Superfluid LDA (SLDA)
Local Density Approximation for Systems with Superfluid Correlations

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Transparencies will be available shortly at http://www.phys.washington.edu/~bulgac
Contents

- Rather lengthy introduction, motivating the LDA approach
- Description of the LDA for systems with pairing correlations.
- Results of application of this new LDA approach to a rather large number of spherical nuclei in a fully self-consistent approach with continuum correctly accounted for.
- Description of the new features of the vortex state in low density neutron matter (neutron stars)
- Summary
References


A. Bulgac and Y. Yu, nucl-th/0109083 (Lectures)
Y. Yu and A. Bulgac, nucl-th/0302007 (Appendix to PRL)
A. Bulgac and Y. Yu, nucl-th/0310066
A. Bulgac and Y. Yu in preparation
A rather incomplete list of major questions still left unanswered in nuclear physics concerning pairing correlations:

- Do nuclear pairing correlations have a volume or/and surface character? Phenomenological approaches give no clear answer as anything fits equally well.
  - The density dependence of the pairing gap (partially related to the previous topic), the role of higher partial waves (p-wave etc.) especially in neutron matter.

- The role of the isospin symmetry in nuclear pairing. Routinely the isospin symmetry is broken in phenomenological approaches with really very lame excuses.
  - Role of collective modes, especially surface modes in finite nuclei, role of “screening effects.”

- Is pairing interaction momentum or/and energy dependent at any noticeable level?
  - Pairing in $T = 0$ channel?

- Does the presence or absence of neutron superfluidity have any influence on the presence and/or character of proton superfluidity and vice versa. New question raised recently: are neutron stars type I or II superconductors?
  - We should try to get away from the heavily phenomenological approach which dominated nuclear pairing studies most of last 40 years and put more effort in an *ab initio* and many-body theory of pairing and be able to make reliable predictions, especially for neutron stars. The studies of dilute atomic gases with tunable interactions could serve as an extraordinary testing ground of theories.
To tell me how to describe pairing correlations in nuclei and nuclear/neutron matter?

Most likely you will come up with one of the standard doctrines, namely:

• BCS within a limited single-particle energy shell (the size of which is chosen essentially arbitrarily) and with a coupling strength chosen to fit some data. Theoretically it makes no sense to limit pairing correlations to a single shell only. This is a pragmatic limitation.

• HFB theory with some kind of “effective” interaction, e.g. Gogny interaction.

Many would (or used to) argue that the Gogny interaction in particular is realistic, as, in particular, its matrix elements are essentially identical to those of the Bonn potential or some Other realistic bare NN-interaction

• In neutron stars often the Landau-Ginzburg theory was used (for the lack of a more practical theory mostly).
How does one decide if one or another theoretical approach is meaningful?

Really, this is a very simple question. One has to check a few things.

😊 Is the theoretical approach based on a sound approximation scheme?
   Well,…, maybe!

😊 Does the particular approach chosen describe known key experimental results, and moreover, does this approach predict new qualitative features, which are later on confirmed experimentally?

😊 Are the theoretical corrections to the leading order result under control, understood and hopefully not too big?
Let us check a simple example, homogeneous dilute Fermi gas with a weak attractive interaction, when pairing correlations occur in the ground state.

\[
\Delta = \frac{8}{e^2} \frac{\hbar^2 k_F^2}{2m} \exp\left(\frac{\pi}{2k_F a}\right)
\]

BCS result

\[
\Delta = \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left(\frac{\pi}{2k_F a}\right)
\]

An additional factor of \(1/(4e)^{1/3} \approx 0.45\) is due to induced interactions Gorkov and Melik-Barkhudarov in 1961.

BCS/HFB in error even when the interaction is very weak, unlike HF!

from Heiselberg et al

“Screening effects” are significant!

s-wave pairing gap in infinite neutron matter with realistic NN-interactions

from Lombardo and Schulze astro-ph/0012209

These are major effects beyond the naïve HFB when it comes to describing pairing correlations.
\[
\Delta = \left( \frac{2}{\pi} \right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp \left[ -\frac{\pi}{2 \tan \delta(k_F)} \right]
\]

NB! Extremely high relative $T_c$

Corrected Emery formula (1960)

RG- renormalization group calculation
Density Functional Theory (DFT)
Hohenberg and Kohn, 1964

\[ E_{gs} = \int d^3 r \varepsilon[\rho(\vec{r})] \]

Local Density Approximation (LDA)
Kohn and Sham, 1965

\[ E_{gs} = \int d^3 r \varepsilon[\rho(\vec{r}), \tau(\vec{r})] \]

\[ \rho(\vec{r}) = \sum_{i=1}^{N} |v_i(\vec{r})|^2 \]

\[ \tau(\vec{r}) = \sum_{i=1}^{N} |\nabla v_i(\vec{r})|^2 \]

Normal Fermi systems only!
Assume that there are two different many-body wave functions, corresponding to the same number particle density!

\[ \Psi_A(\vec{r}_1, \ldots, \vec{r}_N) \Rightarrow \rho(\vec{r}) \]
\[ \Psi_B(\vec{r}_1, \ldots, \vec{r}_N) \Rightarrow \rho(\vec{r}) \]
\[ \Psi_A(\vec{r}_1, \ldots, \vec{r}_N) \neq \Psi_B(\vec{r}_1, \ldots, \vec{r}_N) \]

\[ H = \sum_i T_i + \sum_{i<j} V_{ij} + \sum_{i<j<k} V_{ijk} + \ldots \]

\[ E_A = \left\langle \Psi_A \mid H + \sum_i V_i \mid \Psi_A \right\rangle = \left\langle \Psi_A \mid H \mid \Psi_A \right\rangle + \text{Tr}(V\rho) \]

\[ E_B = \left\langle \Psi_B \mid H + \sum_i U_i \mid \Psi_B \right\rangle = \left\langle \Psi_B \mid H \mid \Psi_B \right\rangle + \text{Tr}(U\rho) \]

\[ E_A < \left\langle \Psi_B \mid H \mid \Psi_B \right\rangle + \text{Tr}(V\rho) \]
\[ E_B < \left\langle \Psi_A \mid H \mid \Psi_A \right\rangle + \text{Tr}(U\rho) \]

\[ E_A + E_B < E_A + E_B \]
LDA (Kohn-Sham) for superfluid fermi systems
(Bogoliubov-de Gennes equations)

\[ E_{gs} = \int d^3 r \varepsilon(\rho(\mathbf{r}), \tau(\mathbf{r}), \nu(\mathbf{r})) \]

\[ \rho(\mathbf{r}) = 2 \sum_k | v_k(\mathbf{r}) |^2, \quad \tau(\mathbf{r}) = 2 \sum_k | \nabla v_k(\mathbf{r}) |^2 \]

\[ \nu(\mathbf{r}) = \sum_k u_k(\mathbf{r}) v_k^*(\mathbf{r}) \]

\[
\begin{pmatrix}
T + U(\mathbf{r}) - \lambda & \Delta(\mathbf{r}) \\
\Delta^*(\mathbf{r}) & -(T + U(\mathbf{r}) - \lambda)
\end{pmatrix}
\begin{pmatrix}
u_k(\mathbf{r}) \\
v_k(\mathbf{r})
\end{pmatrix} = E_k
\begin{pmatrix}
u_k(\mathbf{r}) \\
v_k(\mathbf{r})
\end{pmatrix}
\]

Mean-field and pairing field are both local fields!
(for sake of simplicity spin degrees of freedom are not shown)

There is a little problem! The pairing field \( \Delta \) diverges.
Why would one consider a local pairing field?

✓ Because it makes sense physically!
✓ The treatment is so much simpler!
✓ Our intuition is so much better also.

\[ r_0 \approx \frac{\hbar}{p_F} = k_F^{-1} \]

radius of interaction inter-particle separation

\[ \Delta = \omega_D \text{Exp} \left( -\frac{1}{|V|N} \right) \ll \varepsilon_F \]

\[ \xi \approx \frac{1}{k_F \Delta} \gg r_0 \]

coherence length size of the Cooper pair
**Nature of the problem**

\[ \nu(\vec{r}_1, \vec{r}_2) = \sum_{E_k > 0} v_k^*(\vec{r}_1) u_k(\vec{r}_2) \propto \frac{1}{|\vec{r}_1 - \vec{r}_2|} \]

\[ \Delta(\vec{r}_1, \vec{r}_2) = \frac{1}{2} V(\vec{r}_1, \vec{r}_2) \nu(\vec{r}_1, \vec{r}_2) \]

It is easier to show how this singularity appears in infinite homogeneous matter (BCS model)

\[ \nu_k(\vec{r}) = v_k \exp(i\vec{k} \cdot \vec{r}), \quad u_k(\vec{r}) = u_k \exp(i\vec{k} \cdot \vec{r}) \]

\[ v_k^2 = \frac{1}{2} \left( 1 - \frac{\varepsilon_k - \lambda}{\sqrt{(\varepsilon_k - \lambda)^2 + \Delta^2}} \right), \quad u_k^2 + v_k^2 = 1, \quad \varepsilon_k = \frac{\hbar^2 k^2}{2m} + U, \quad \Delta = \frac{\hbar^2 \delta}{2m} \]

\[ \nu(r) = \frac{\Delta m}{2\pi^2 \hbar^2} \int_0^\infty dk \frac{\sin(kr)}{kr} \frac{k^2}{\sqrt{(k^2 - k^F)^2 + \delta^2}} \]
A (too) simple case

\[ k_F \to 0, \delta \to 0 \]

\[ \nu(|\vec{r}_1 - \vec{r}_2|) \to \frac{\Delta m}{2\pi^2 \hbar^2} \int_0^\infty dk \frac{\sin kr}{kr} = \frac{\Delta m}{2\pi^2 \hbar^2} \frac{\pi}{2|\vec{r}_1 - \vec{r}_2|} \]

The integral converges (conditionally) at \( k > 1/r \) \((\text{iff } r > 0)\)

The divergence is due to high momenta and thus its nature is independent of whether the system is finite or infinite
If one introduces an explicit momentum cut-off one has to deal with this integral iff $r > 0$. If $r = 0$ then the integral is simply:

$$h(x) = \frac{2}{\pi} \int_{0}^{x} dy \frac{\sin y}{y}$$

In the final analysis all is an issue of the order of taking various limits: $r \to 0$ versus cut-off $x \to \infty$. 

$$h(x) = \frac{2}{\pi} \int_{0}^{x} dy = \frac{2x}{\pi}$$
Solution of the problem in the case of the homogeneous matter
(Lee, Huang and Yang and others)

Gap equation

\[ V(\vec{r}_1 - \vec{r}_2) = g \delta(\vec{r}_1 - \vec{r}_2) \]

Lippmann-Schwinger equation
(zero energy collision)
\[ T = V + VGT \]

Now combine the two equations and the divergence is (magically) removed!

\[ \frac{m}{4\pi\hbar^2 a} = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{\sqrt{(\varepsilon_k - \lambda)^2 + \Delta^2}} - \frac{1}{\varepsilon_k} \right\} \]

\[ 1 = -\frac{g}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{(\varepsilon_k - \lambda)^2 + \Delta^2}} \]

\[ -\frac{mg}{4\pi\hbar^2 a} + 1 = -\frac{g}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\varepsilon_k} \]
How people deal with this problem in finite systems?

- Introduce an explicit energy cut-off, which can vary from 5 MeV to 100 MeV (sometimes significantly higher) from the Fermi energy.
- Use a particle-particle interaction with a finite range, the most popular one being Gogny’s interaction.

Both approaches are in the final analysis equivalent in principle, as a potential with a finite range $r_0$ provides a (smooth) cut-off at an energy $E_c = \hbar^2/mr_0^2$.

- The argument that nuclear forces have a finite range is superfluous, because nuclear pairing is manifest at small energies and distances of the order of the coherence length, which is much smaller than nuclear radii.
- Moreover, LDA works pretty well for the regular mean-field.
- A similar argument fails as well in case of electrons, where the radius of the interaction is infinite and LDA is fine.
**Pseudo-potential approach**
(appropriate for very slow particles, very transparent
but somewhat difficult to improve)

Lenz (1927), Fermi (1931), Blatt and Weiskopf (1952)
Lee, Huang and Yang (1957)

\[-\frac{\hbar^2 \Delta \mathbf{r}}{m} \psi (\mathbf{r}) + V (\mathbf{r}) \psi (\mathbf{r}) = E \psi (\mathbf{r}), \quad V (\mathbf{r}) \approx 0 \text{ if } r > R\]

\[\psi (\mathbf{r}) = \exp (i \mathbf{k} \cdot \mathbf{r}) + \frac{f}{r} \exp (ikr) \approx 1 + \frac{f}{r} + \ldots \approx 1 - \frac{a}{r} + O(kr)\]

\[f^{-1} = -\frac{1}{a} + \frac{1}{2} r_0 k^2 - ik, \quad g = \frac{4\pi \hbar^2 a}{m(1 + ika)} + \ldots\]

if \(kr_0 \ll 1\) then \(V (\mathbf{r}) \psi (\mathbf{r}) \Rightarrow g \delta (\mathbf{r}) \frac{\partial}{\partial r} [r \psi (\mathbf{r})]\)

Example : \(\psi (\mathbf{r}) = \frac{A}{r} + B + \ldots \Rightarrow \delta (\mathbf{r}) \frac{\partial}{\partial r} [r \psi (\mathbf{r})] = \delta (\mathbf{r}) B\)
How to deal with an inhomogeneous/finite system?

\[ v_{\text{reg}} (\vec{r}) = \sum_i \left[ v_i^* (\vec{r}) u_i (\vec{r}) + \frac{\Delta(\vec{r}) \psi_i^* (\vec{r}) \psi_i (\vec{r})}{2(\lambda - \epsilon_i)} \right] - \frac{\Delta(\vec{r})}{2} G_{\text{reg}} (\lambda, \vec{r}) \]

\[ G_{\text{reg}} (\lambda, \vec{r}) \overset{\text{def}}{=} \lim_{\vec{r}' \to \vec{r}} \left[ G(\vec{r}, \vec{r}', \lambda) + \frac{m}{2\pi \hbar^2 |\vec{r} - \vec{r}'|} \right] \]

\[ [h(\vec{r}) - \epsilon_i] \psi_i (\vec{r}) = 0 \]

\[ [\lambda - h(\vec{r})] G(\vec{r}, \vec{r}', \lambda) = \delta (\vec{r} - \vec{r}') \]

There is complete freedom in choosing the Hamiltonian \( h \) and we are going to take advantage of this!
We shall use a “Thomas-Fermi” approximation for the propagator $G$.

$$
G(\vec{r}, \vec{r}', \lambda) = -\frac{m \exp(ik_F(\vec{r})|\vec{r} - \vec{r}'|)}{2\pi\hbar^2|\vec{r} - \vec{r}'|} \\
\approx -\frac{m}{2\pi\hbar^2|\vec{r} - \vec{r}'|} - \frac{ik_F(\vec{r})m}{2\pi\hbar^2} + O(|\vec{r} - \vec{r}'|)
$$

$$
\hbar^2 k_F^2(\vec{r}) \frac{2}{2m} + U(\vec{r}) = \lambda, \quad \hbar^2 k_c^2(\vec{r}) \frac{2}{2m} + U(\vec{r}) = \lambda + E_c
$$

$$
\nu_{reg}(\vec{r}) \overset{\text{def}}{=} \sum_{E_i \leq E_c} \nu_i^*(\vec{r})u_i(\vec{r}) + \frac{\Delta(\vec{r})}{4\pi^2} \int_{\lambda - \frac{\hbar^2 k^2}{2m} - U(\vec{r}) + i\gamma}^{k_c(\vec{r})} \frac{1}{k^2} dk \left\{ \frac{i\Delta(\vec{r})k_F(\vec{r})m}{4\pi\hbar^2} \right\}
$$

Regularized anomalous density  
Regular part of $G$
FIG. 2. The gap $\Delta$ and the effective coupling constant $g_{\text{eff}}$ as a function of the cut-off energy $E_c$ for three regularization schemes. The full lines correspond to calculations using Eqs. (15–17). Circles correspond to the regularization scheme presented in Ref. [5] (when only terms with $k_c$ are present). The pentagrams correspond to the vacuum regularization scheme [16]. The calculation was performed for homogeneous neutron matter with $\rho = 0.08 \text{ fm}^{-3}$ and $g = -250 \text{ MeV} \cdot \text{fm}^3$. 


FIG. 1. The neutron pairing field (17) as a function of the radial coordinate and of the cut-off energy $E_c$. Upward various curves correspond to $E_c = 20$, $30$, $35$, $40$, $45$ and $50$ MeV respectively. On the scale of the figure the last two curves are indistinguishable.
The SLDA (renormalized) equations

\[ E_{gs} = \int d^3 r \{ \varepsilon_N [\rho(\vec{r}), \tau(\vec{r})] + \varepsilon_S [\rho(\vec{r}), \nu(\vec{r})] \} \]

\[ \varepsilon_S [\rho(\vec{r}), \nu(\vec{r})] \overset{\text{def}}{=} -\Delta(\vec{r})\nu_c(\vec{r}) = g_{\text{eff}}(\vec{r})\nu_c(\vec{r})^2 \]

\[ \begin{cases} 
[ h(\vec{r}) - \mu ]u_i(\vec{r}) + \Delta(\vec{r})v_i(\vec{r}) = E_i u_i(\vec{r}) \\
\Delta^*(\vec{r}) u_i(\vec{r}) - [ h(\vec{r}) - \mu ] v_i(\vec{r}) = E_i v_i(\vec{r}) \end{cases} \]

\[ \begin{cases} 
 h(\vec{r}) = -\vec{\nabla} \frac{\hbar^2}{2m(\vec{r})} \vec{\nabla} + U(\vec{r}) \\
 \Delta(\vec{r}) = -g_{\text{eff}}(\vec{r})\nu_c(\vec{r}) \end{cases} \]

\[ \frac{1}{g_{\text{eff}}(\vec{r})} = \frac{1}{g[\rho(\vec{r})]} - \frac{m(\vec{r})k_c(\vec{r})}{2\pi^2\hbar^2} \left\{ 1 - \frac{k_F(\vec{r})}{k_c(\vec{r})} \ln \frac{k_c(\vec{r}) + k_F(\vec{r})}{k_c(\vec{r}) - k_F(\vec{r})} \right\} \]

\[ \rho_c(\vec{r}) = 2 \sum_{E_i \geq 0} \left| v_i(\vec{r}) \right|^2, \quad v_c(\vec{r}) = \sum_{E_i \geq 0} v_i^*(\vec{r}) u_i(\vec{r}) \]

\[ E_c + \mu = \frac{\hbar^2 k_c^2(\vec{r})}{2m(\vec{r})} + U(\vec{r}), \quad \mu = \frac{\hbar^2 k_F^2(\vec{r})}{2m(\vec{r})} + U(\vec{r}) \]

Position and momentum dependent running coupling constant
Observables are (obviously) independent of cut-off energy (when chosen properly).
A few notes:

- The cut-off energy $E_c$ should be larger than the Fermi energy.
- It is possible to introduce an even faster converging scheme for the pairing field with $E_c$ of a few $\Delta$'s only.
- Even though the pairing field was renormalized, the total energy should be computed with care, as the “pairing” and “kinetic” energies separately diverge.

\[
E_{gs} = \int d^3r [\mathcal{E}_N(r) + \mathcal{E}_S(r)],
\]
\[
\mathcal{E}_S(r) := -\Delta(r)\nu_c(r) = g_{eff}(r)|\nu_c(r)|^2
\]

Still diverges!

- One should now introduce the normal and the superfluid contributions to the bare/unrenormalized Energy Density Functional (EDF).

\[
\mathcal{E}_S(r) = g_0(r)|\nu_p(r) + \nu_n(r)|^2 + g_1(r)|\nu_p(r) - \nu_n(r)|^2
\]

We considered so far only the case $g_0 = g_1$. 

Isospin symmetry
Peculiarity of the finite systems:
deep hole states are continuum states.

\[
\begin{pmatrix}
\hbar - \lambda & \Delta \\
\Delta^+ & - (\hbar^* - \lambda)
\end{pmatrix}
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix} = E_k
\begin{pmatrix}
u_k \\
v_k
\end{pmatrix}
\]

Andreev reflection

\[
\begin{pmatrix}
E - (h - \lambda) & -\Delta \\
\star & E + (h^* - \lambda)
\end{pmatrix}
\begin{pmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
The nuclear landscape and the models

The isotope and isotone chains treated by us are indicated with red numbers.

Courtesy of Mario Stoitsov
The Rare Isotope Accelerator (RIA) is our highest priority for major new construction. RIA will be the world-leading facility for research in nuclear structure and nuclear astrophysics.
Priority Near-Term

1  FES  International Thermonuclear Experimental Reactor
2  ASCR  UltraScale Scientific Computing Capability

Tie for

3  HEP  Joint Dark Energy Mission
     BES  Linac Coherent Light Source
     BER  Protein Production and Tags
     NP  Rare Isotope Accelerator

Tie for

5  BER  Characterization & Imaging
     NP  Continuous Electron Beam Accelerator Facility 12GeV Upgrade

Tie for

7  ASCR  Esnet Upgrade
     ASCR  NERSC Upgrade
     BES  Transmission Electron Achromatic Microscope

12  HEP  BTeV

U.S. Department of Energy
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Facilities for the Future of Science: A Twenty-Year Outlook
Pairing correlations show prominently in the staggering of the binding energies.

*Systems with odd particle number are less bound than systems with even particle number.*
How well does the new approach work?

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</tr>
</tbody>
</table>

Ref. 23, S.Q. Zhang et al. nucl-th/0302032. - RMF
One-neutron separation energies

\[
S_n = \left( V + \frac{1}{A} \rho^2 \frac{\partial}{\partial \rho} \right)_{\rho = \rho_c} \left( V + \frac{1}{A} \rho^2 \frac{\partial}{\partial \rho} \right)_{\rho = 0}
\]

Volume pairing
\[
g(\vec{r}) = g
\]

Volume + Surface pairing
\[
g(\vec{r}) = V_0 \left( 1 - \frac{\rho(\vec{r})}{\rho_c} \right)
\]

Normal EDF:
- **SLy4** - Chabanat et al.
- **FaNDF\(^0\)** – Fayans
  JETP Lett. 68, 169 (1998)
Two-neutron separation energies

- $S_{2n}$ [MeV]
- $A$

- Sn
  - SLy4
  - FaNDF$^0$

- Pb
  - SLy4
  - FaNDF$^0$

- Exp.
- Volume
One-nucleon separation energies

- $N = 50$
  - FaNDF$^0$

- $N = 82$
  - FaNDF$^0$

- $N = 126$
  - FaNDF$^0$
  - Ca

$S_p$ [MeV]
- We use the same normal EDF as Fayans et al. volume pairing only with one universal constant.
Spatial profiles of the pairing field for tin isotopes and two different (normal) energy density functionals

Sn
SLy4

Sn
FaNDF$^0$
Charge radii

Let me backtrack a bit and summarize some of the ingredients of the LDA to superfluid nuclear correlations.

**Energy Density (ED) describing the normal system**

\[
E_{gs} = \int d^3 r \left\{ \varepsilon_N[\rho_n(\vec{r}), \rho_p(\vec{r})] + \varepsilon_S[\rho_n(\vec{r}), \rho_p(\vec{r}), v_n(\vec{r}), v_p(\vec{r})] \right\}
\]

**ED contribution due to superfluid correlations**

\[
\begin{align*}
\varepsilon_N[\rho_n(\vec{r}), \rho_p(\vec{r})] &= \varepsilon_N[\rho_p(\vec{r}), \rho_n(\vec{r})] \\
\varepsilon_S[\rho_n(\vec{r}), \rho_p(\vec{r}), v_n(\vec{r}), v_p(\vec{r})] &= \varepsilon_S[\rho_p(\vec{r}), \rho_n(\vec{r}), v_p(\vec{r}), v_n(\vec{r})]
\end{align*}
\]

**Isospin symmetry**

(Coulomb energy and other relatively small terms not shown here.)

Let us consider the simplest possible ED compatible with nuclear symmetries and with the fact that nuclear pairing correlations are relatively weak.

\[
\varepsilon_S[\rho_p, \rho_n, v_p, v_n] = g_0 \left| v_p + v_n \right|^2 + g_1 \left| v_p - v_n \right|^2
\]

Like \( \rho_p + \rho_n \)

Like \( \rho_p - \rho_n \)

\( g_0 \) and \( g_1 \) could depend as well on \( \rho_p \) and \( \rho_n \)
Let us stare at this part of the ED for a moment, … or two.

SU(2) invariant

\[
\varepsilon_S(\nu_p, \nu_n) = g_0 \left| \nu_p + \nu_n \right|^2 + g_1 \left| \nu_p - \nu_n \right|^2 \\
= g \left[ \left| \nu_p \right|^2 + \left| \nu_n \right|^2 \right] + g' \left[ \nu_p^* \nu_n + \nu_n^* \nu_p \right]
\]

\[g = g_0 + g_1 \quad g' = g_0 - g_1\]

NB I am dealing here with s-wave pairing only (S=0 and T=1)!

The last term could not arise from a two-body bare interaction.
• Dumitrescu and Horoi, Nuovo Cimento A 103, 635 (1990)

considered various mechanisms to couple the proton and neutron superfluids in nuclei, in particular a zero range four-body interaction which could lead to terms like $\propto |\nu_n|^2 |\nu_p|^2$

• Buckley, Metlitski and Zhitnitsky, astro-ph/0308148 considered an SU(2) – invariant Landau-Ginsburg description of neutron stars in order to settle the question of whether neutrons and protons superfluids form a type I or type II superconductor. However, I have doubts about the physical correctness of the approach.
In the end one finds that a suitable superfluid nuclear EDF has the following structure:

\[
\mathcal{E}_S[v_p, v_n] = g(\rho_p, \rho_n)[|v_p|^2 + |v_n|^2] + f(\rho_p, \rho_n)[|v_p|^2 - |v_n|^2] \frac{\rho_p - \rho_n}{\rho_p + \rho_n}
\]

where \( g(\rho_p, \rho_n) = g(\rho_n, \rho_p) \)

and \( f(\rho_p, \rho_n) = f(\rho_n, \rho_p) \)

Isospin symmetric

Charge symmetric
Goriely et al, Phys. Rev. C 66, 024326 (2002) in the most extensive and by far the most accurate fully self-consistent description of all known nuclear masses (2135 nuclei with $A \geq 8$) with an rms better than 0.7 MeV use for pairing couplings:

$$
\begin{align*}
V_{pp}^+ &= -265.3 \text{ MeV} \\
V_{nn}^+ &= -237.6 \text{ MeV} \\
V_{pp}^- &= -277.8 \text{ MeV} \\
V_{nn}^- &= -246.9 \text{ MeV} \\
E_c &= 15 \text{ MeV}
\end{align*}
$$

for even systems

While no other part of their nuclear EDF violates isospin symmetry, and moreover, while they were unable to incorporate any contribution from CSB-like forces, this fact remains as one of the major drawbacks of their results and it is an embarrassment and needs to be resolved. Without that the entire approach is in the end a mere interpolation, with limited physical significance.
Let us now remember that there are more neutron rich nuclei and let me estimate the following quantity from all measured nuclear masses:

\[
\frac{N-Z}{A} = 0.1473
\]

Conjecturing now that Goriely et al, Phys. Rev. C 66, 024326 (2002) have as a matter of fact replaced in the “true” pairing EDF the isospin density dependence simply by its average over all masses, one can easily extract from their pairing parameters the following relation:

\[
\mathcal{E}_S[\nu_p, \nu_n] = g \left[ |\nu_p|^2 + |\nu_n|^2 \right] + f \left[ |\nu_p|^2 - |\nu_n|^2 \right] \frac{\rho_p - \rho_n}{\rho_p + \rho_n}
\]

where \( f \approx -0.39 \) \( g > 0 \) and \( g < 0 \)
The most general form of the superfluid contribution (s-wave only) to the LDA energy density functional, compatible with known nuclear symmetries.

\[ \varepsilon_S[\nu_p, \nu_n] = g(\rho_p, \rho_n) |\nu_p|^2 + g(\rho_n, \rho_p) |\nu_n|^2 \]

✓ In principle one can consider as well higher powers terms in the anomalous densities, but so far I am not aware of any need to do so, if one considers binding energies alone.

✓ There is so far no clear evidence for gradient corrections terms in the anomalous density or energy dependent effective pairing couplings.
How can one determine the density dependence of the coupling constant $g$? I know two methods.

- In homogeneous low density matter one can compute the pairing gap as a function of the density. **NB this is not a BCS or HFB result!**

$$
\Delta = \left( \frac{2}{e} \right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left( \frac{\pi}{2k_F a} \right)
$$

- One can also compute the energy of the normal and superfluid phases as a function of density, as was recently done by Carlson et al, Phys. Rev. Lett. 91, 050401 (2003) for a Fermi system interacting with an infinite scattering length (Bertsch’s MBX 1999 challenge).

In both cases one can extract from these results the superfluid contribution to the LDA energy density functional in a straightforward manner.
A NEUTRON STAR: SURFACE and INTERIOR

CORE:
Homogeneous Matter

CRUST:
Nuclei
Neutron Superfluid

ATMOSPHERE
ENVELOPE
CRUST
OUTER CORE
INNER CORE

Magnetic field

Neutron Superfluid
Neutron Vortex
Nuclei in a lattice

Neutron Superfluid + Proton Superconductor

Neutron Vortex
Magnetic Flux Tube

Polar cap
Cone of open magnetic field lines

“meat balls”
“lasagna”

Borrowed from http://www.lsw.uni-heidelberg.de/~mcamenzi/NS_Mass.html
Landau criterion for superflow stability
(flow without dissipation)

Consider a superfluid flowing in a pipe with velocity $v_s$:

$$E_0 + \frac{Nm v_s^2}{2} < E_0 + \varepsilon_p + \vec{v}_s \cdot \vec{p} + \frac{Nm v_s^2}{2} \implies v_s < \frac{\varepsilon_p}{p}$$

no internal excitations

One single quasi-particle excitation with momentum $p$

In the case of a Fermi superfluid this condition becomes

$$v_s < \frac{\Delta}{\hbar k_F}$$
Vortex in neutron matter

\[
\begin{pmatrix}
    u_{\alpha kn}(\vec{r}) \\
    v_{\alpha kn}(\vec{r})
\end{pmatrix} =
\begin{pmatrix}
    u_{\alpha}(r) \exp[i(n + 1/2)\phi - ikz] \\
    v_{\alpha}(r) \exp[i(n - 1/2)\phi - ikz]
\end{pmatrix}, \quad n \text{ - half - integer}
\]

\[\Delta(\vec{r}) = \Delta(r) \exp(i\phi), \quad \vec{r} = (r, \phi, z) \text{ [cylindrical coordinates]}\]

Oz - vortex symmetry axis

Ideal vortex, Onsager's quantization (one \(\hbar\) per Cooper pair)

\[
\mathbf{\nabla}_v(\vec{r}) = \frac{\hbar}{2mr^2} (y,-x,0) \quad \Leftarrow \quad \frac{1}{2\pi} \oint_C \mathbf{\nabla}_v(\vec{r}) \cdot d\vec{r} = \frac{\hbar}{2m}
\]
Fayans’s FaNDF$^0$

$$\Delta = \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left(-\frac{\pi}{2 \tan \delta(k_F)}\right)$$

An additional factor of $1/(4e)^{1/3}$ is due to induced interactions. Again, HFB not valid.

Distances scale with $l_F$.
Dramatic structural changes of the vortex state naturally lead to significant changes in the energy balance of a neutron star.

- In low density region $\varepsilon(\rho_{out})\rho_{out} > \varepsilon(\rho_{in})\rho_{in}$, which thus leads to a large anti-pinning energy $E_{pin}^{V} > 0$:

  $$E_{pin}^{V} = [\varepsilon(\rho_{out})\rho_{out} - \varepsilon(\rho_{in})\rho_{in}]V$$

- The energy per unit length is going to be changed dramatically when compared to previous estimates, by

  $$\frac{\Delta E_{vortex}}{L} \approx [\varepsilon(\rho_{out})\rho_{out} - \varepsilon(\rho_{in})\rho_{in}]\pi R^2$$

- Specific heat, transport properties are expected to significantly affected as well.
Conclusions

- An LDA-DFT formalism for describing pairing correlations in Fermi systems has been developed. This represents the first genuinely local extension of the Kohn-Sham LDA from normal to superfluid systems - \textbf{SLDA}.

- Nuclear symmetries lead to a relatively simple form of the superfluid contributions to the energy density functional.

- Phenomenological analysis of a relatively large number of nuclei (more than 200) indicates that with a single coupling constant one can describe very accurately proton and neutron pairing correlations in both odd and even nuclei. However, there seems to be a need to introduce a consistent isospin dependence of the pairing EDF.

- There is a need to understand the behavior of the pairing as a function of density, from very low to densities several times nuclear density, in particular pairing in higher partial waves, in order to understand neutron stars.

- It is not clear so far whether proton and neutron superfluids do influence each other in a direct manner, if one considers binding energies alone.

- The formalism has been applied as well to vortices in neutron stars and to describe various properties of dilute atomic Fermi gases and there is also an extension to 2-dim systems due to Yu, Aberg and Reinman.