Part A  - Comments on the papers of Burovski et al.

Part B  - On Superfluid Properties of Asymmetric Dilute Fermi Systems
Comments on papers of

E. Burovski, N. Prokof’ev, B. Svistunov and M. Troyer


by A. Bulgac, J.E. Drut and P. Magierski
Determinant Diagrammatic Monte Carlo

The partition function is expanded in a power series in the interaction.

\[
Z = \sum_{n=0}^{\infty} (-U)^n \sum_{x_1 \ldots x_n} \int_{0<\tau_1<\tau_2<\ldots<\beta} \prod_{j=1}^{n} d\tau_j \operatorname{Tr} \left[ e^{-\beta H_0} \prod_{j=1}^{n} c_{\uparrow}(x_j \tau_j) c_{\downarrow}(x_j \tau_j) c_{\downarrow}^\dagger(x_j \tau_j) c_{\uparrow}^\dagger(x_j \tau_j) \right]
\]

It is notoriously known that the pairing gap is a non-analytical function of the interaction strength and that no power expansion of pairing gap exists. It is completely unclear why an expansion of this type should describe correctly the pairing properties of a Fermi gas at unitarity.
Extrapolation prescription used by Burovski et al.

Argument based on comparing the continuum and lattice $T$-matrix at unitarity.

\[
\frac{\Gamma(\xi, p) - \Gamma_{\text{cont}}(\xi, p)}{\Gamma(\xi, p)} \sim \nu^{1/3}
\]

However:

- $T$-matrix is governed by the scattering length, which is infinite at unitarity
- many Fermion system is governed by Fermi wave length, which is finite at unitarity
- large error bars and clear non-linear dependence
Energy at critical temperature
Notice the strong size dependence!

It is clearly seen that the presence of the lattice suppresses the critical temperature considerably, nearly by a factor of 4, depending on the filling factor. Strong dependence of $T_c$ on $\nu$ is in apparent contradiction with Ref. [24], which claims weak or no $\nu$-dependence. This disagreement might be due to the difference in the single-particle spectra $\varepsilon_k$ used: Ref. [24] employs the parabolic spectrum with spherically symmetric...
Single particle kinetic energy and occupation probabilities

We have found that in order to have a reasonable accuracy, the highest momentum states should have an occupation probability of less than 0.01!

Notice the large difference, and the spread of values, between the kinetic energy of the free particle and the kinetic energy in the Hubbard model, even at half the maximum momentum (one quarter of the maximum energy).

Occupation probabilities are from our results, where we treat the kinetic energy exactly. Burovski et al. have not considered this quantity. Since both groups have similar filling factors, we expect large deviations from continuum limit.
Finite size scaling

Burovski et al.

\[ T_c = 0.157(7) \, \varepsilon_F \]

These authors however never displayed the order parameter as function of T and we have to assume that the phase transition really exists in their simulation.

Bulgac et al.

(new, preliminary results)

\[ T_c \approx 0.23(3) \, \varepsilon_F \]

Value consistent with behavior of other thermodynamic quantities
Preliminary new data! Finite size scaling consistent with our previously determined value

\[ T_c \approx 0.23 \varepsilon_F \]

temperature. The intersection of scaled curves turns out to be inconsistent with the estimate for \( T_c \) derived from the caloric curve inspection.

Burovski et al.

However, see next slide
Two-body correlation function, condensate fraction

System dependent scale
Here the Fermi wavelength $\frac{L}{2}$
Green symbols, $T=0$ results of Astrakharchick et al, PRL 95, 230405 (2005)

Power law critical scaling expected between the Fermi wavelength and $\frac{L}{2}$ !!!  Clearly $L$ is in all cases too small!
The energy of a Fermi gas at unitarity in a trap at the critical temperature is determined experimentally, even though T is not.

\[
\frac{E(T_c)}{E(0)} - 1 = \begin{cases} 
0.8 & \text{Kinast et al, Science 307, 1296(2005)} \\
0.2 & \text{Burovski et al. cond-mat/0605350} \\
0.6 & \text{Bulgac et al. PRL 96, 090404 (2006)} 
\end{cases}
\]

Our conclusions based on our results

Below the transition temperature the system behaves as a free condensed Bose gas (!), which is superfluid at the same time! No thermodynamic hint of Fermionic degrees of freedom!

Above the critical temperature one observes the thermodynamic behavior of a free Fermi gas! No thermodynamic trace of bosonic degrees of freedom!

New type of fermionic superfluid.
Part B
On Superfluid Properties of Asymmetric Dilute Fermi Systems

Aurel Bulgac, Michael McNeil Forbes and Achim Schwenk
Department of Physics, University of Washington
Outline:

- Induced $p$-wave superfluidity in asymmetric Fermi gases

- $T=0$ thermodynamics in asymmetric Fermi gases at unitarity
  Bulgac and Forbes, cond-mat/0606043
Green – spin up
Yellow – spin down

If \( |\mu_\uparrow - \mu_\downarrow| < \frac{\Delta}{\sqrt{2}} \) the same solution as for \( \mu_\uparrow = \mu_\downarrow \)

LOFF (1964) solution
Pairing gap becomes a spatially varying function
Translational invariance broken

Muether and Sedrakian (2002)
Translational invariant solution
Rotational invariance broken
Son and Stephanov, cond-mat/0507586

Parish, Marchetti, Lamacraft, Simons
cond-mat/0605744

Sheeny and Radzihovsky, PRL 96, 060401(2006)

Pao, Wu, and Yip, PR B 73, 132506 (2006)
FIG. 1: (Color online) The dependence of the pairing gaps in the LOFF phase (upper panel) and the DFS phase (lower panel) on the asymmetry parameter for several values of the the total momentum $P/k_F$ and deformation parameter $\delta \varepsilon$ which are indicated in the panels.

FIG. 4: (Color online) The dependence of the free energy of the LOFF (upper panel) and the plane-wave DFS phase (lower panel) on the asymmetry parameter for several values of the deformation parameter $\delta \varepsilon$ and the total momentum $P/k_F$ which are indicated in the panels.

Sedrakian, Mur-Petit, Polls, Muether
What we predict?

**Induced $p$-wave superfluidity in asymmetric Fermi gases**

Two new superfluid phases where before they were not expected

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Bulgac, Forbes, Schwenk

One Bose superfluid coexisting with one $P$-wave Fermi superfluid

Two coexisting $P$-wave Fermi superfluids
**BEC regime**

- all minority (spin-down) fermions form dimers and the dimers organize themselves in a Bose superfluid

- the leftover/un-paired majority (spin-up) fermions will form a Fermi sea

- the leftover spin-up fermions and the dimers coexist and, similarly to the electrons in a solid, the leftover spin-up fermions will experience an attraction due to exchange of Bogoliubov phonons of the Bose superfluid
FIG. 1: The ratio $\Delta/\varepsilon_F$ ($\varepsilon_F = \hbar^2 k_F^2/2m$) as a function of $n_f/n_b$, for a fixed boson number density $n_b = 10^{13} \text{ cm}^{-3}$ and $n_b a^3 = 0.064$ (solid line) and $n_b a^3 = 0.037$ (dashed line) respectively. The dots show the value of the gap in the case of p-wave pairing for $n_b a^3 = 0.064$.

Bulgac, Bedaque, Fonseca, cond-mat/030602

\[
\Delta_p \sim \varepsilon_F \exp \left( -0.44 \frac{n_b}{n_f} \right), \quad \text{if } \frac{n_f}{n_b} \ll k_F a \ll 1
\]

\[
\Delta_p \sim \varepsilon_F \exp \left( -\frac{6\pi^2}{\alpha^2_{fb} (k_F a)^2 \ln(x^2) n_f} \frac{n_b}{n_f} \right), \quad \text{if } \frac{n_f}{n_b} \gg k_F a, \quad x^2 = \left( \frac{\hbar k_F}{m_b c} \right)^2
\]

\[
\Delta_p \bigg|_{\text{max}} \sim \varepsilon_F \exp \left( -\frac{5.6}{k_F a} \right), \quad \text{if } \frac{n_f}{n_b} \approx 0.44 k_F a \ll 1
\]
**BCS regime:**

\[
U_{ind}^{\uparrow\uparrow}(0, \vec{p}_1 - \vec{p}_2) = \begin{pmatrix} \vec{p}_2, \uparrow \\ -\vec{p}_2, \uparrow \end{pmatrix} - \begin{pmatrix} \vec{p}_1, \uparrow \\ -\vec{p}_1, \uparrow \end{pmatrix}
\]

\[
= -N_F \left( \frac{4\pi a\hbar^2}{m} \right)^2 \frac{L(|\vec{p}_1 - \vec{p}_2|/(2\hbar k_F))}{2}.
\]

The same mechanism works for the minority/spin-down component.
\[ \Delta^\uparrow_p \sim \varepsilon_F^\uparrow \exp \left( \frac{1}{N_F^\uparrow U_p^\uparrow} \right) = \varepsilon_F^\uparrow \exp \left( -\frac{\pi^2}{4k_F^\uparrow k_F^\downarrow a^2 L_p^\uparrow \left( \frac{k_F^\uparrow}{k_F^\downarrow} \right)} \right) \]

\[ L_p(z) = \frac{5z^2 - 2}{15z^4} \ln|1 - z^2| - \frac{z^2 + 5}{30z} \ln \left| \frac{1 - z}{1 + z} \right| - \frac{z^2 + 2}{15z^2} \]

\[ \Delta^\uparrow_p \bigg|_{\text{max}} \sim \varepsilon_F^\uparrow \exp \left( -\frac{\pi^2}{0.11 \left( 2k_F^\uparrow a \right)^2} \right), \quad \text{for } k_F^\downarrow \approx 0.77k_F^\uparrow \text{ and fixed } k_F^\downarrow \]

\[ \Delta^\uparrow_p \sim \varepsilon_F^\uparrow \exp \left( -\frac{3\pi^2}{2 \left( 2k_F^\uparrow a \right)^2 \ln \left( \frac{k_F^\uparrow}{k_F^\downarrow} \right)} \frac{k_F^\uparrow}{k_F^\downarrow} \right) \quad \text{for } k_F^\uparrow \gg k_F^\downarrow \]

\[ \Delta^\downarrow_p \sim \varepsilon_F^\downarrow \exp \left( -\frac{18\pi^2}{\left( 2k_F^\uparrow a \right)^2 k_F^\downarrow} \frac{k_F^\uparrow}{k_F^\downarrow} \right) \]
At unitarity the equation of state of a two-component fermion system is subject to rather tight theoretical constraints, which lead to well defined predictions for the spatial density profiles in traps and the grand canonical phase diagram is:

In the grand canonical ensemble there are only two dimensionfull quantities
We use both micro-canonical and grand canonical ensembles

\[ x = \frac{n_b}{n_a} \leq 1, \quad y = \frac{\mu_b}{\mu_a} \leq 1 \]

\[ E(n_a, n_b) = \frac{3}{5} \alpha \left[ n_a g(x) \right]^{5/3} \]

\[ P(\mu_a, \mu_b) = \frac{2}{5} \beta \left[ \mu_a h(y) \right]^{5/2} = \mu_a n_a + \mu_b n_b - E(n_a, n_b) = \frac{2}{3} E(n_a, n_b) \]

\[ y = \frac{g'(x)}{g(x) - xg'(x)}, \quad h(y) = \frac{1}{g(x) - xg'(x)} \]

\[ x = \frac{h'(y)}{h(y) - yh'(y)}, \quad g(x) = \frac{1}{h(y) - yh'(y)} \]

The functions \( g(x) \) and \( h(y) \) determine fully the thermodynamic properties and only a few details are relevant.
Both $g(x)$ and $h(y)$ are convex functions of their argument.

Non-trivial regions exist!

$$h(y) = \begin{cases} 
1 & \text{if } y \leq y_0 \\
\frac{1 + y}{(2\xi)^{3/5}} & \text{if } y \in [y_1, 1] 
\end{cases}$$

$h''(y) \geq 0$

$y_0 \leq Y_0 < y_c < Y_1 \leq 1, \quad y_c = (2\xi)^{3/5} - 1$

$g(0) = 1, \quad g(x) = (2\xi)^{3/5}$

$g''(x) \geq 0$

$g'(0) \leq Y_0$ and $g'(1) \in \left[\frac{g(1)}{1 + Y_1^{-1}}, \frac{g(1)}{2}\right]$

Bounds given by GFMC

Bounds from the energy required to add a single spin-down particle to a fully polarized Fermi sea of spin-up particles
Now put the system in a trap

\[ \mu_{a,b}(\vec{r}) = \lambda_{a,b} - V(\vec{r}), \quad y(\vec{r}) = \frac{\mu_b(\vec{r})}{\mu_a(\vec{r})} \]

\[ 2\mu_- = \lambda_a - \lambda_b \]

\[ n_a(\vec{r}) = \beta \left[ \mu_a(\vec{r})h(y(\vec{r})) \right]^{3/2} \left[ h(y(\vec{r})) - y(\vec{r})h'(y(\vec{r})) \right] \]

\[ n_b(\vec{r}) = \beta \left[ \mu_a(\vec{r})h(y(\vec{r})) \right]^{3/2} h'(y(\vec{r})) \]
- **blue** - $P = 0$ region
- **green** - $0 < P < 1$ region
- **red** - $P = 1$ region
Column densities (experiment)

Zweirlein et al. cond-mat/0605258
\[ \gamma = \frac{y_1 - 1}{y_0 - 1} = \frac{R_0^2 - R_{\text{vac}}^2}{R_1^2 - R_{\text{vac}}^2} \approx 0.70(5) \]

Experimental data from Zwierlein et al. cond-mat/0605258
Main conclusions:

• At T=0 a two component fermion system is always superfluid, irrespective of the imbalance and a number of unusual phases should exists.

• At T=0 and unitarity an asymmetric Fermi gas has non-trivial partially polarized phases