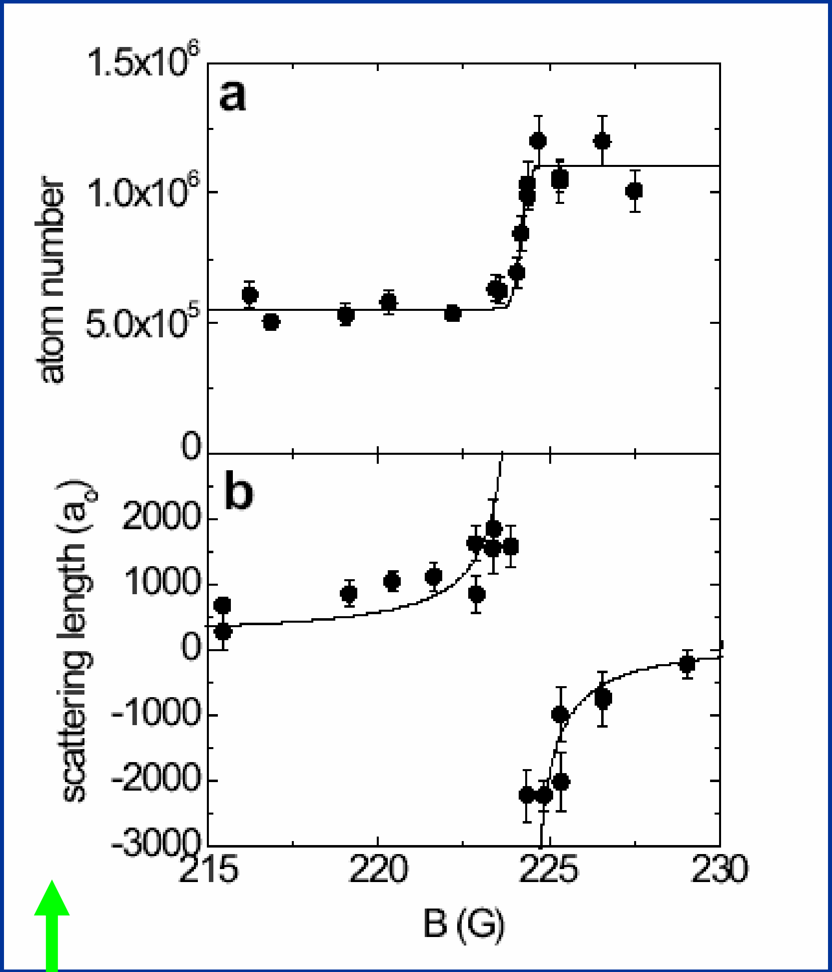
If $a>0$ ($a \gg r_0$) and na^3 á 1 the system is a dilute BEC of tightly bound dimers

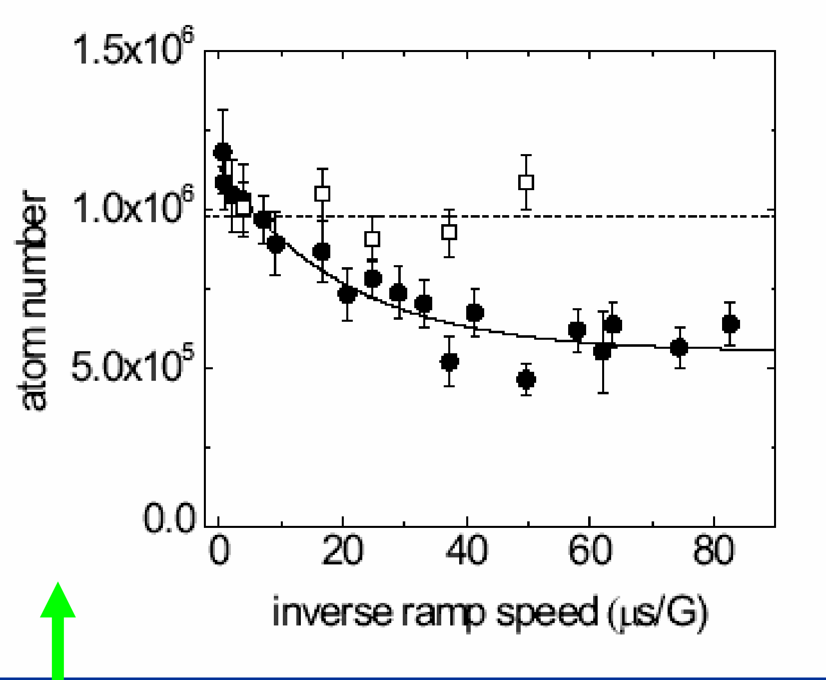
$$\varepsilon_2 = -\frac{\hbar^2}{ma^2} \quad \text{and} \quad n_b a^3 << 1, \quad \text{where} \quad n_b = \frac{n_f}{2} \text{ and } a_{bb} = 0.600(3)a > 0$$

Expected phases of a two species dilute Fermi system



Regal, Ticknor, Bohm and Jin, Nature 424, 47 (2003)





Number of atoms after ramping B from 228.25 G to 216.15 (black dots) and for ramping B down (at 40 ms/G) and up at various rates (squares).

a) Loss of atoms |9/2,-9/2> and |9/2,-5/2> as a function of final B. The initial value of B=227.81 G.

b) Scattering length between hyperfine states |9/2,-9/2> and |9/2,-5/2> as a function of the magnetic field B.

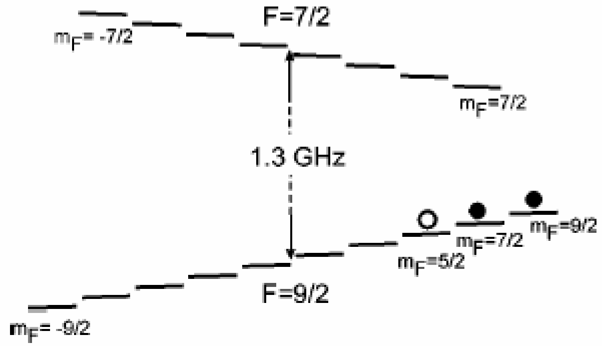
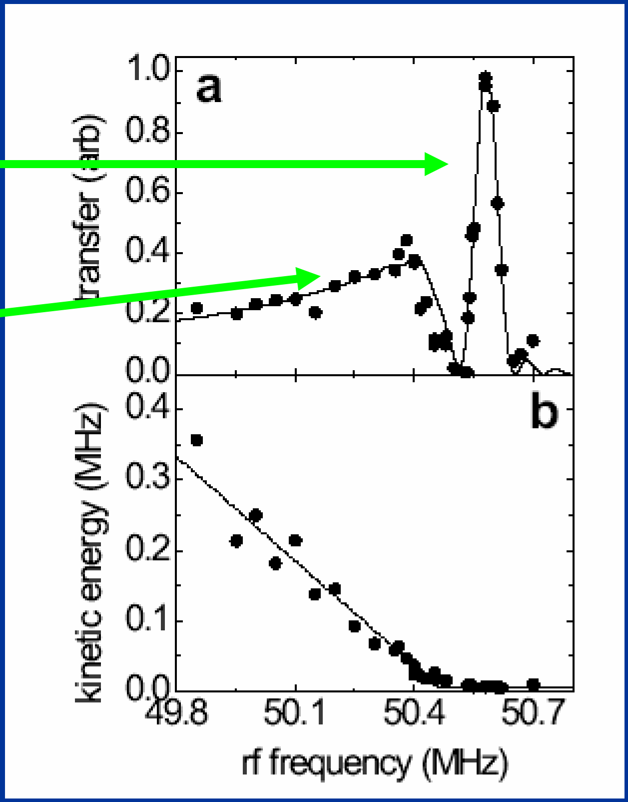


FIG. 1. Schematic of the ground-state hyperfine levels of ^{40}K (shown with exaggerated Zeeman splittings).

DeMarco and Jin, Phys. Rev. A 58, R4267 (1998)

Regal, Ticknor, Bohm and Jin, Nature 424, 47 (2003)

Symmetric peak is near the atomic |9/2,-5/2> to |9/2,-7/2> transition. The total number of |9/2,-5/2> and |9/2,-7/2> atoms is constant.



Asymmetric peak corresponds to dissociation of molecules into free |9/2,-5/2> and |9/2,-7/2> atoms. The total number of |9/2,-5/2> and |9/2,-7/2> atoms increases.

$$h\nu_{rf} = h\nu_{atom} - E_{binding} - \Delta E$$

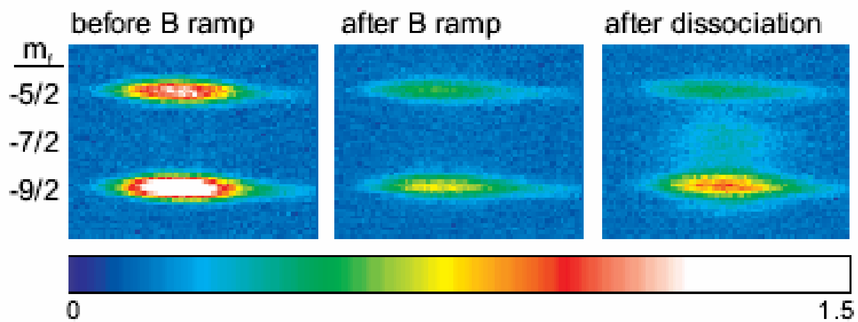
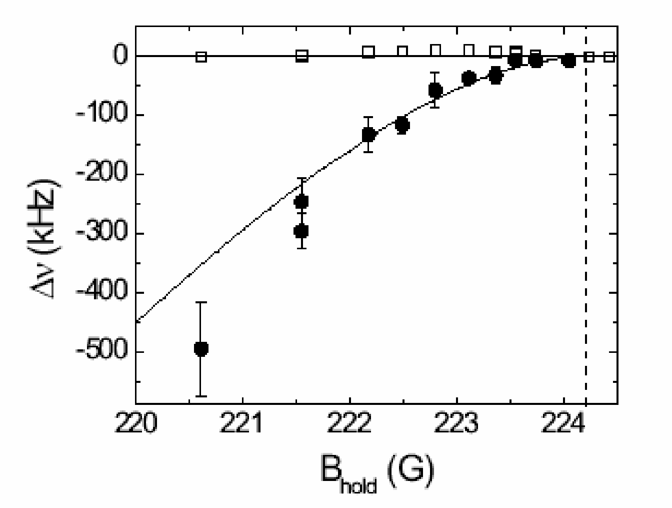


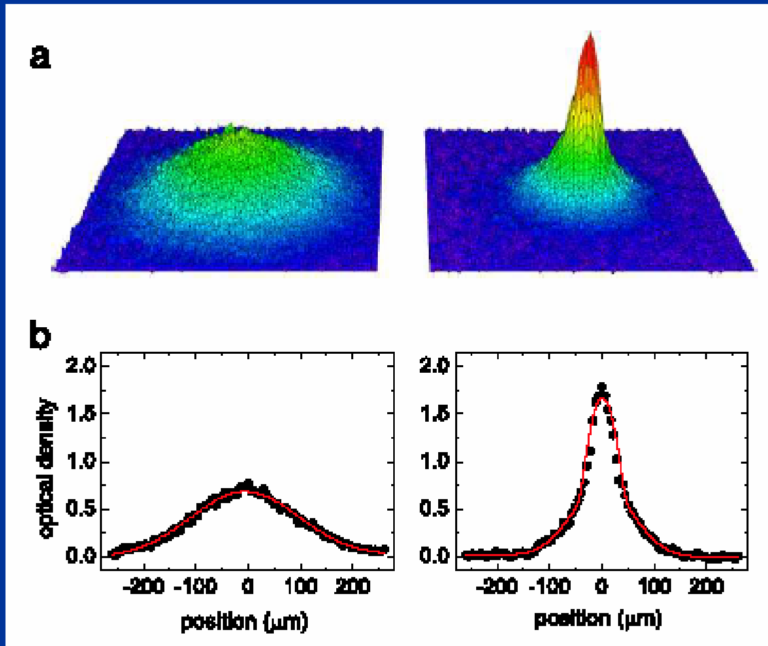
FIG. 4: Absorption images of the quantum gas using a Stern-Gerlach technique. We start with ultracold fermionic atoms in the $|9/2,-5/2\rangle$ and $|9/2,-9/2\rangle$ states of ^{40}K. A magnetic field ramp through the Feshbach resonance causes 50% atom loss, due to adiabatic conversion of atoms to diatomic molecules. To directly detect these bosonic molecules we apply an rf photodissociation pulse; the dissociated molecules then appear in the $|9/2,-7/2\rangle$ and $|9/2,-9/2\rangle$ atom states. The shaded bar indicates the optical depth.

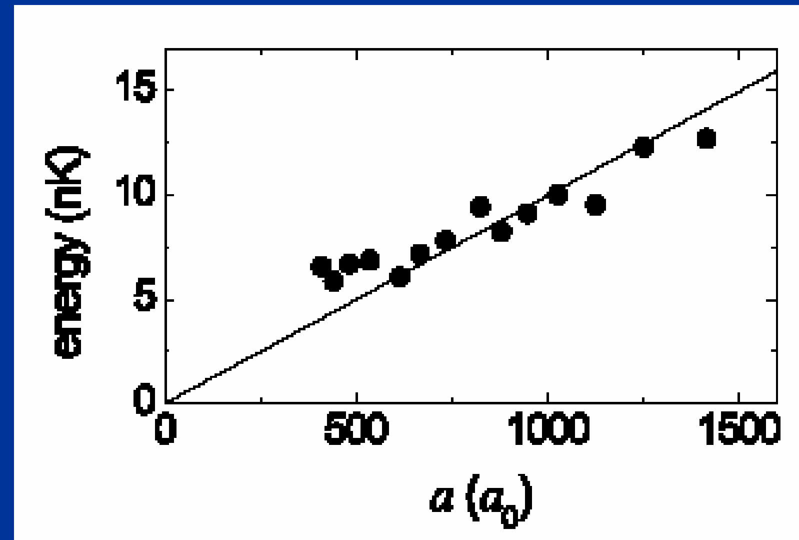


Dimer/molecule binding energy

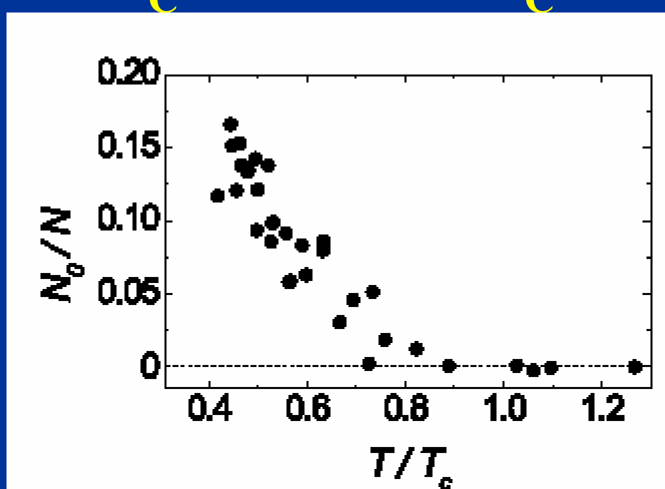
Molecular BEC in a cloud of ^{40}K atoms (fermions)

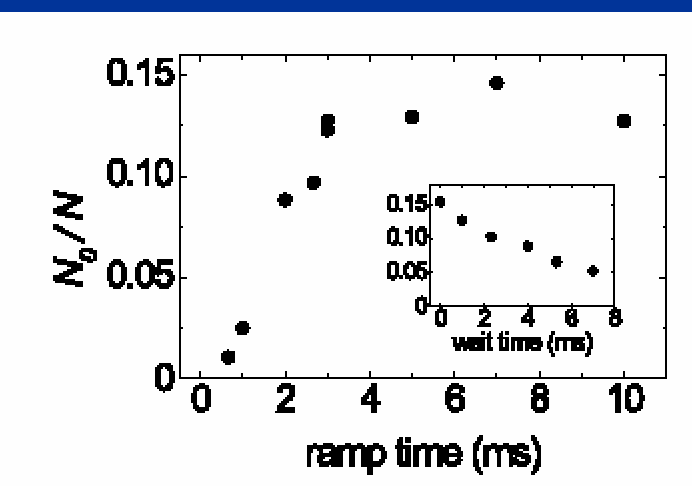
Greiner, Regal and Jin, Nature 426, 537 (2003)



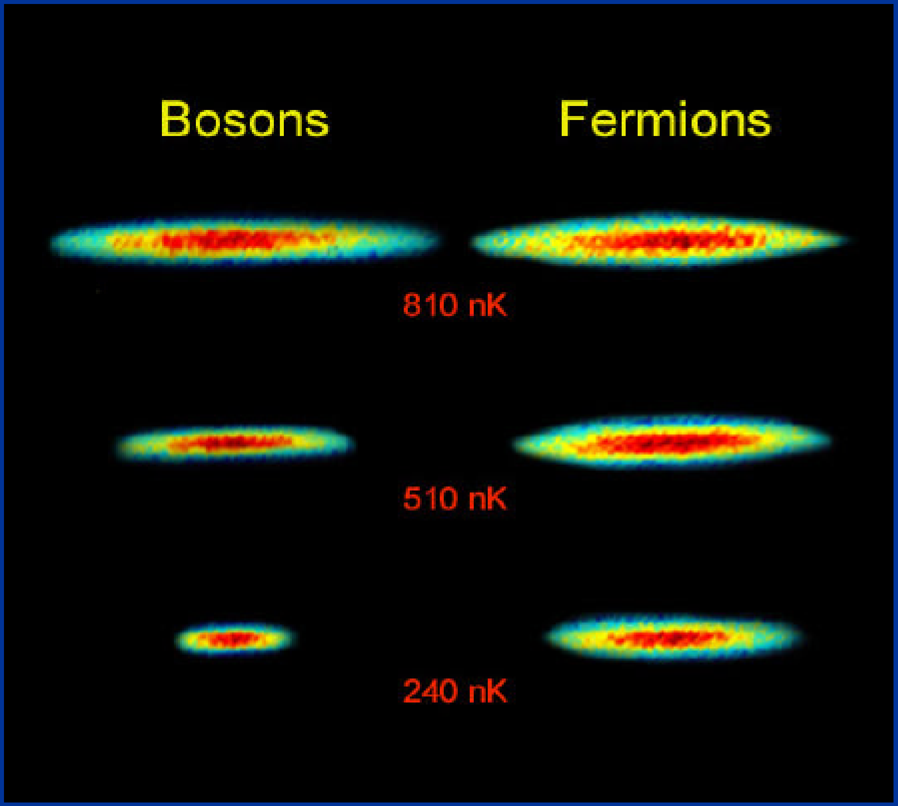


$T > T_C$ $T<T_C$





Size of the atomic cloud as a function of temperature around the critical temperature



“Fundamental” and “effective” Hamiltonians

$$H_{\rm fund} = -\sum_s \int d^3r \psi_s^\dagger(\boldsymbol{r}) \frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} \psi_s(\boldsymbol{r}) + \frac{1}{2} \sum_{s_1, s_2} \int d^3r_1 d^3r_2 \psi_{s_1}^\dagger(\boldsymbol{r}_1) \psi_{s_2}^\dagger(\boldsymbol{r}_2) \psi_{s_2}(\boldsymbol{r}_2) \psi_{s_1}(\boldsymbol{r}_1) V_{s_1 s_2}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|),$$

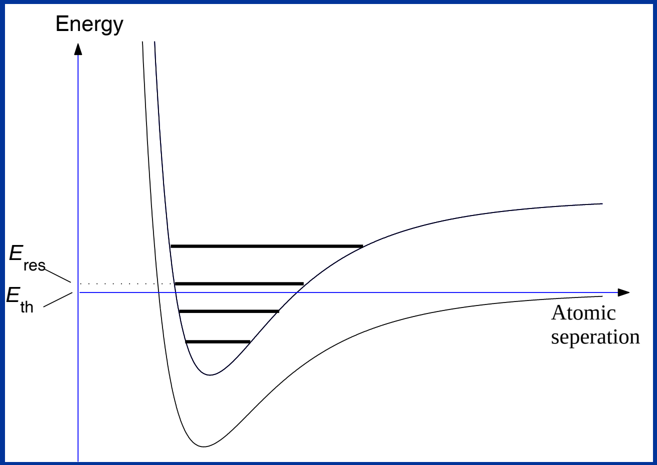
If one is interested in phenomena with momenta $p = \hbar k \ll \hbar/r_0$, where r_0 is the typical range of the interaction, the “fundamental” Hamiltonian is too complex.

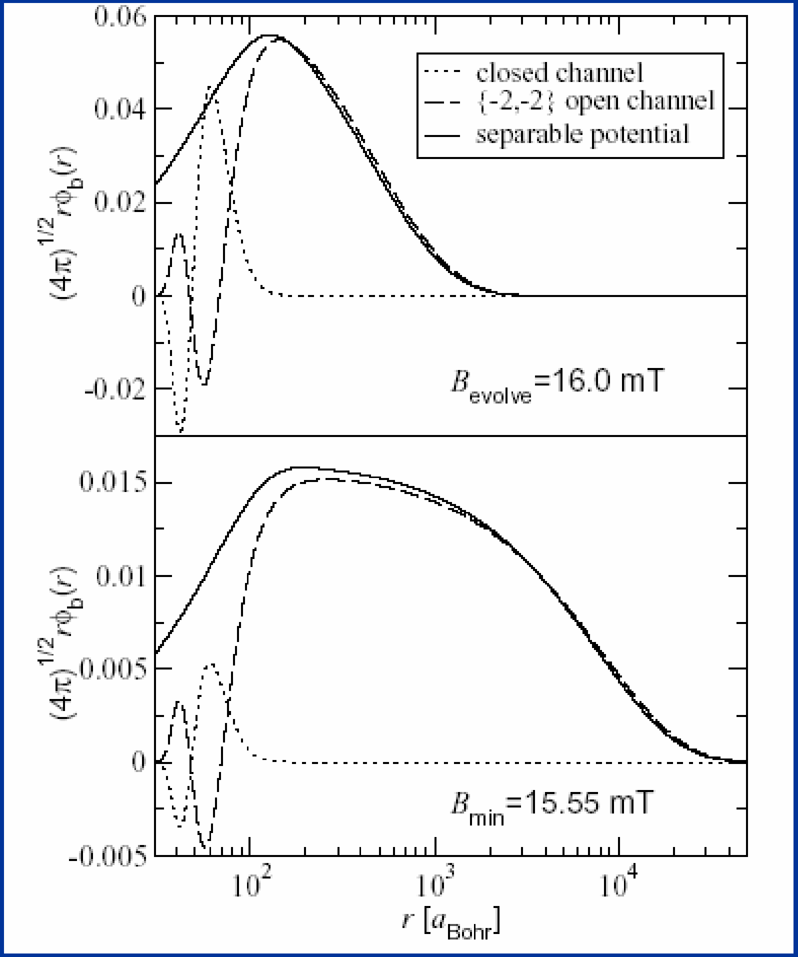
$$\begin{aligned} H_{\rm eff}(\boldsymbol{r}) = &-\psi^\dagger(\boldsymbol{r}) \frac{\hbar^2 \boldsymbol{\nabla}_a^2}{2m} \psi_a(\boldsymbol{r}) + \frac{1}{2} \lambda_{aa} \psi_a^\dagger(\boldsymbol{r}) \psi_a^\dagger(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \\ &-\psi^\dagger(\boldsymbol{r}) \frac{\hbar^2 \boldsymbol{\nabla}_m^2}{4m} \psi_m(\boldsymbol{r}) + \varepsilon \psi_m^\dagger(\boldsymbol{r}) \psi_m(\boldsymbol{r}) + \frac{1}{2} \lambda_{mm} \psi_m^\dagger(\boldsymbol{r}) \psi_m^\dagger(\boldsymbol{r}) \psi_m(\boldsymbol{r}) \psi_m(\boldsymbol{r}) \\ &+\lambda_{am} \psi_m^\dagger(\boldsymbol{r}) \psi_a^\dagger(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \psi_m(\boldsymbol{r}) + \boldsymbol{\alpha} \psi_m^\dagger(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \psi_a(\boldsymbol{r}) + \boldsymbol{\alpha} \psi_a^\dagger(\boldsymbol{r}) \psi_a^\dagger(\boldsymbol{r}) \psi_m(\boldsymbol{r}), \end{aligned}$$

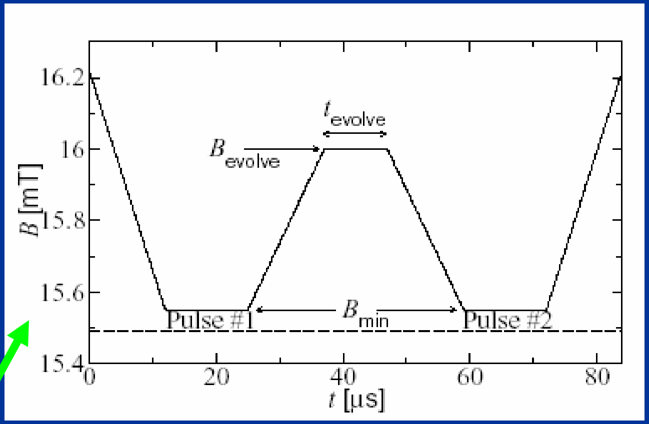
Working with contact couplings requires regularization and renormalization, which can be done in several different, but equivalent ways.

We will show that $H_{\rm eff}$ is over-determined.

Köhler, Gasenzer, Jullienne and Burnett, cond-mat/0305028.







Feshbach resonance

NB The size of the “Feshbach molecule” (closed channel state) is largely B-independent and smaller than the interparticle separation.

Some simple estimates in case $a > 0$ and $a \gg r_0$

wf in open channel at $r > r_0$

$$r\psi_1(r) = Ar_0\left[1 + O\left(\frac{r_0}{a}\right)\right]\exp\left(-\frac{r}{a}\right)$$

wfs in region $r < r_0$

$$r\psi_1(r) \approx r\psi_2(r) \approx r_0 A$$

Probability to find two atoms:

$$P(r < r_0) = \int_0^{r_0} r^2 dr\left[\psi_1(r)^2 + \psi_2(r)^2\right] \approx \frac{2A^2 r_0^3}{3} \quad \left(\text{or } \frac{A^2 r_0^3}{3} \text{ if oscillate}\right)$$

$$P(r > r_0) = \int_{r_0}^{\infty} r^2 dr \psi_1(r)^2 \approx \frac{A^2 a r_0^2}{2}$$

$$\boxed{\frac{P(r > r_0)}{P(r < r_0)} \propto \frac{a}{r_0} \quad >> \ 1}$$

Most of the time the two atoms spend at large separations,

$y_1(r)$ — open channel (dimer), $y_2(r)$ — closed channel (Feshbach molecule)

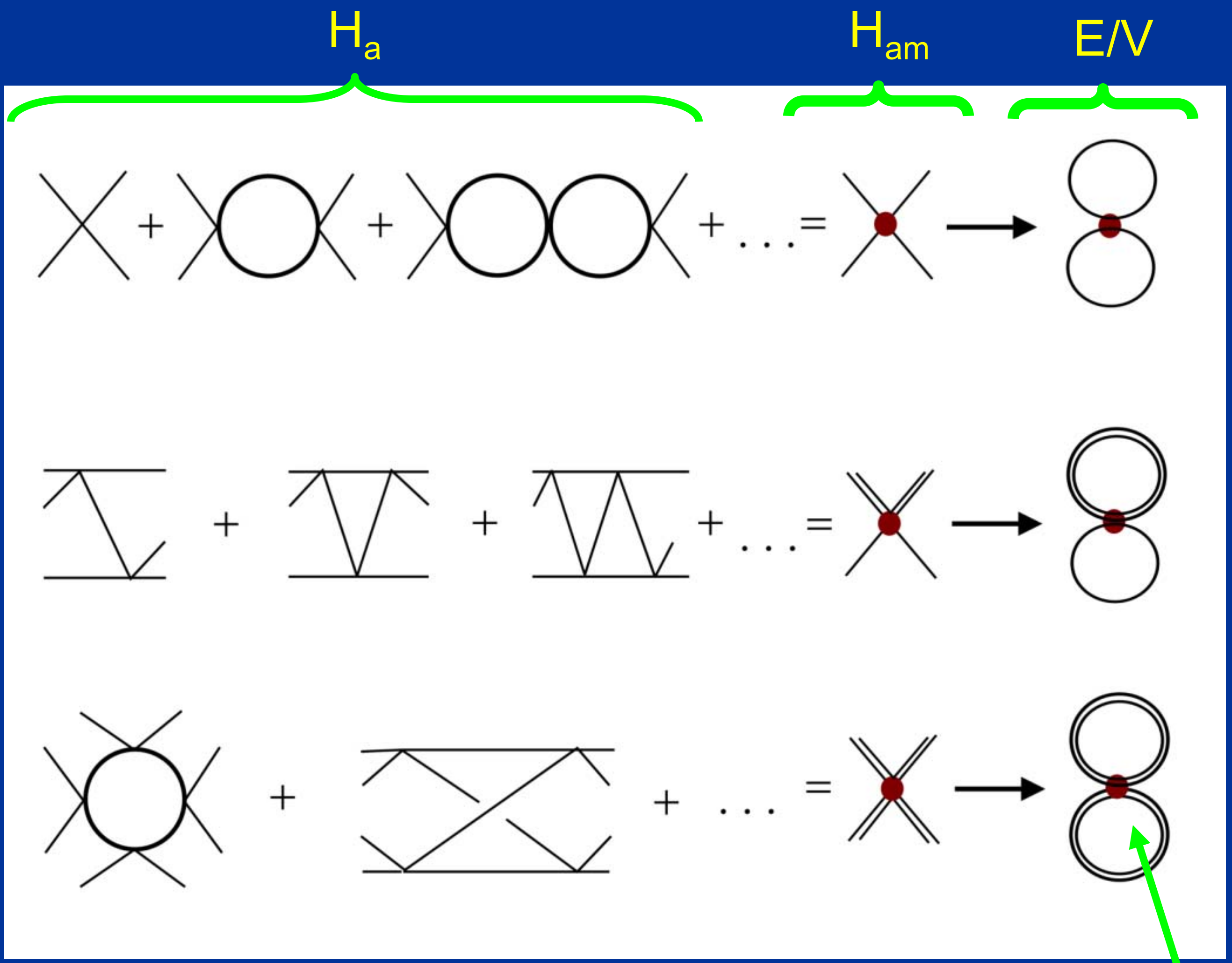
In order to develop our program we have at first to have a well defined procedure for constructing an effective Hamiltonian for interacting atoms and dimers starting from the "fundamental" Hamiltonian describing bare interacting atoms.

$$H_a = -\psi_a^+ \frac{\hbar^2\nabla^2}{2m}\psi_a + \frac{1}{2}\lambda_2\psi_a^+\psi_a^+\psi_a\psi_a + \frac{1}{3}\lambda_3\psi_a^+\psi_a^+\psi_a^+\psi_a\psi_a\psi_a$$

$$H_{am} = \psi_a^+\left(-\frac{\hbar^2\nabla^2}{2m}\right)\psi_a + \psi_m^+\left(-\frac{\hbar^2\nabla^2}{4m}+\varepsilon_2\right)\psi_m + \frac{1}{2}\lambda_{aa}\psi_a^+\psi_a^+\psi_a\psi_a + \lambda_{am}\psi_a^+\psi_m^+\psi_m\psi_a + \frac{1}{2}\lambda_{mm}\psi_m^+\psi_m^+\psi_m\psi_m$$

H_a is a low energy reduction of the "fundamental" Hamiltonian, λ_2 and λ_3 are determined by the scattering length a and a three-body characteristic (denoted below by a_3'). Interaction terms with derivatives are small as long as $kr_0 \ll 1$.

H_{am} is determined by the matching" to be briefly described below.



atom-atom vertex
(Lippmann-Schwinger eq.)

atom-dimer vertex
(Faddeev eqs.)

dimer-dimer vertex
(Yakubovsky eqs.)

Matching between the 2--, 3-- and 4--particle amplitudes computed with H_a and H_{am}.
Only diagrams containing l_2--vertices are shown.

The effective vertices thus defined (right side) can then be used to compute the ground state interaction energy in the leading order terms in an na^3 expansion, which is given by the diagrams after the arrows.

Fermi atoms

$$\lambda_{aa} = \frac{4\pi\hbar^2 a}{m}, \qquad \varepsilon_2 = -\frac{\hbar^2}{ma^2}$$

$$\lambda_{am} = \frac{3\pi\hbar^2 a_{am}}{m} = \frac{3.537\pi\hbar^2 a}{m}, \qquad a_{am} = 1.179a$$

$$\lambda_{mm} = \frac{2\pi\hbar^2 a_{mm}}{m} = \frac{1.2\pi\hbar^2 a}{m}, \qquad a_{mm} = 0.600(3)a$$

a_{am} was first computed first by Skornyakov and Ter-Martirosian (1957) who studied neutron-deuteron scattering.

a_{mm} was computed by Petrov (2003) and Fonseca (2003) .

Consider now a dilute mixture of fermionic atoms and (bosonic) dimers at temperatures smaller than the dimer binding energy ($a>0$ and $a \gg r_0$)

$$\frac{E}{V} = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m} n_f + \frac{\pi\hbar^2 a}{m} n_f^2 + \frac{3.537\pi\hbar^2 a}{m} n_f n_b + \frac{0.6\pi\hbar^2 a}{m} n_b^2 + \varepsilon_2 n_b + \text{corrections}$$

$$n_f = \frac{k_F^3}{3\pi^2}, \qquad \varepsilon_2 = -\frac{\hbar^2}{ma^2}$$

$$U_{fbf}(q,\omega) = U_{fb}^2 \frac{2n_b\varepsilon_q}{\hbar^2\omega^2 - \varepsilon_q(\varepsilon_q + 2n_b U_{bb})}$$

$$U_{bb} = \frac{4\pi\hbar^2 a_{bb}}{m_b}, \qquad \varepsilon_q = \frac{\hbar^2 q^2}{2m_b}$$

in coordinate representation at $\omega = 0$

$$U_{fbf}(r) = -\frac{U_{fb}^2}{U_{bb}} \frac{1}{4\pi\xi_b^2 r} \exp\left(-\frac{r}{\xi_b}\right)$$

$$\xi_b = \frac{\hbar}{2m_b s_s} = \frac{a_{bb}}{\sqrt{16\pi n_b a_{bb}^3}} >> a_{bb}, \qquad s_b^2 = \frac{n_b U_{bb}}{m_b}$$

One can show that pairing is typically weak!

Induced fermion-fermion interaction

Bardeen *et al.* (1967),
Heiselberg *et al.* (2000),
Bijlsma *et al.* (2000)
Viverit (2000),
Viverit and Giorgini (2000)

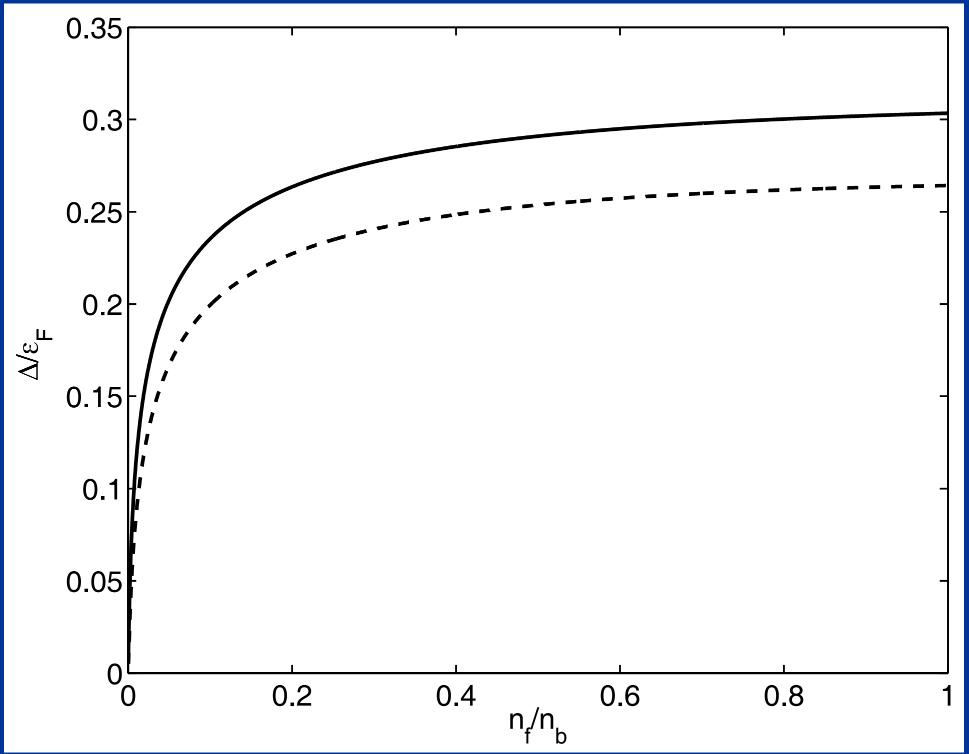
coherence/healing length and speed of sound

The atom-dimer mixture can potentially be a system where relatively strong coupling pairing can occur.

$$\Delta = \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left[\frac{2}{\pi k_F a} \left(1 - 5.21 \frac{\ln\left(1 + 4k_F^2 \xi_b^2\right)}{4k_F^2 \xi_b^2} \right)^{-1} \right]$$

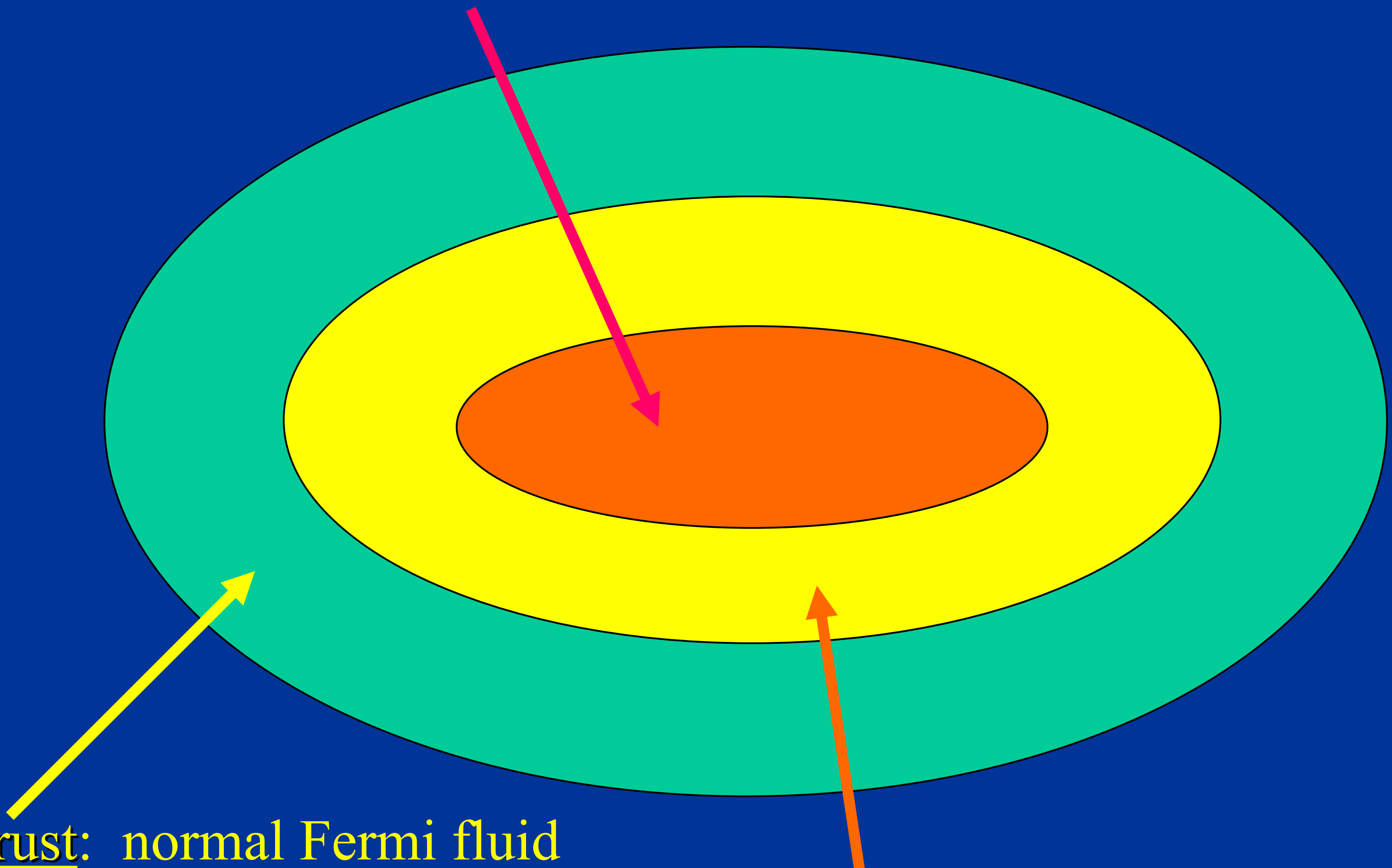
$a = n_b^{-1/3}/2.5$ (solid line)

$a = n_b^{-1/3}/3$ (dashed line)



How this atomic-molecular cloud really looks like in a trap?

Core: Molecular BEC



Crust: normal Fermi fluid

Mantle: Molecular BEC + Atomic Fermi Superfluid

Everything s made of one kind of atoms only, in two different hyperfine states.

All this follows by solving the Thomas-Fermi equations:

$$\left.\begin{aligned}&\frac{\hbar^2 k_F^2(\vec{r})}{2m}+\left(U_{ff}-\frac{U_{fb}^2}{U_{bb}}\right)n_f(\vec{r})=\mu_f-\frac{U_{fb}}{U_{bb}}\mu_b-\left(1-\frac{2U_{fb}}{U_{bb}}\right)V(\vec{r})\\ &n_b(\vec{r})=\frac{\mu_b-2V(\vec{r})-U_{fb}n(\vec{r})}{U_{bb}}\end{aligned}\right\}\ \mathrm{F\ \&\ B}$$

<u>The mantle</u>, the layer between the core and the crust.
Molecular BEC + Fermi BCS

$$\left.\frac{\hbar^2 k_F^2(\vec{r})}{2m}+U_{ff}n_f(\vec{r})=\mu_f-V(\vec{r})\right\}\ \mathrm{F}\quad \text{only for } V(\vec{R}_2)=const<V(\vec{r})<V(\vec{R}_3)=const$$

<u>The crust</u>, the outside layer.
Normal Fermi gas

$$\left.n_b(\vec{r})=\frac{\mu_b-2V(\vec{r})}{U_{bb}}\right\}$$

B only for $V(\vec{r})<V(\vec{R}_1)=const$

<u>The core</u>, the central region.
Molecular BEC

So far we dealt with the relatively simple cases, when the system is dilute, in the sense that the particles are on average at separations significantly larger than the interaction radius $\approx r_0$.

However, in these systems the scattering length a really plays the role of interaction radius.

What happens when |a|=¶ ?

Consider Bertsch's MBX challenge (1999): "Find the ground state of infinite homogeneous neutron matter interacting with an infinite scattering length."

$$r_0 \to 0 \quad << \quad \lambda_F \quad << \quad |a| \to \infty$$

- Carlson, Morales, Pandharipande and Ravenhall, PRC 68, 025802 (2003), with Green Function Monte Carlo (GFMC)

$$\frac{E_N}{N} = \alpha_N \frac{3}{5} \varepsilon_F, \qquad \alpha_N = 0.54$$

normal state

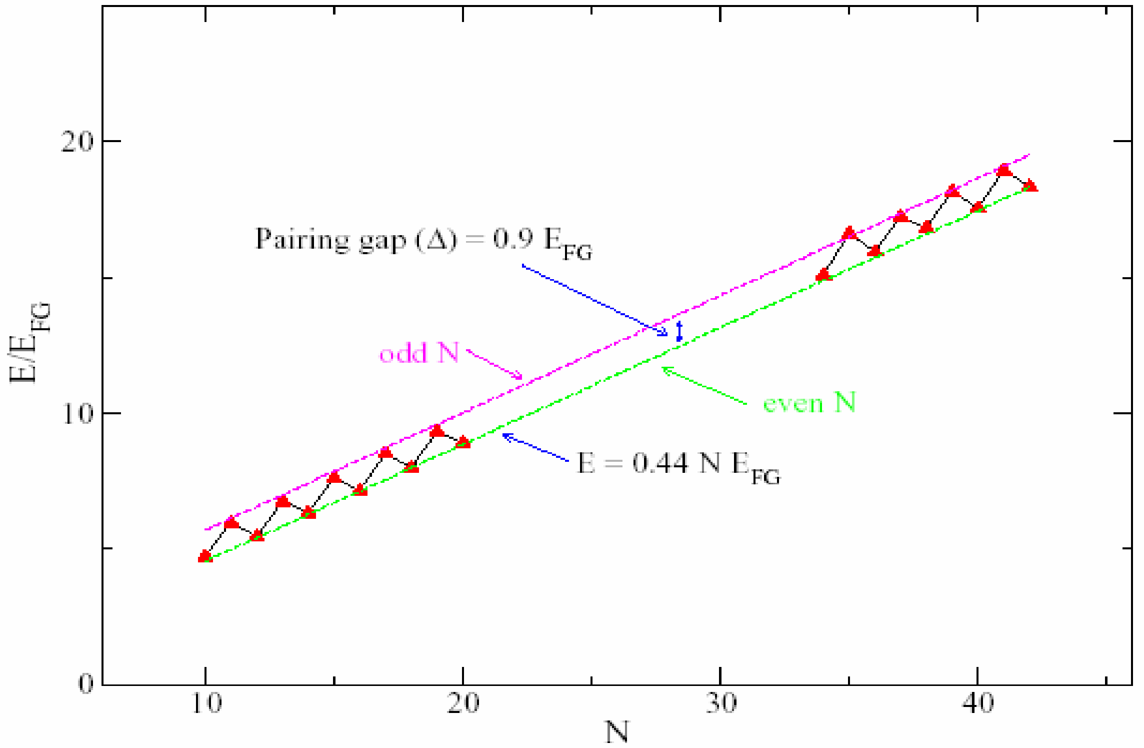
- Carlson, Chang, Pandharipande and Schmidt, PRL 91, 050401 (2003), with GFMC

$$\frac{E_S}{N} = \alpha_S \frac{3}{5} \varepsilon_F, \qquad \alpha_S = 0.44$$

superfluid state

This state is half the way from BCS→BEC crossover, the pairing correlations are in the strong coupling limit and HFB invalid again.

$$\Delta(2n+1) = E(2n+1) - \frac{1}{2}(E(2n) + E(2n+2))$$

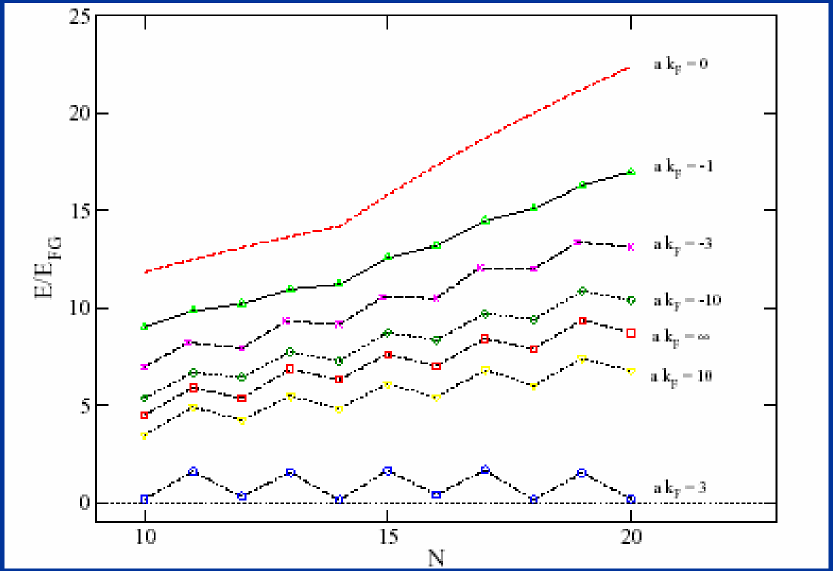


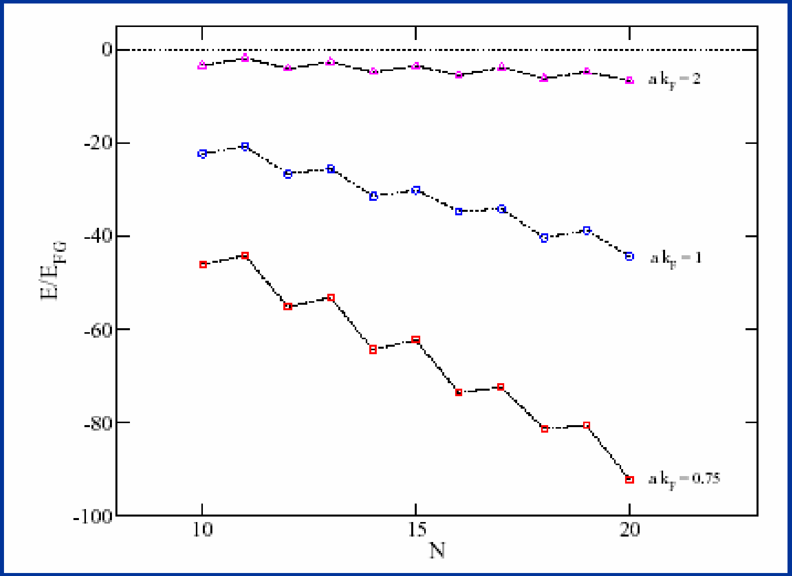
Result for $ak_F = -\infty$

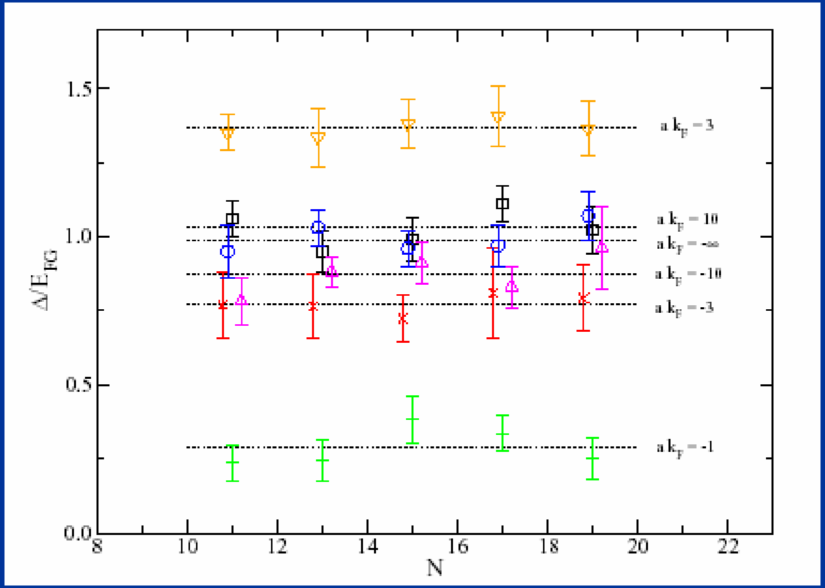
$$E_{FG} = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}$$

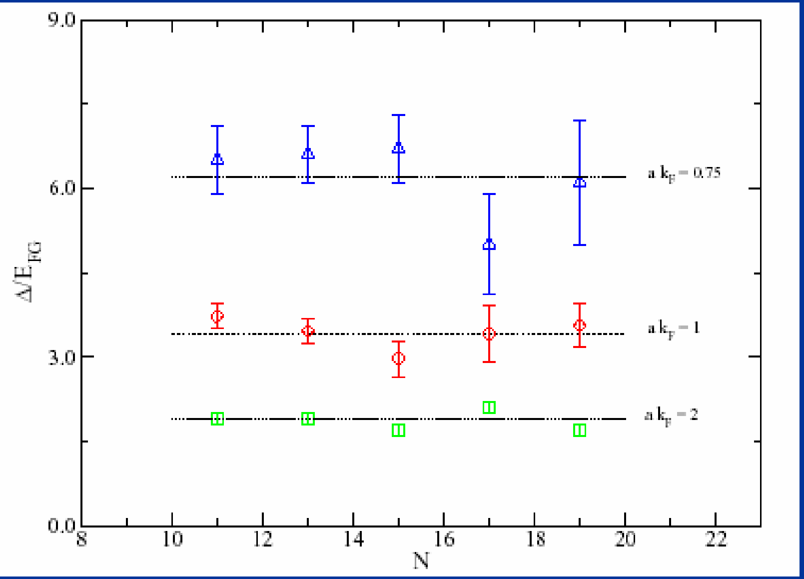
Green Function Monte Carlo with Fixed Nodes
J. Carlson, S.-Y. Chang, V. Pandharipande and K. Schmidt
private communication (2003)

Fixed node GFMC results, J. Carlson *et al.* (2003)



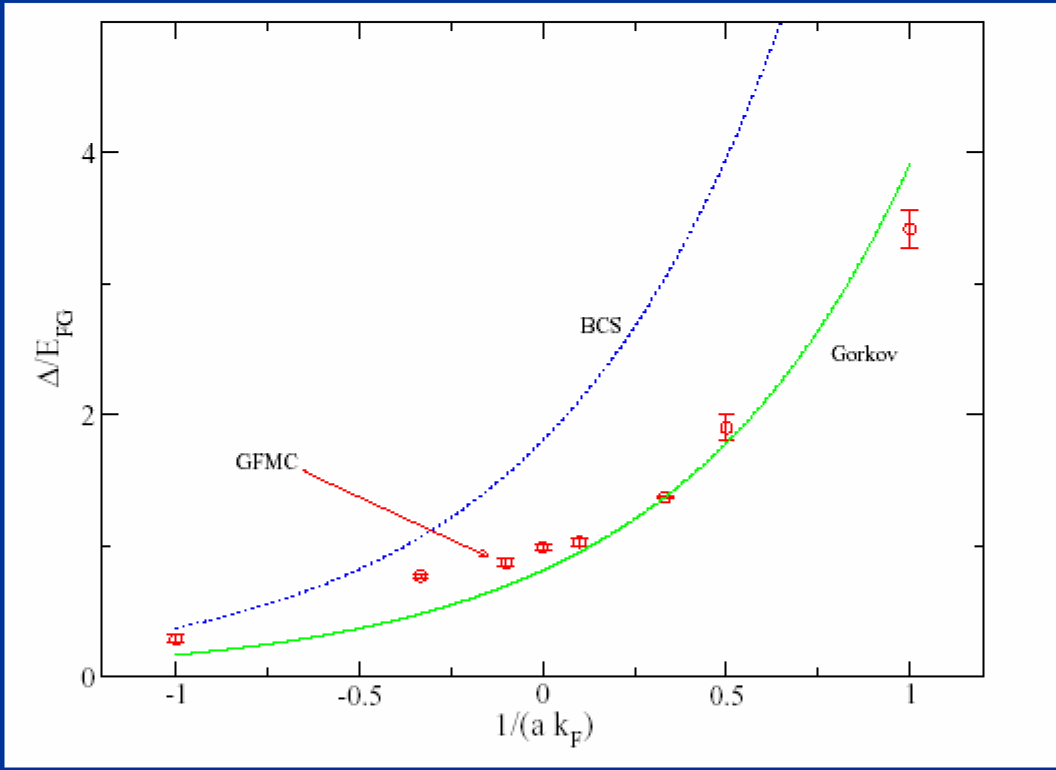






a<0 (apart from two cases)

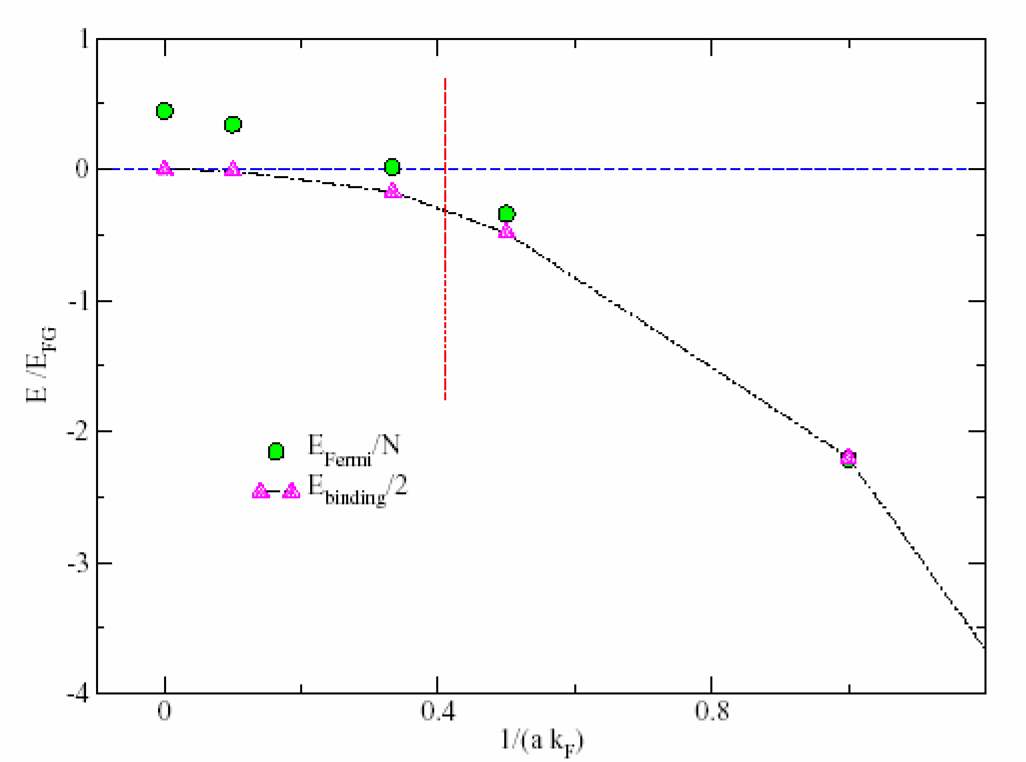
a>0



$$\Delta_{Gorkov} = \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left(\frac{\pi}{2k_F a}\right)$$

$$\Delta_{BCS} = \frac{8}{e^2} \frac{\hbar^2 k_F^2}{2m} \exp\left(\frac{\pi}{2k_F a}\right)$$

Fixed node GFMC results, J. Carlson *et al.* (2003)



Even though two atoms can bind,
there is no binding among dimers!

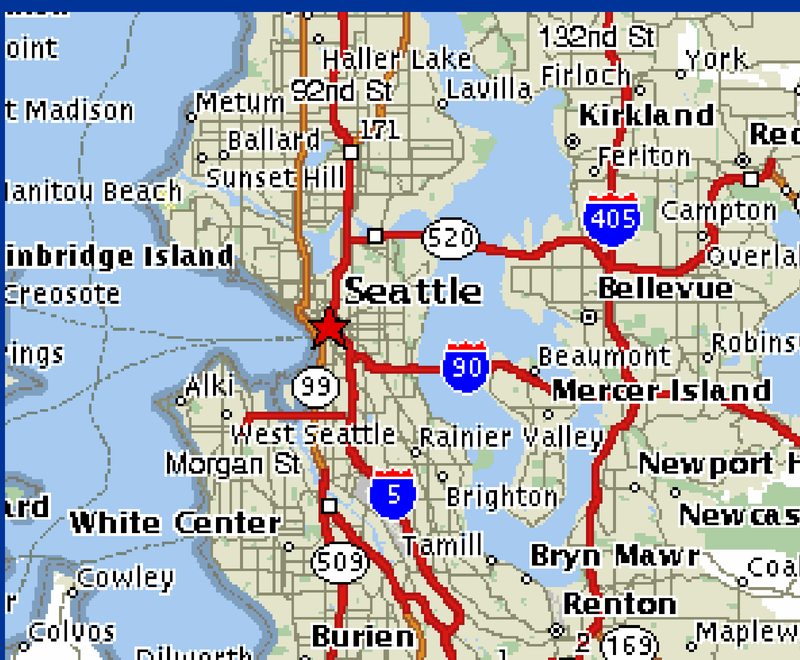
Fixed node GFMC results, J. Carlson *et al.* (2003)

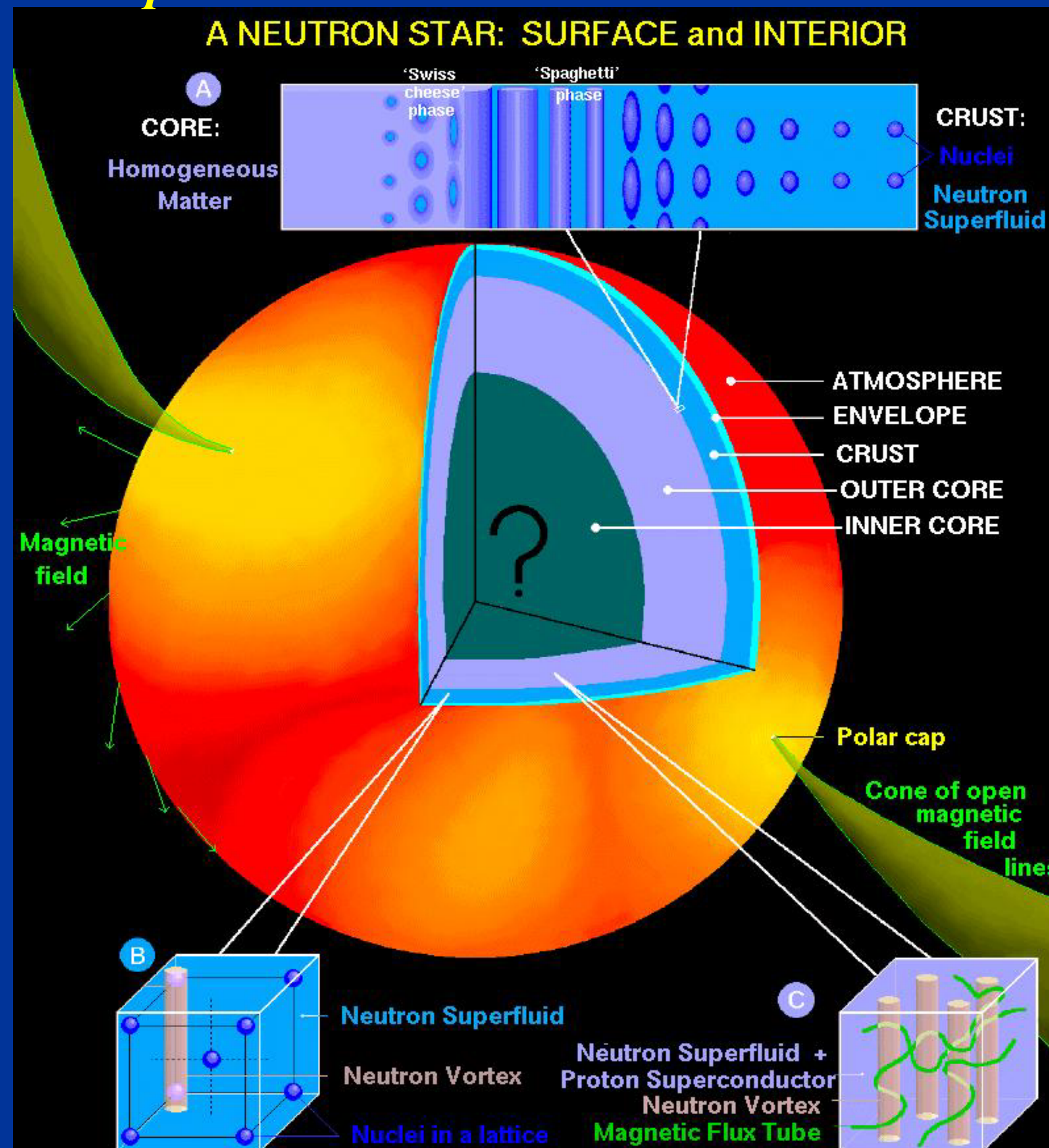
Anderson and Itoh, Nature, 1975

"Pulsar glitches and restlessness as a hard superfluidity phenomenon"

The crust of neutron stars is the only other place in the entire Universe where one can find solid matter, except planets.

- **A neutron star will cover the map at the bottom**
- **The mass is about 1.5 solar masses**
- **Density 10^{14} g/cm^3**





Author: Dany Page

Landau criterion for superflow stability
(flow without dissipation)

Consider a superfluid flowing in a pipe with velocity v_s:

$$E_0 + \frac{Nm\mathrm{v}_s^2}{2} < E_0 + \varepsilon_{\vec{p}} + \vec{\mathrm{v}}_s \cdot \vec{p} + \frac{Nm\mathrm{v}_s^2}{2} \Rightarrow \mathrm{v}_s < \frac{\varepsilon_{\vec{p}}}{p}$$

no internal excitations

One single quasi-particle excitation with momentum p

In the case of a Fermi superfluid this condition becomes

$$\mathrm{v}_s < \frac{\Delta}{\hbar k_F}$$

Density Functional Theory (DFT)
Hohenberg and Kohn, 1964

$$E_{gs} = \mathrm{E}[\rho(\vec{r})]$$

particle density

Local Density Approximation (LDA)
Kohn and Sham, 1965

$$E_{gs} = \int d^3 r \varepsilon[\rho(\vec{r}), \tau(\vec{r})]$$

$$\rho(\vec{r}) = \sum_{i=1}^{N} |\, \mathrm{v}_i(\vec{r})\,|^2$$

$$\tau(\vec{r}) = \sum_{i=1}^{N} |\, \vec{\nabla} \mathrm{v}_i(\vec{r})\,|^2$$

The energy density is typically determined in *ab initio* calculations of infinite homogeneous matter.

The main reason for the A in LDA is due to the inaccuracies of the gradient corrections.

Single-particle wave functions

Normal Fermi systems only!

LDA (Kohn-Sham) for superfluid fermi systems
(Bogoliubov-de Gennes equations)

$$E_{gs} = \int d^3 r \varepsilon(\rho(\vec{r}), \tau(\vec{r}), \nu(\vec{r}))$$

$$\rho(\vec{r}) = 2\sum_k |\mathrm{v}_\mathrm{k}(\vec{r})|^2, \quad \tau(\vec{r}) = 2\sum_k |\vec{\nabla}\mathrm{v}_\mathrm{k}(\vec{r})|^2$$

$$\nu(\vec{r}) = \sum_k \mathrm{u}_\mathrm{k}(\vec{r})\mathrm{v}_\mathrm{k}^*(\vec{r})$$

$$\begin{pmatrix} T + U(\vec{r}) - \mu & \Delta(\vec{r}) \\ \Delta^*(\vec{r}) & -(T + U(\vec{r}) - \mu) \end{pmatrix} \begin{pmatrix} \mathrm{u}_k(\vec{r}) \\ \mathrm{v}_k(\vec{r}) \end{pmatrix} = E_k \begin{pmatrix} \mathrm{u}_k(\vec{r}) \\ \mathrm{v}_k(\vec{r}) \end{pmatrix}$$

Mean-field and pairing field are both local fields!

(for sake of simplicity spin degrees of freedom are not shown)

There is a little problem! The pairing field D diverges.

Nature of the problem

$$\nu(\vec{r}_1,\vec{r}_2) = \sum_{E_k>0} \mathrm{v}_\mathrm{k}^*(\vec{r}_1)\mathrm{u}_\mathrm{k}(\vec{r}_2) \propto \frac{1}{|\vec{r}_1-\vec{r}_2|}$$

at small separations

$$\Delta(\vec{r}_1,\vec{r}_2) = \frac{1}{2}V(\vec{r}_1,\vec{r}_2)\nu(\vec{r}_1,\vec{r}_2)$$

It is easier to show how this singularity appears
in infinite homogeneous matter (BCS model)

$$\mathrm{v}_\mathrm{k}(\vec{r}_1) = \mathrm{v}_\mathrm{k}\exp(i\vec{k}\cdot\vec{r}_1), \qquad \mathrm{u}_\mathrm{k}(\vec{r}_2) = \mathrm{u}_\mathrm{k}\exp(i\vec{k}\cdot\vec{r}_2)$$

$$\mathrm{v}_\mathrm{k}^2 = \frac{1}{2}\left(1-\frac{\varepsilon_\mathrm{k}-\mu}{\sqrt{(\varepsilon_\mathrm{k}-\mu)^2+\Delta^2}}\right), \quad \mathrm{u}_\mathrm{k}^2+\mathrm{v}_\mathrm{k}^2 = 1, \quad \varepsilon_\mathrm{k} = \frac{\hbar^2\vec{k}^2}{2m}+U, \quad \Delta = \frac{\hbar^2\delta}{2m}$$

$$\nu(r) = \frac{\Delta m}{2\pi^2\hbar^2}\int_0^\infty dk\,\frac{\sin(kr)}{kr}\frac{k^2}{\sqrt{(k^2-k_F^2)^2+\delta^2}}, \qquad r = |\vec{r}_1-\vec{r}_2|$$

Pseudo-potential approach

(appropriate for very slow particles, very transparent but somewhat difficult to improve)

Lenz (1927), Fermi (1931), Blatt and Weiskopf (1952)
Lee, Huang and Yang (1957)

$$-\frac{\hbar^2 \Delta_{\vec{r}}}{m}\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\psi(\vec{r}), \quad V(\vec{r}) \approx 0 \text{ if } r > R$$

$$\psi(\vec{r}) = \exp(i\vec{k}\cdot\vec{r}) + \frac{f}{r}\exp(ikr) \approx 1 + \frac{f}{r} + \ldots \approx 1 - \frac{a}{r} + O(kr)$$

$$f^{-1} = -\frac{1}{a} + \frac{1}{2}r_0 k^2 - ik, \qquad g = \frac{4\pi\hbar^2 a}{m(1+ika)} + \ldots$$

$$\text{if } kr_0 << 1 \quad \text{then} \quad V(\vec{r})\psi(\vec{r}) \Rightarrow g\delta(\vec{r})\frac{\partial}{\partial r}\left[r\psi(\vec{r})\right]$$

$$\text{Example}: \quad \psi(\vec{r}) = \frac{A}{r} + B + \ldots \Rightarrow \delta(\vec{r})\frac{\partial}{\partial r}\left[r\psi(\vec{r})\right] = \delta(\vec{r})B$$

The SLDA (renormalized) equations

$$E_{gs} = \int d^3r \{\varepsilon_N[\rho(\vec{r}),\tau(\vec{r})] + \varepsilon_S[\rho(\vec{r}),\nu(\vec{r})]\}$$

$$\varepsilon_S[\rho(\vec{r}),\nu(\vec{r})] \overset{def}{=} -\Delta(\vec{r})\nu_c(\vec{r}) = g_{\text{eff}}(\vec{r})|\nu_c(\vec{r})|^2$$

$$\begin{cases} [h(\vec{r})-\mu]\mathrm{u}_i(\vec{r}) + \Delta(\vec{r})\mathrm{v}_i(\vec{r}) = E_i\mathrm{u}_i(\vec{r}) \\ \Delta^*(\vec{r})\mathrm{u}_i(\vec{r}) - [h(\vec{r})-\mu]\mathrm{v}_i(\vec{r}) = E_i\mathrm{v}_i(\vec{r}) \end{cases} \qquad \begin{cases} h(\vec{r}) = -\vec{\nabla}\dfrac{\hbar^2}{2m(\vec{r})}\vec{\nabla} + U(\vec{r}) \\ \Delta(\vec{r}) = -g_{\text{eff}}(\vec{r})\nu_c(\vec{r}) \end{cases}$$

$$\boxed{\frac{1}{g_{\mathit{eff}}(\vec{r})} = \frac{1}{g[\rho(\vec{r})]} - \frac{m(\vec{r})k_c(\vec{r})}{2\pi^2\hbar^2}\left\{1 - \frac{k_F(\vec{r})}{2k_c(\vec{r})}\ln\frac{k_c(\vec{r})+k_F(\vec{r})}{k_c(\vec{r})-k_F(\vec{r})}\right\}}$$

$$\rho_c(\vec{r}) = 2\sum_{E_i\geq 0}^{E_c}|\mathrm{v}_i(\vec{r})|^2, \qquad \nu_c(\vec{r}) = \sum_{E_i\geq 0}^{E_c}\mathrm{v}_i^*(\vec{r})\mathrm{u}_i(\vec{r})$$

$$E_c + \mu = \frac{\hbar^2 k_c^2(\vec{r})}{2m(\vec{r})} + U(\vec{r}), \qquad \mu = \frac{\hbar^2 k_F^2(\vec{r})}{2m(\vec{r})} + U(\vec{r})$$

Position and momentum dependent running coupling constant
Observables are (obviously) independent of cut-off energy (when chosen properly).

How can one determine the density dependence of the coupling constant g? I know two methods.

$$\varepsilon_S\left[\rho(\vec{r}),\nu(\vec{r})\right]=g[\rho(\vec{r})]\,|\,\nu(\vec{r})\,|^2$$ ← Superfluid contribution to EDF

✓ In homogeneous low density matter one can compute the pairing gap as a function of the density. NB this is not a BCS/HFB result!

$$\Delta=\left(\frac{2}{e}\right)^{7/3}\frac{\hbar^2k_F^2}{2m}\exp\left(\frac{\pi}{2k_Fa}\right)$$

✓ One computes also the energy of the normal and superfluid phases as a function of density, as was recently done by Carlson et al, Phys. Rev. Lett. 91, 050401 (2003) for a Fermi system interacting with an infinite scattering length (Bertsch's MBX 1999 challenge)

In both cases one can extract from these results the superfluid contribution to the LDA energy density functional in a straight forward manner.

Vortex in neutron matter

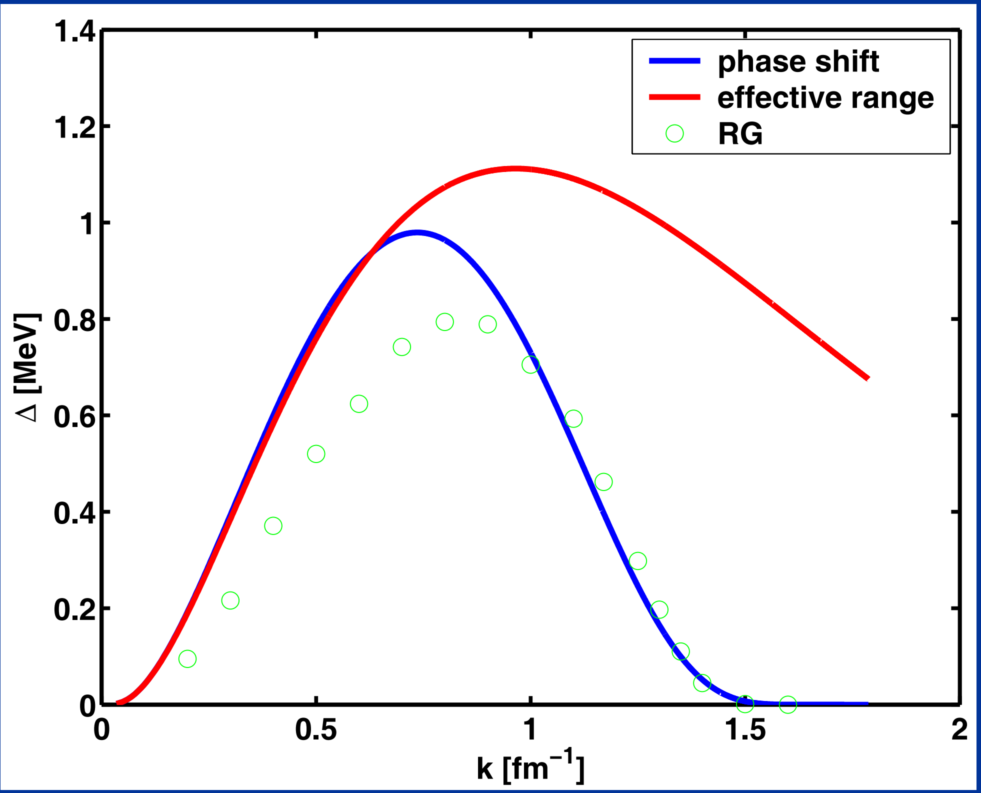
$$\begin{pmatrix} \mathrm{u}_{\alpha\,\mathrm{kn}}(\vec{r}) \\ \mathrm{v}_{\alpha\,\mathrm{kn}}(\vec{r}) \end{pmatrix} = \begin{pmatrix} \mathrm{u}_{\alpha}(r)\exp[i(n+1/2)\phi - ikz] \\ \mathrm{v}_{\alpha}(r)\exp[i(n-1/2)\phi - ikz] \end{pmatrix}, \quad \text{n - half-integer}$$

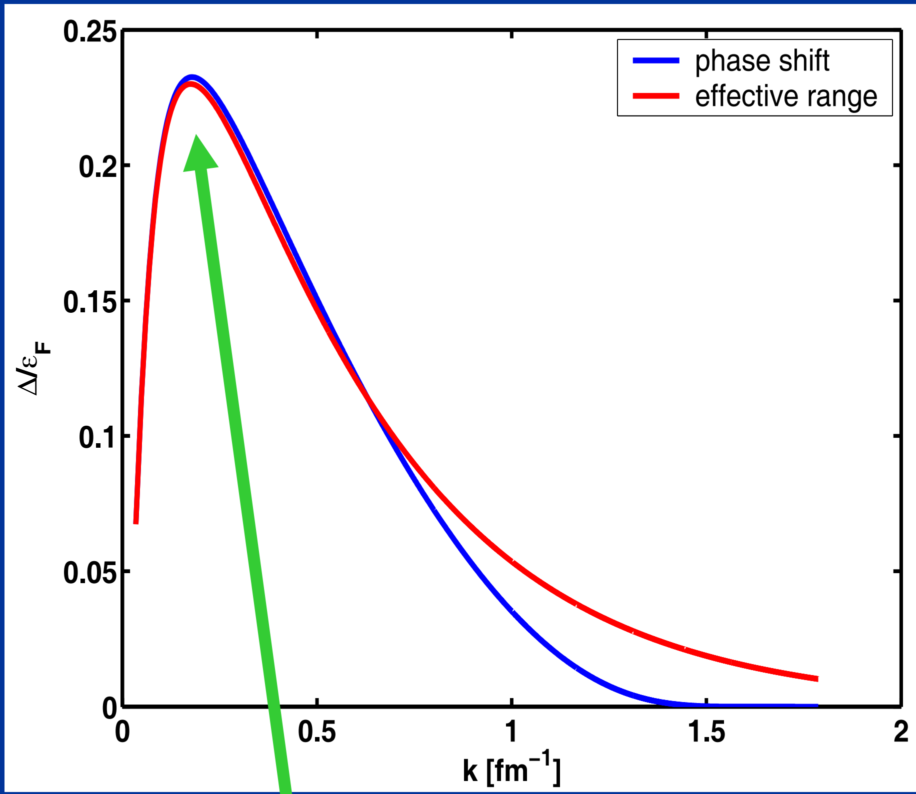
$$\Delta(\vec{r}) = \Delta(r)\exp(i\phi), \qquad \vec{r} = (r,\phi,z) \ \text{[cyllindrical coordinates]}$$

Oz - vortex symmetry axis

Ideal vortex, Onsager's quantization (one $\hbar$ per Cooper pair)

$$\vec{\mathrm{V}}_{\mathrm{v}}(\vec{r}) = \frac{\hbar}{2mr^2}(y,-x,0) \ \Leftarrow \ \frac{1}{2\pi}\oint_C \vec{\mathrm{V}}_{\mathrm{v}}(\vec{r})\cdot d\vec{r} = \frac{\hbar}{2m}$$





NB! Extremely high relative T_c

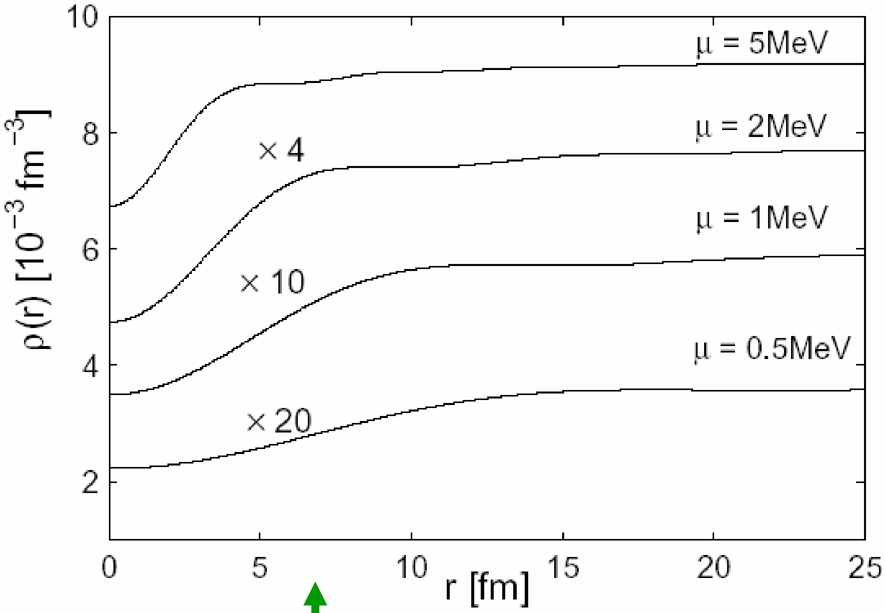
$$\Delta = \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left[-\frac{\pi}{2\tan\delta(k_F)}\right]$$

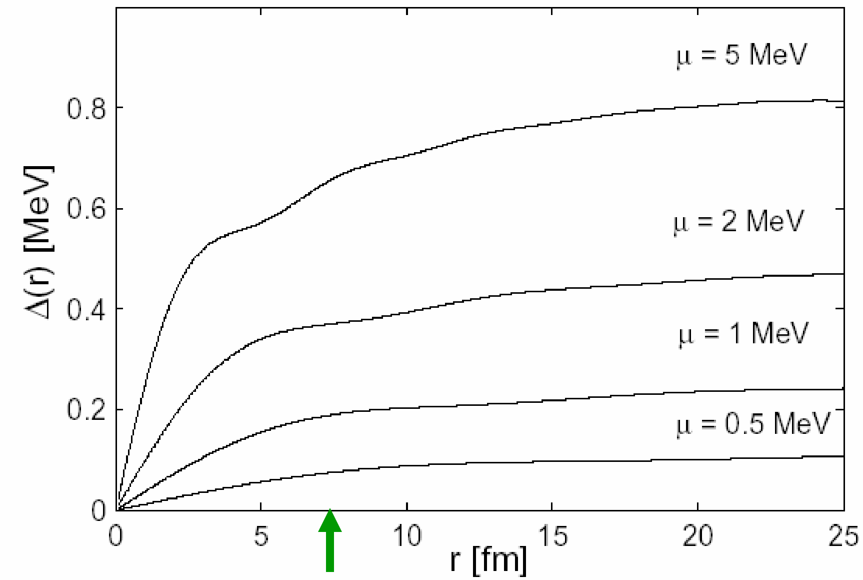
Corrected Emery formula (1960)

NN-phase shift

RG- renormalization group calculation

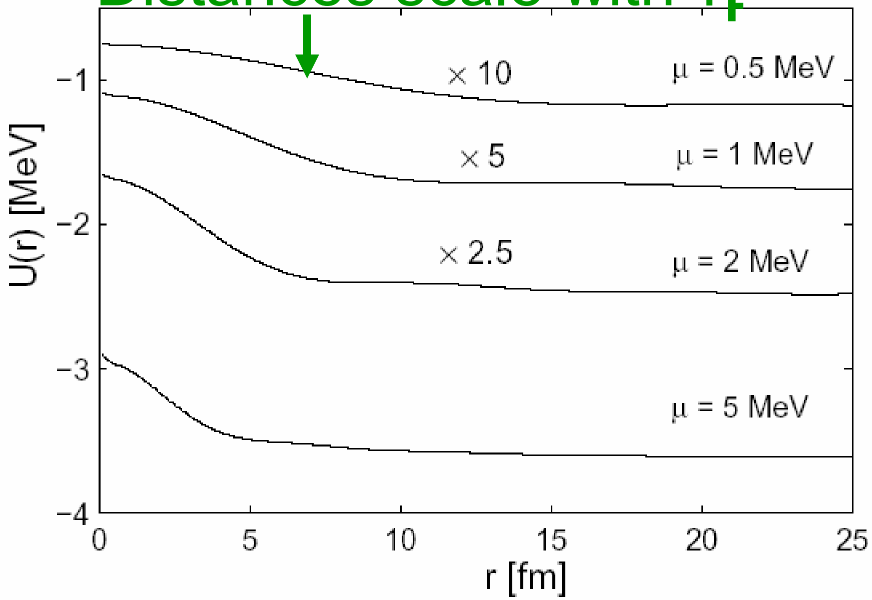
Schwenk, Friman, Brown, Nucl. Phys. A713, 191 (2003)

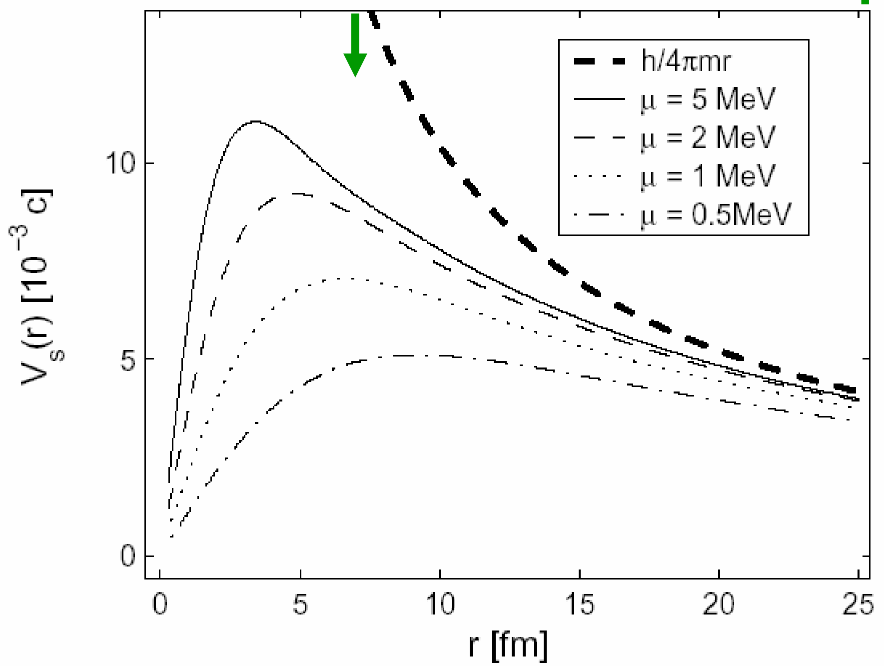




Distances scale with x>> l_F

Distances scale with l_F





Dramatic structural changes of the vortex state naturally lead to significant changes in the energy balance of a neutron star

- $\frac{v_S}{v_F} \leq \left.\frac{\Delta}{2\varepsilon_F}\right|_{max} \approx 0.12,$ extremely fast vortical motion,

$$\frac{\lambda_F}{\xi} \propto \frac{\Delta}{\varepsilon_F}$$

- In low density region $\varepsilon(\rho_{out})\rho_{out} > \varepsilon(\rho_{in})\rho_{in}$

which thus leads to a large anti - pinning energy $E_{pin}^V > 0$:

$$E_{pin}^V = [\varepsilon(\rho_{out})\rho_{out} - \varepsilon(\rho_{in})\rho_{in}]V$$

- The energy per unit length is going to be changed dramatically when compared to previous estimates, by

$$\frac{\Delta E_{vortex}}{L} \approx [\varepsilon(\rho_{out})\rho_{out} - \varepsilon(\rho_{in})\rho_{in}]\pi R^2$$

- Specific heat, transport properties are expected to significantly affected as well.

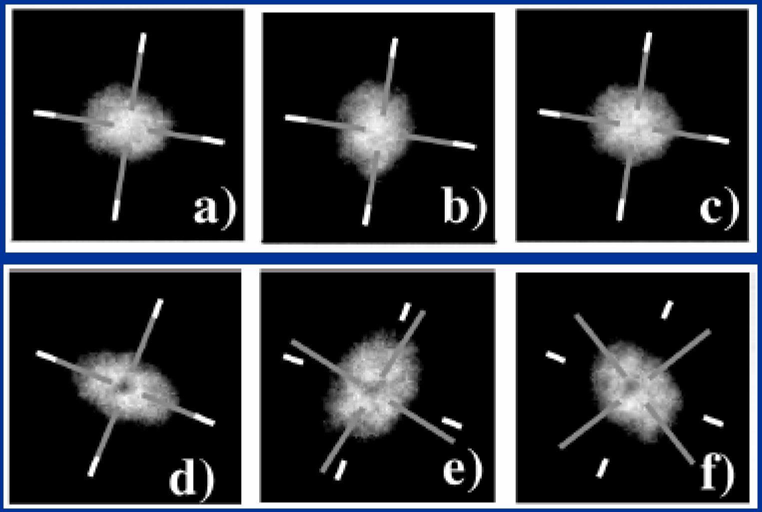
Some similar conclusions have been reached recently also by Donati and Pizzochero, Phys. Rev. Lett. 90, 211101 (2003).

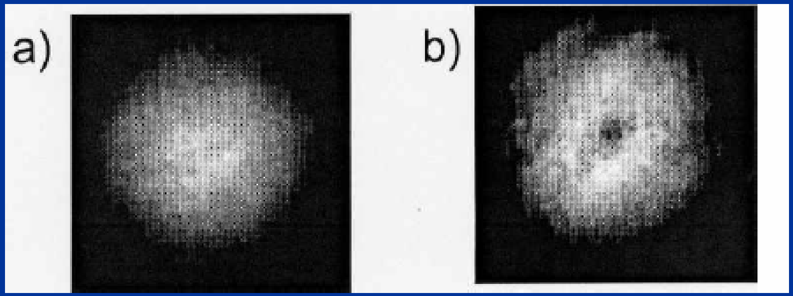
Vortices in dilute atomic Fermi systems in traps

- ✓ 1995 BEC was observed.
- ✓ 2000 vortices in BEC were created, thus BEC confirmed un-ambiguously.
- ✓ In 1999 DeMarco and Jin created a degenerate atomic Fermi gas.
- ✓ 2002 O'Hara, Hammer, Gehm, Granada and Thomas observed expansion of a Fermi cloud compatible with the existence of a superfluid fermionic phase.

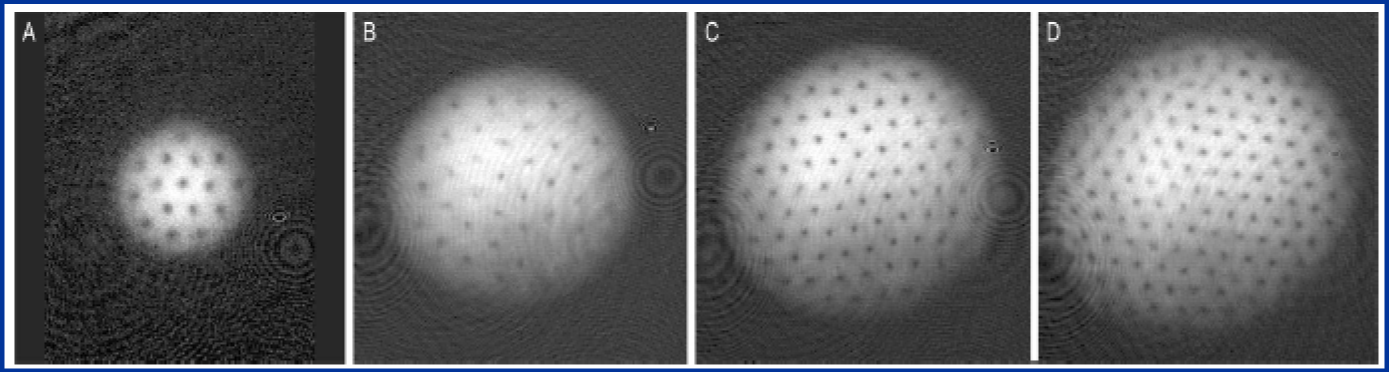
Observation of stable/quantized vortices in Fermi systems would provide the ultimate and most spectacular proof for the existence of a Fermionic superfluid phase.

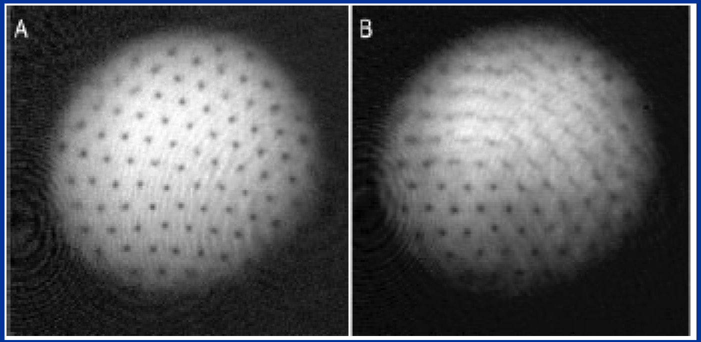
BEC Vortices



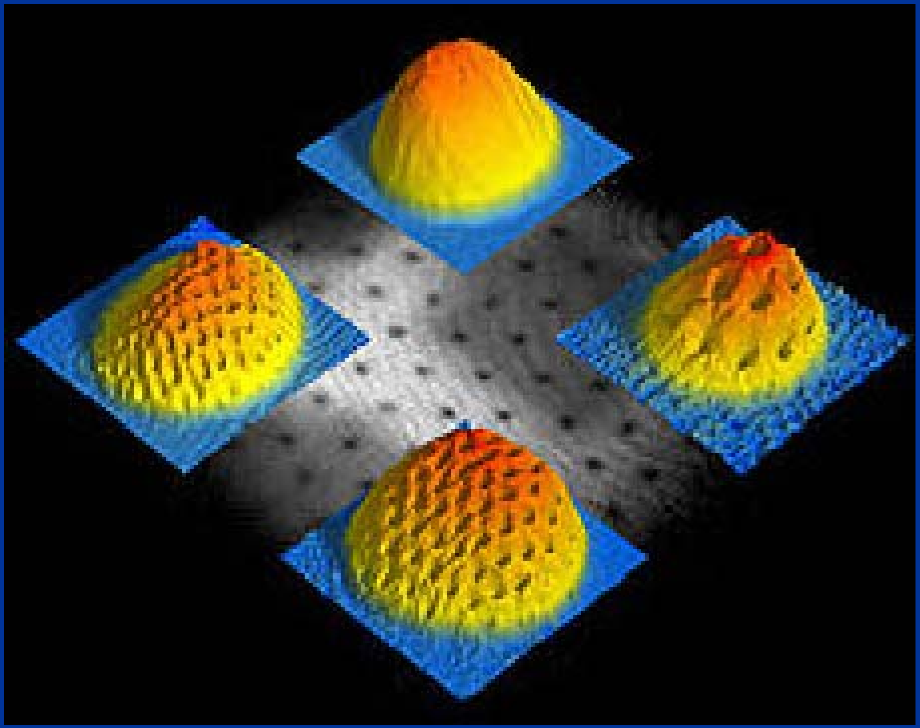


K.W. Madison *et al*, J. Mod. Opt. 47, 2715 (2000),
F. Chevy *et al*, Phys. Rev. Lett. 85, 2223 (2000).





J.R. Abo-Shaeer *et al*, Science, 285, 1703 (2001)



Why would one study vortices in neutral Fermi superfluids?

They are perhaps just about the only phenomenon in which one can have a true stable superflow!

How can one put in evidence a vortex in a Fermi superfluid?

Hard to see, since density changes are not expected, unlike the case of a Bose superfluid.

What we learned from the structure of a vortex in low density neutron matter can help however.

If the gap is not small one can expect a noticeable density depletion along the vortex core, and the bigger the gap the bigger the depletion.

One can change the magnitude of the gap by altering the scattering length between two atoms with magnetic fields by means of a Feshbach resonance.

Now one can construct an LDA functional to describe this new state of Fermionic matter

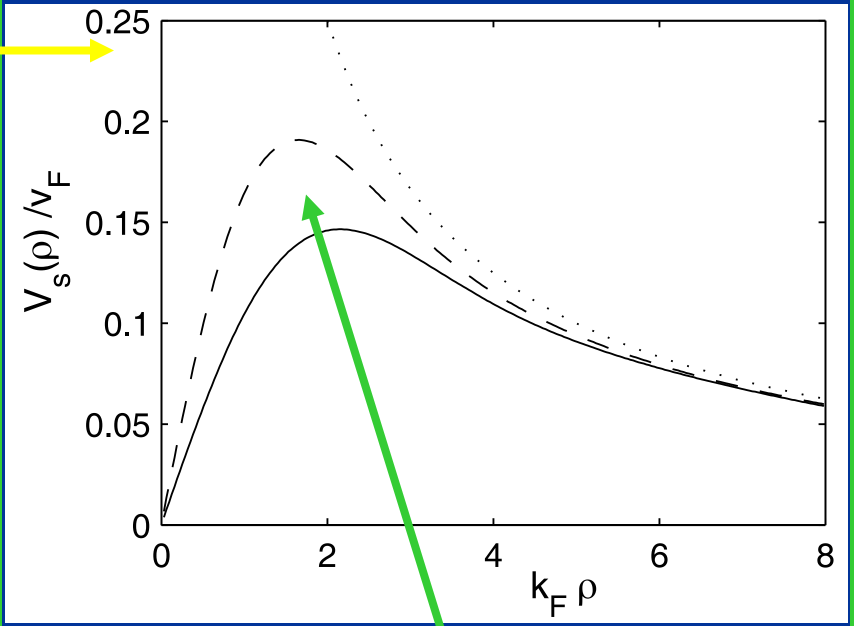
$$\mathcal{E}(\boldsymbol{r})n(\boldsymbol{r}) = \frac{\hbar^2}{m}\left[\frac{m}{2m^*}\tau(\boldsymbol{r}) + \beta n(\boldsymbol{r})^{5/3} + \gamma\frac{|\nu(\boldsymbol{r})|^2}{n(\boldsymbol{r})^{1/3}}\right],$$

$$n(\boldsymbol{r}) = \sum_\alpha |v_\alpha(\boldsymbol{r})|^2, \quad \tau(\boldsymbol{r}) = \sum_\alpha |\boldsymbol{\nabla} v_\alpha(\boldsymbol{r})|^2,$$

$$\nu(\boldsymbol{r}) = \sum_\alpha v_\alpha^*(\boldsymbol{r})u_\alpha(\boldsymbol{r}).$$

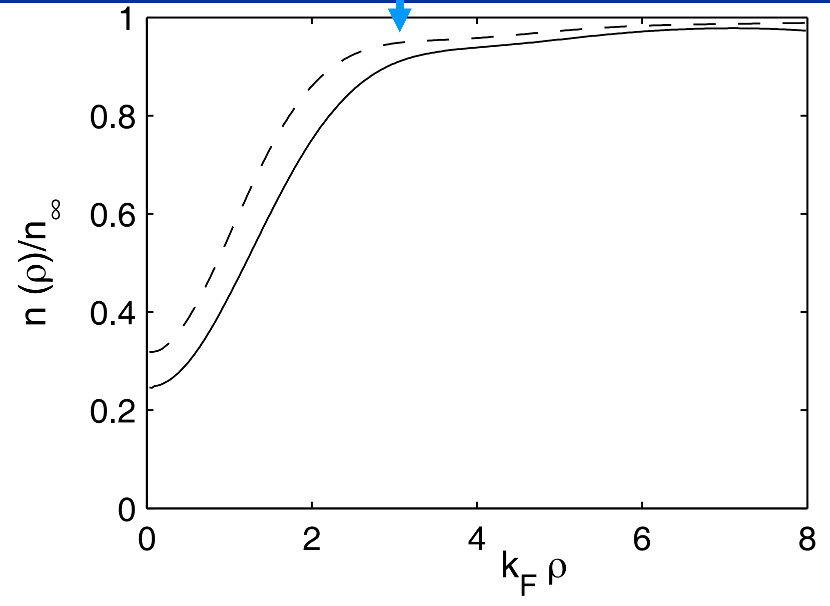
- This form is not unique, as one can have either: $b=0$ (set I) or $b\neq0$ and $m^*=m$ (set II).
- Gradient terms not determined yet (expected minor role).

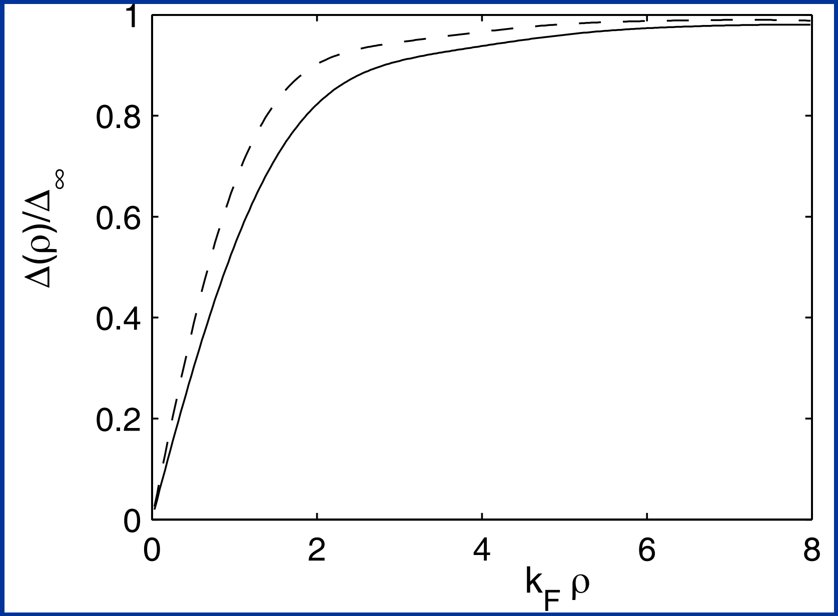
Solid lines - parameter set I,
Dashed lines for parameter set II
Dots – velocity profile for ideal vortex



The depletion along the vortex core is reminiscent of the corresponding density depletion in the case of a vortex in a Bose superfluid, when the density vanishes exactly along the axis for 100% BEC.

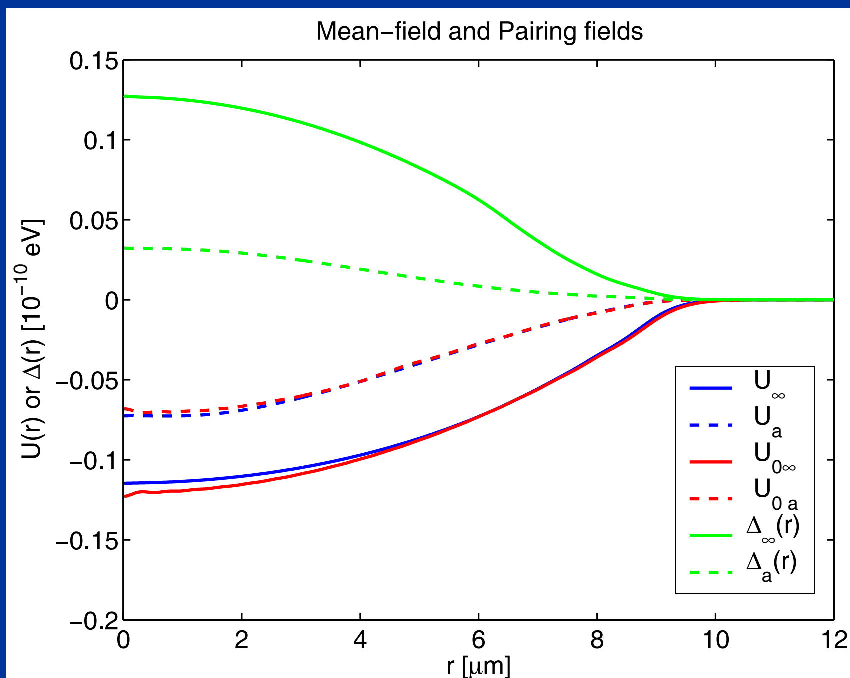
Extremely fast quantum vortical motion!

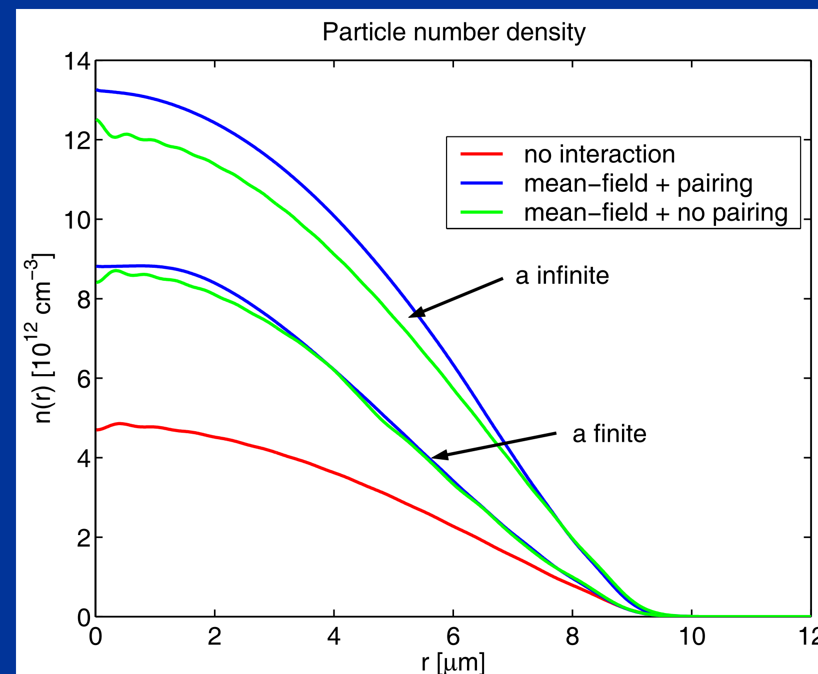




40K (Fermi) atoms in a spherical harmonic trap

Effect of interaction, with and without weak and strong pairing correlations with fixed chemical potential.

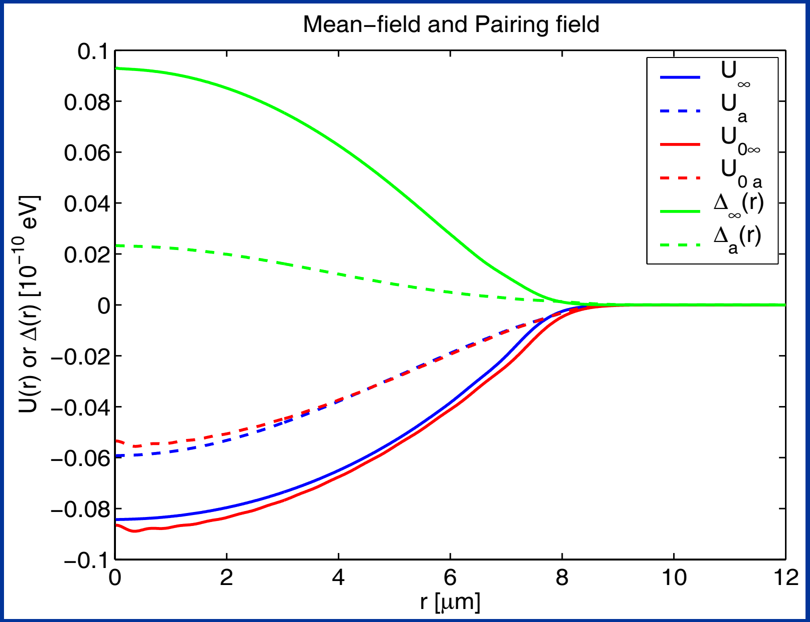


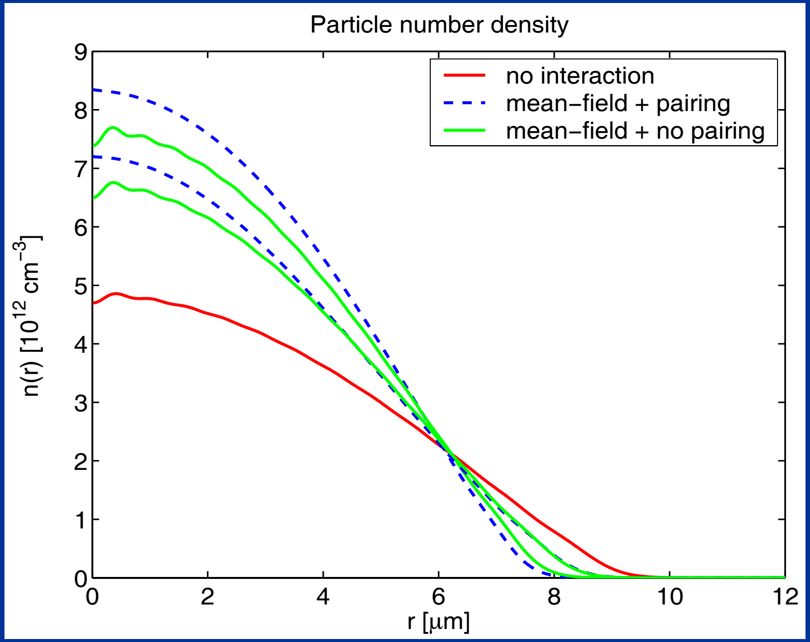


m= 0.14µ10^{-10}eV, ħw=0.568 µ10^{-12}eV,
a = -12.63nm (when finite)

40K (Fermi) atoms in a spherical harmonic trap

Effect of interaction, with and without weak and strong pairing correlations with fixed particle number, N = 5200.





ħw=0.568 μ10^{-12}eV, a = -12.63nm (when finite)

Conclusions:

✓ The field of dilute atomic systems is going to be for many years to come one of the most exciting fields in physics, with lots surprises at every corner.