Dilute atomic gases with large positive scattering length

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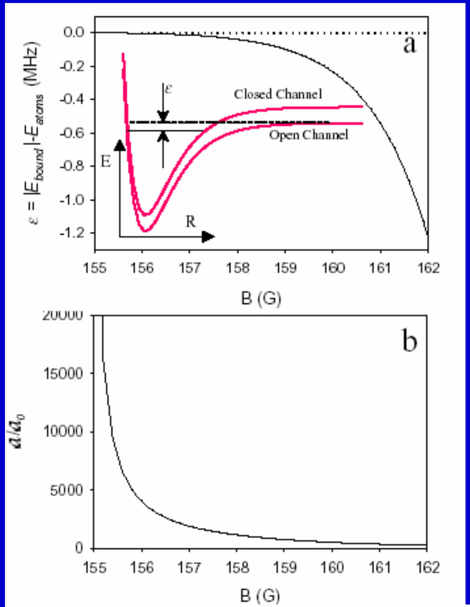
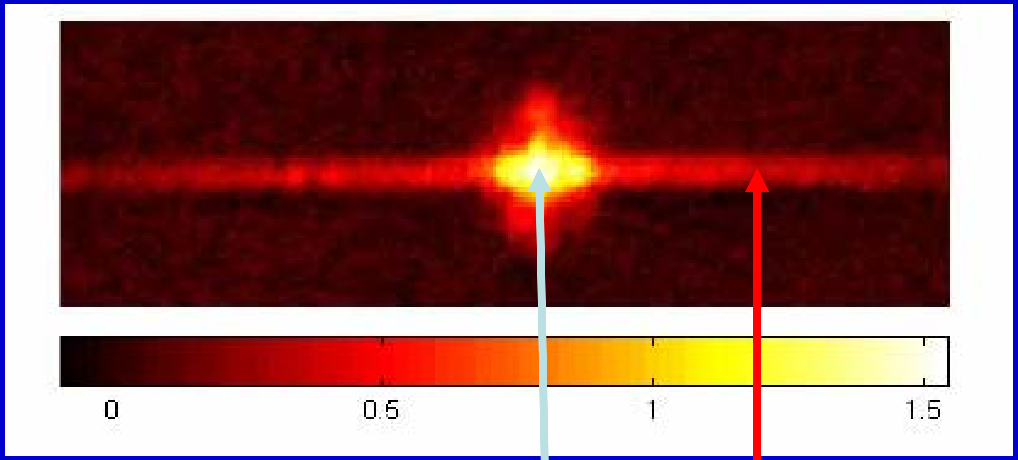
Atom-Molecule Coherence in a Bose-Einstein Condensate

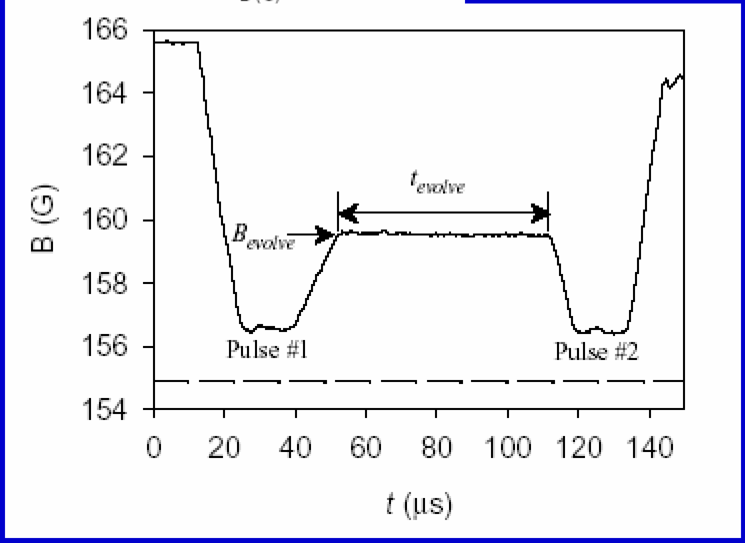
Donley, Clausen, Thompson and Wieman, Nature, 417, 529 (2002).

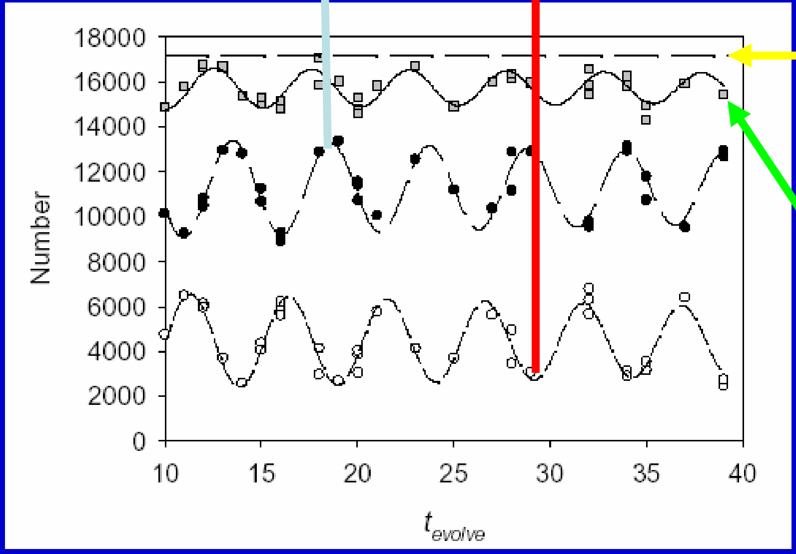




Remnant atoms (T≈3 nK)

Burst atoms (T≈150 nK)





$N_{initial}$

$N_{burst}+N_{remnant}$

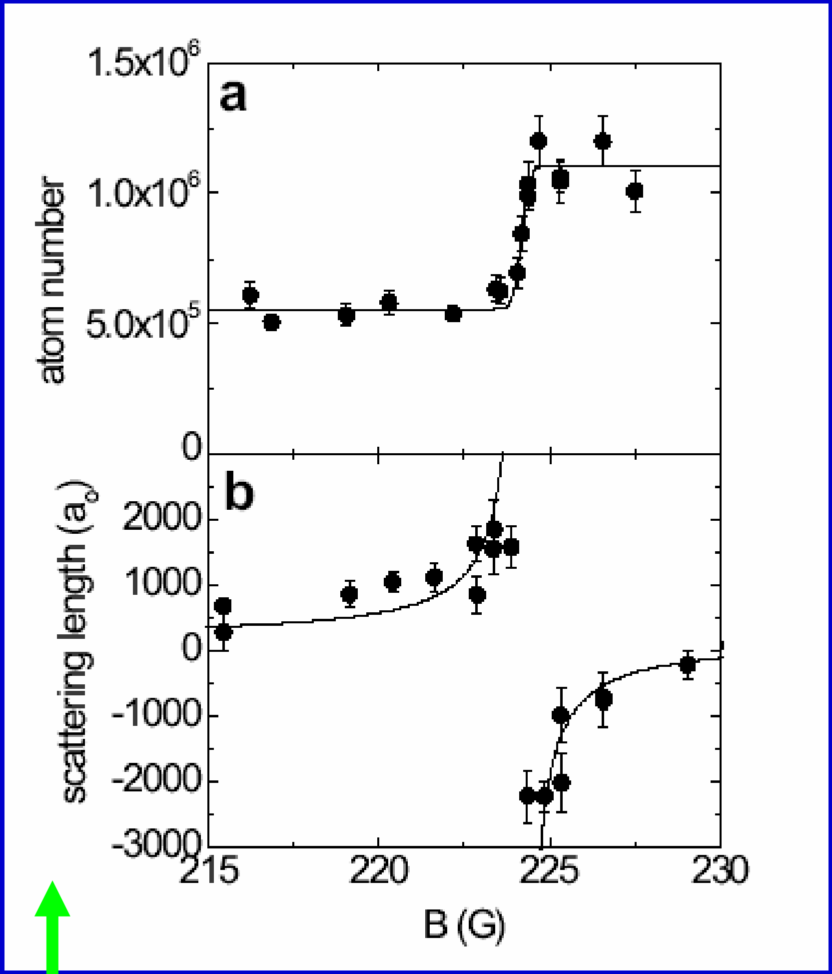
The difference $DN=N_{initial} - (N_{burts}+N_{remnant})$ is likely the number of molecules formed. $DN/N_{initial}$ changes widely with the pulse length and the cloud density.

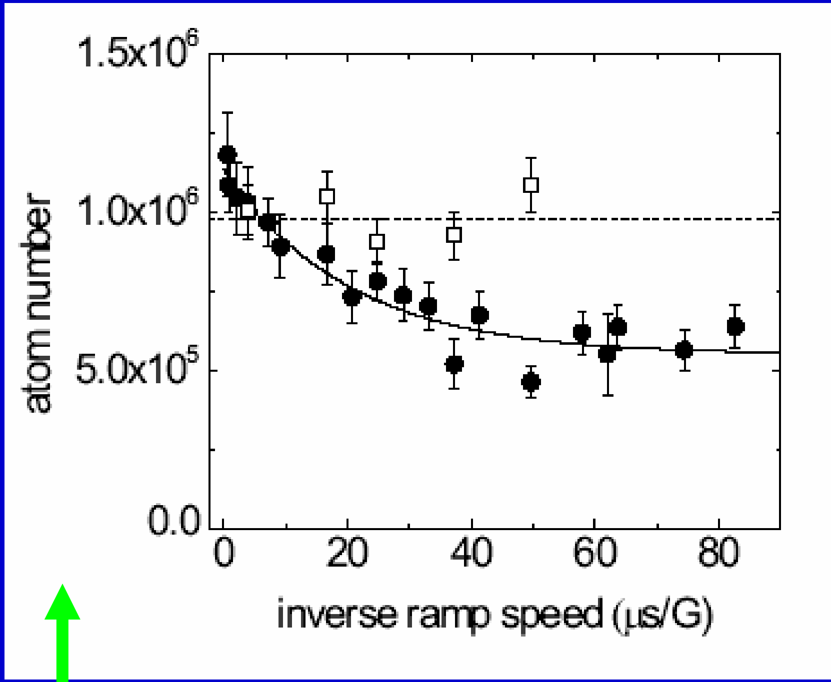
Creation of ultracold molecules from a Fermi gas of atoms

Regal, Ticknor, Bohm and Jin, cond-mat/0305028

- **Atomic cloud of ^{40}K at T = 0.13 … 0.33 T_F initially in hyperfine state |9/2,-9/2>.**
- **Prepare a nearly equal incoherent mixture of |9/2,-9/2> and |9/2,-5/2> states at B=227.81 G.**
- **Use a new Feshbach resonance between |9/2,-9/2> and |9/2,-5/2> states at B= 224.21 G.**
- **Ramp the magnetic field across the resonance (from high to low B) at rates of $(40\ \mathrm{ms/G})^{-1}$.**
- **Observe the number of atoms left from absorption image of the cloud after expansion. The light is resonant with atomic transitions only and thus only atoms but not molecules (dimers) are observed.**
- **The molecules/dimers temperature is below the BEC transition temperature.**
- **Using radio frequency (rf) spectroscopy with photon energy near the |9/2,-5/2> and |9/2-7/2> atomic energy splitting populate the |9/2,-7/2> state at various B_{hold}.**
- **Use a Stern-Gerlach imaging to separate various hyperfine states.**

Reagal, Ticknor, Bohm and Jin, cond-mat/0305028





Number of atoms after ramping B from 228.25 G to 216.15 (black dots) and for ramping B down (at 40 ms/G) and up at various rates (squares).

a) Loss of atoms |9/2,-9/2> and |9/2,-5/2> as a function of final B. The initial value of B=227.81 G.

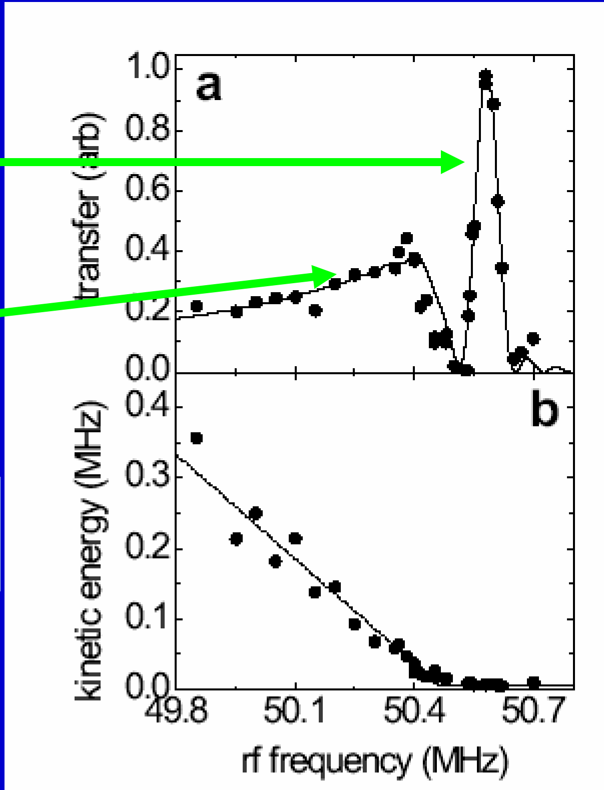
b) Scattering length between hyperfine states |9/2,-9/2> and |9/2,-5/2> as a function of the magnetic field B.

Reagal, Ticknor, Bohm and Jin, cond-mat/0305028

Symmetric peak is near the atomic |9/2,-5/2> to |9/2,-7/2> transition. The total number of |9/2,-5/2> and |9/2,-7/2> atoms is constant.

Asymmetric peak corresponds to dissociation of molecules into free |9/2,-5/2> and |9/2,-7/2> atoms. The total number of |9/2,-5/2> and |9/2,-7/2> atoms increases.

$$h\nu_{rf} = h\nu_{atom} - E_{binding} - \Delta E$$



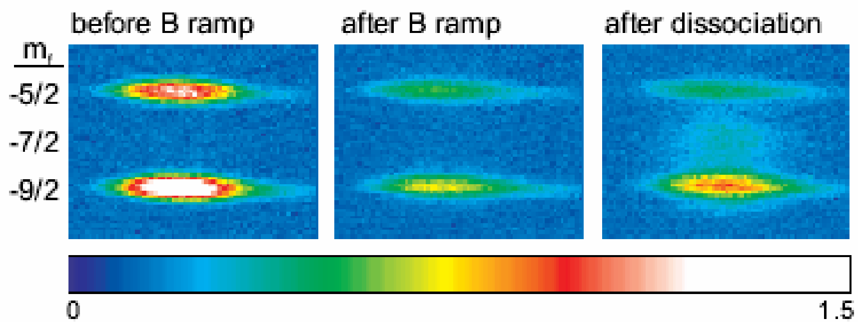
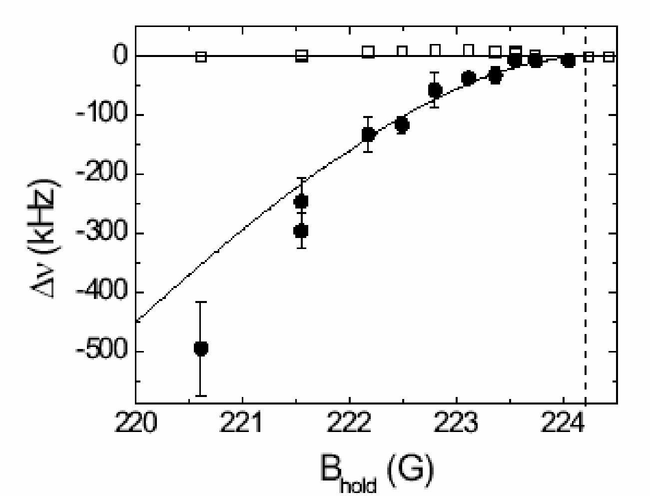


FIG. 4: Absorption images of the quantum gas using a Stern-Gerlach technique. We start with ultracold fermionic atoms in the $|9/2,-5/2\rangle$ and $|9/2,-9/2\rangle$ states of ^{40}K. A magnetic field ramp through the Feshbach resonance causes 50% atom loss, due to adiabatic conversion of atoms to diatomic molecules. To directly detect these bosonic molecules we apply an rf photodissociation pulse; the dissociated molecules then appear in the $|9/2,-7/2\rangle$ and $|9/2,-9/2\rangle$ atom states. The shaded bar indicates the optical depth.



Dimer/molecule binding energy

What have we learned from this experiment?

✓ **A rather stable and cold mixture of fermionic atoms and bosonic molecules (dimers), the latter likely in a BEC state can be formed.**

✓ **The ratio of atoms to dimers can apparently be varied almost at will.**

✓ **The formation of dimers and their dissociation by undoing the change in the magnetic field is likely a reversible process, entropy does not seem to be created at a noticeable rate and heating is apparently small. Or is it?**

What is the nature of this new object?
Can we describe it?

What else is there theoretically?

VOLUME 83, NUMBER 14 PHYSICAL REVIEW LETTERS 4 OCTOBER 1999

Rarified Liquid Properties of Hybrid Atomic-Molecular Bose-Einstein Condensates

Eddy Timmermans,[1] Paolo Tommasini,[2] Robin Côté,[2,*] Mahir Hussein,[2,3] and Arthur Kerman[4]

$$\hat{H} = \int d^3r\, \hat{\psi}_a^\dagger\left[-\frac{\hbar^2\nabla^2}{2m} + \frac{\lambda_a}{2}\hat{\psi}_a^\dagger\hat{\psi}_a + \lambda\hat{\psi}_m^\dagger\hat{\psi}_m\right]\hat{\psi}_a + \int d^3r\, \hat{\psi}_m^\dagger\left[-\frac{\hbar^2\nabla^2}{4m} + \frac{\lambda_m}{2}\hat{\psi}_m^\dagger\hat{\psi}_m + \epsilon\right]\hat{\psi}_m + \frac{\alpha}{\sqrt{2}}\int d^3r\,\{\hat{\psi}_m^\dagger\hat{\psi}_a\hat{\psi}_a + \hat{\psi}_m\hat{\psi}_a^\dagger\hat{\psi}_a^\dagger\}, \qquad (2)$$

$$e = \frac{N}{\Omega}\left\{\frac{\lambda_a}{2}f_a^2 + \frac{\lambda_m}{2}f_m^2 + \lambda f_m f_a\right\} - \sqrt{\frac{N}{\Omega}}\{\alpha\sqrt{2f_m}\,f_a\} + \epsilon f_m\,, \qquad (6)$$

where f_a (f_m) denotes the fraction of atoms (molecules), $f_a = \Omega n_a/N$, ($f_m = \Omega n_m/N$). Since $f_a = 1 - 2f_m$, the energy is a function of the molecule fraction, the ground state value of which is determined by minimizing e.

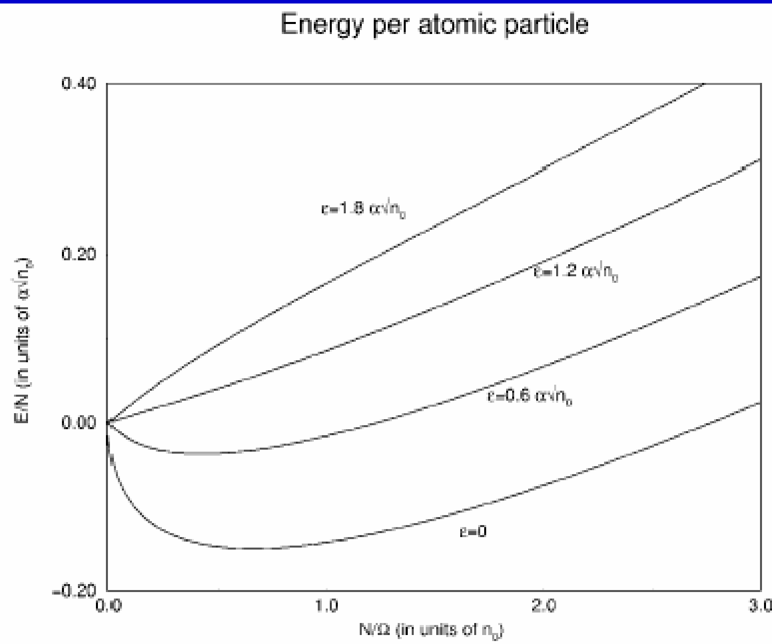


FIG. 2. Ground state energy per atomic particle as a function of the density at different detunings, ϵ. The curves were calculated using realistic values: a reference density $n_0 \sim 10^{14}$ cm^{-3}, and interaction strengths $\lambda_a = \alpha/\sqrt{n_0}$, $\lambda_m = 2\alpha/\sqrt{n_0}$, and $\lambda = 0.2\alpha/\sqrt{n_0}$. The densities at which the minima occur for the two curves of lowest detuning are the self-determined densities that a "free" condensate would adopt in the ground state.

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Signatures of Resonance Superfluidity in a Quantum Fermi Gas

M. L. Chiofalo,* S. J. J. M. F. Kokkelmans, J. N. Milstein, and M. J. Holland

Feshbach resonance between the lowest hyperfine states of ^{40}K

$$H = \sum_{k\sigma} \epsilon_k a^\dagger_{k\sigma} a_{k\sigma} + \nu \sum_k b^\dagger_k b_k + U \sum_{qkk'} a^\dagger_{q/2+k\uparrow} a^\dagger_{q/2-k\downarrow} a_{q/2-k'\downarrow} a_{q/2+k'\uparrow} + \left(g \sum_{kq} b^\dagger_q a_{q/2-k\downarrow} a_{q/2+k\uparrow} + \text{H.c.} \right),$$

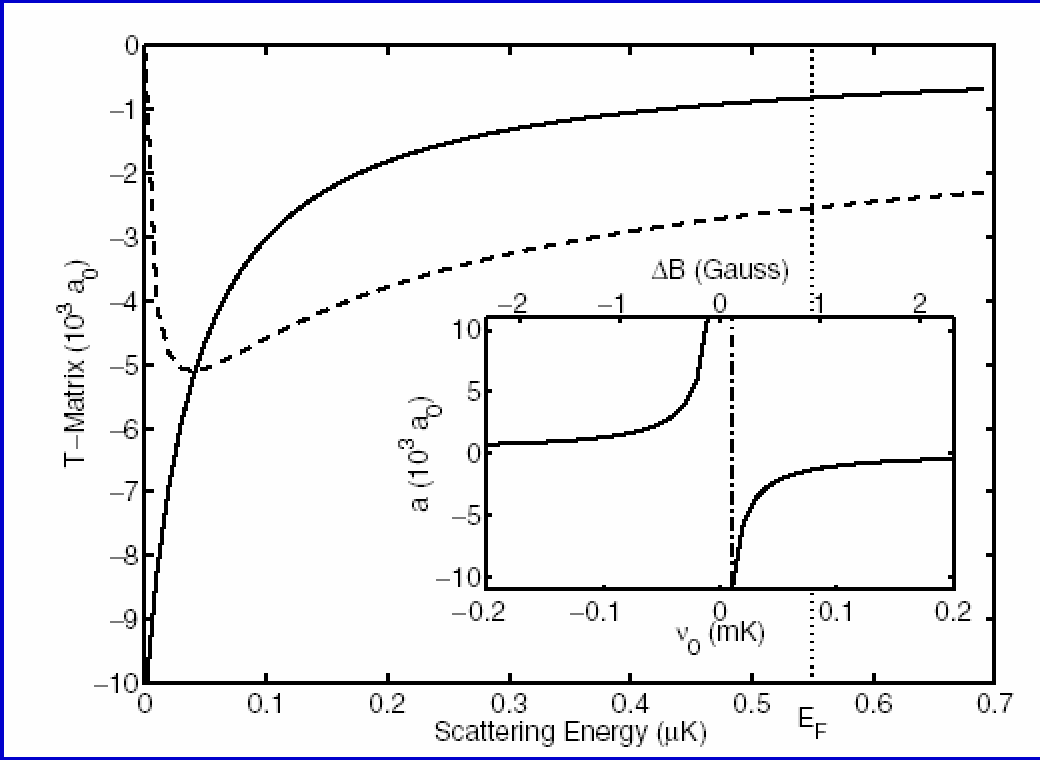
$$a = a_{bg}\left(1 - \frac{\kappa}{\nu_0}\right)$$

$$a_{bg} = 176\ a_0,\quad \kappa = 0.657\ \text{mK},\quad \nu_0 = 20 E_F$$

$$U_0 = \frac{4\pi\hbar^2 a_{bg}}{m},\quad g_0 = \sqrt{\kappa U_0}$$

$$\begin{cases} U = U_0\Gamma,\quad g = g_0\Gamma,\quad \nu = \nu_0 + \alpha g g_0 \\ \Gamma^{-1} = 1 - \dfrac{m k_c}{2\pi^2\hbar^2} U_0 \end{cases}$$

Cut-off regularization



Solid line – *Re* T, dashed line *Im* T
With high accuracy →

$$T = \frac{4\pi\hbar^2}{m}\frac{1}{-\frac{1}{a} - ik}$$

No (bare) molecule-molecule interaction (unlike Timmermans *et al.*)

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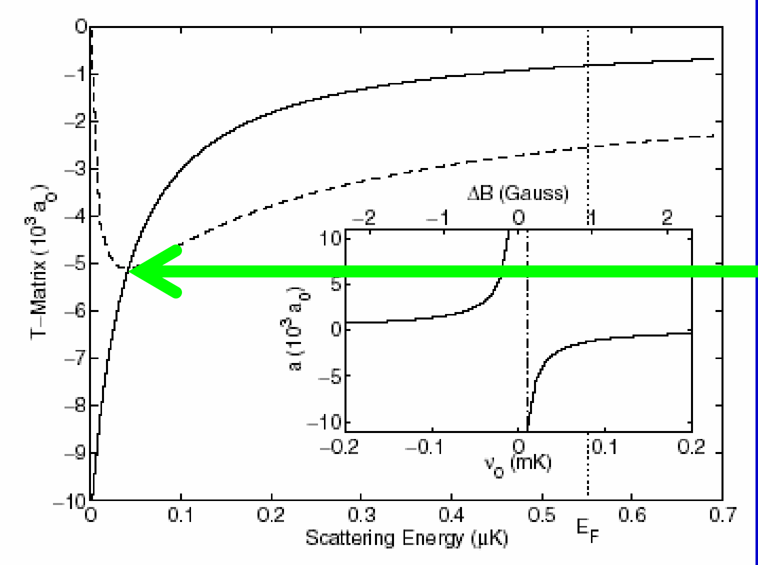


$$T = \frac{4\pi\hbar^2}{m} \frac{1}{-\frac{1}{a} - ik}$$

FIG. 1. Real (solid line) and imaginary (dashed line) components of the T matrix for collisions of the lowest two spin states of ^{40}K at a detuning of $20E_F$, shown in length dimensions, i.e., $T_k/(4\pi\hbar^2/m)$. The scattering length is the intercept at zero scattering energy which for this case is approximately $-10\,000a_0$, where a_0 is the Bohr radius. The large variation in the T matrix over the relevant energy range indicates that a quantum field theory developed from this microscopic basis will in general need to account for physics beyond the scattering length approximation. The inset shows the scattering length as a function of detuning, with $20E_F$ detuning indicated by the dash-dotted line. This curve obeys the following form: $a = a_{bg}(1 - \kappa/\nu_0)$, where $a_{bg} = 176a_0$ and $\kappa = 0.657$ mK [20]. The quasipotentials to be renormalized are then $U_0 = 4\pi\hbar^2 a_{bg}/m$ and $g_0 = \sqrt{\kappa U_0}$.

With goods accuracy over the entire energy range, no effective range corrections appear necessary on this plot!

Really this means only that $ka = O(1)$.

What do we know about dilute Bose systems?

For details see Braaten, Hammer and Hermans Phys. Rev. A 63, 063609 (2001)

$$\frac{E}{N} = \frac{2\pi\hbar^2 a}{m}\left\{1 + \frac{128}{15}\sqrt{\frac{na^3}{\pi}} + \left(16\pi na^3\right)\left[c_E + \frac{4\pi - 3\sqrt{3}}{6\pi}\ln(16\pi na^3)\right]\right\}$$

$$+ \frac{2\pi\hbar^2 a}{m}\left(16\pi na^3\right)^{3/2}\left[\frac{16}{\pi}\left(c_E + \frac{4\pi - 3\sqrt{3}}{6\pi}\ln(16\pi na^3) - \frac{16}{\pi}\frac{r_s}{a} + b'\right)\right]$$

Bogoliubov (1947) Lee, Huang and Yang (1957)

Effective range corrections appear only at this order!

$na^3 << 1$
$a>0$ - s-wave scattering length
r_s - s-wave effective range
c_E - potential dependent parameter (three-body collisions)
b' - universal constant

What do we know about dilute normal Fermi systems?

(For a recent review see Hammer and Furnstahl, Nucl. Phys. A678, 277 (2000))

$$\frac{E}{N} = \frac{k_F^2}{2M}\left[\frac{3}{5} + (g-1)\left\{\frac{2}{3\pi}(k_F a_s) + \frac{4}{35\pi^2}(11 - 2\ln 2)(k_F a_s)^2 + \frac{1}{10\pi}(k_F r_s)(k_F a_s)^2 + \big(0.076 + 0.057(g-3)\big)(k_F a_s)^3\right\} + (g+1)\frac{1}{5\pi}(k_F a_p)^3 + (g-1)(g-2)\frac{16}{27\pi^3}(4\pi - 3\sqrt{3})(k_F a_s)^4 \ln(k_F a_s) + \cdots\right]. \quad (1)$$

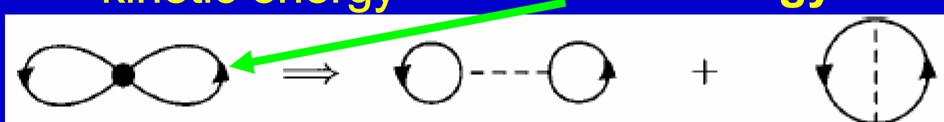
g – spin degeneracy, a_s - s-wave scattering length,
a_p – p-wave scattering length, r_s – s-wave effective range

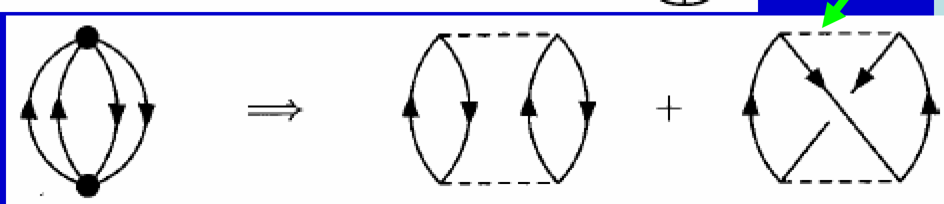
kinetic energy

HF energy

correlation energy

$$\frac{1}{2}\sum_i\left(\omega_i^{RPA} - \omega_i^{HF}\right)$$





effective range corrections appears at this order

Lessons:

- ✓ The first type of correction to be accounted for in both Bose and Fermi systems is the Lee & Yang, Huang, Luttinger (1957) correlation energy, which is still determined by the scattering length.
- ✓ The next type of correction for Bose systems (Hugengoltz & Pines, Wu, Sawada & Brueckner and Sawada, 1959) is a little bit more complicated, but it is still largely controlled by the scattering length and additionaly by a three-body characteristic (Braaten & Nieto, 1999)
- ✓ Effective range corrections appear only much later.
- ✓ More importantly, the corrections to mean-field are <u>always</u> controlled by the parameter na^3
- ✓ When the parameter na^3 becomes large, other methods are required.

"Fundamental" and "effective" Hamiltonians

$$H_{\text{fund}} = -\sum_s \int d^3r\, \psi_s^\dagger(\boldsymbol{r}) \frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} \psi_s(\boldsymbol{r}) + \frac{1}{2} \sum_{s_1,s_2} \int d^3r_1 d^3r_2\, \psi_{s_1}^\dagger(\boldsymbol{r}_1) \psi_{s_2}^\dagger(\boldsymbol{r}_2) \psi_{s_2}(\boldsymbol{r}_2) \psi_{s_1}(\boldsymbol{r}_1) V_{s_1 s_2}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|),$$

Since one is interested in phenomena with momenta $p = \hbar k \ll \hbar/r_0$, where r_0 is the typical range of the interaction, the "fundamental" Hamiltonian is too complex.

$$\begin{aligned} H_{\text{eff}}(\boldsymbol{r}) = & -\psi^\dagger(\boldsymbol{r}) \frac{\hbar^2 \boldsymbol{\nabla}_a^2}{2m} \psi_a(\boldsymbol{r}) + \frac{1}{2}\lambda_{aa} \psi_a^\dagger(\boldsymbol{r}) \psi_a^\dagger(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \\ & -\psi^\dagger(\boldsymbol{r}) \frac{\hbar^2 \boldsymbol{\nabla}_m^2}{4m} \psi_m(\boldsymbol{r}) + \varepsilon \psi_m^\dagger(\boldsymbol{r}) \psi_m(\boldsymbol{r}) + \frac{1}{2}\lambda_{mm} \psi_m^\dagger(\boldsymbol{r}) \psi_m^\dagger(\boldsymbol{r}) \psi_m(\boldsymbol{r}) \psi_m(\boldsymbol{r}) \\ & +\lambda_{am} \psi_m^\dagger(\boldsymbol{r}) \psi_a^\dagger(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \psi_m(\boldsymbol{r}) + \boldsymbol{\alpha} \psi_m^\dagger(\boldsymbol{r}) \psi_a(\boldsymbol{r}) \psi_a(\boldsymbol{r}) + \boldsymbol{\alpha} \psi_a^\dagger(\boldsymbol{r}) \psi_a^\dagger(\boldsymbol{r}) \psi_m(\boldsymbol{r}), \end{aligned}$$

Working with contact couplings requires regularization and renormalization, which can be done in several different, but equivalent ways.

We will show that H_{eff} is over-determined.

A) Pseudo-potential approach

(appropriate for very slow particles, very transparent but somewhat difficult to improve)

Lenz (1927), Fermi (1931), Blatt and Weiskopf (1952)
Lee, Huang and Yang (1957)

$$-\frac{\hbar^2 \Delta_{\vec{r}}}{m}\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\,\psi(\vec{r}), \quad V(\vec{r}) \approx 0 \text{ if } r > R$$

$$\psi(\vec{r}) = \exp(i\vec{k}\cdot\vec{r}) + \frac{f}{r}\exp(ikr) \approx 1 + \frac{f}{r} + \ldots \approx 1 - \frac{a}{r} + O(kr)$$

$$f^{-1} = -\frac{1}{a} + \frac{1}{2}r_0 k^2 - ik, \qquad g = \frac{4\pi\hbar^2 a}{m(1+ika)} + O(k^2)$$

$$\text{if } \; kr_0 << 1 \quad \text{then} \quad V(\vec{r})\psi(\vec{r}) \Rightarrow g\,\delta(\vec{r})\frac{\partial}{\partial r}\left[r\psi(\vec{r})\right]$$

$$\text{Example : } \; \psi(\vec{r}) = \frac{A}{r} + B + \ldots \Rightarrow \delta(\vec{r})\frac{\partial}{\partial r}\left[r\psi(\vec{r})\right] = \delta(\vec{r})B$$

B) Momentum cut-off regularization

$$G_0(\vec{r},\vec{r}\,',E) = \lim_{k_c\to\infty}\int_{l\le k_c}\frac{d^3l}{(2\pi)^3}\frac{\exp[i\vec{l}\cdot(\vec{r}-\vec{r}\,')]}{E-\dfrac{\hbar^2k^2}{m}=i\eta} \qquad \vec{r},\vec{r}\,' \text{ - relative coordinate}$$

$$\stackrel{k_c\to\infty}{=} -\frac{m\exp(ik\,|\vec{r}-\vec{r}\,'|)}{4\pi\hbar^2\,|\vec{r}-\vec{r}\,'|} = -\frac{m}{4\pi\hbar^2\,|\vec{r}-\vec{r}\,'|} - \frac{imk}{4\pi\hbar^2} + O(|\vec{r}-\vec{r}\,'|^2)$$

$$T(\vec{r},\vec{r}\,',E) = \lambda_{aa}\delta(\vec{r})\delta(\vec{r}-\vec{r}\,') + \iint d^3s\,d^3l\,\lambda_{aa}\delta(\vec{r})\delta(\vec{r}-\vec{s})G_0(\vec{s},\vec{t},E)T(\vec{t},\vec{r}\,',E)$$

$$T(k) = \frac{\lambda_{aa}}{1-\lambda_{aa}\Gamma} \stackrel{def}{=} \frac{4\pi\hbar^2a}{m}\frac{1}{1+ika} + O(k^2)$$

$$\Gamma = \int_{l\le k_c}\frac{d^3l}{(2\pi)^3}\frac{1}{E-\dfrac{\hbar^2k^2}{m}+i\eta} = -\frac{mk_c}{2\pi^2\hbar^2} - i\frac{mk}{4\pi\hbar^2} + O(k^2)$$

$$\frac{m}{4\pi\hbar^2a} = \frac{1}{\lambda_{aa}} + \frac{mk_c}{2\pi^2\hbar^2}$$

NB the coupling constant runs with cut-off

C) Dimensional regularization (DR)

'‘t Hooft and Veltman, Nucl. Phys. **B44**, 189 (1972)

$$\int_0^\infty dk\, k^n \overset{def}{=} 0 \qquad \text{iff} \qquad n \neq -1$$

$$G(\vec{r},\vec{r}\,',E)\Big|_{r=r'=0} \overset{def}{=} -i\,\frac{mk}{4\pi\hbar^2}$$

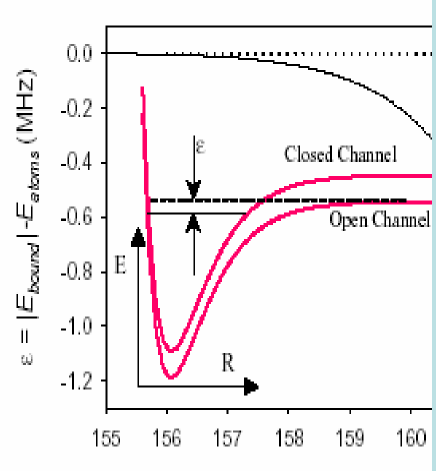
$$\frac{m}{4\pi\hbar^2 a} \overset{def}{=} \frac{1}{\lambda_{aa}} \quad \longleftarrow$$

Practical consequences:

- ✓ most loop diagrams vanish identically
- ✓ coupling constants do not run anymore with cut-offs

Explicit introduction of "Feshbach molecules"

Example: open channel — two ^{85}Rb atoms in $f = 2$, $m_f = -2$ state each
closed channel — two ^{85}Rb atoms in $f = 3$ each and total $M_f = -4$



$$\begin{cases} -\dfrac{\hbar^2}{m}\Delta\psi_1(\vec r) + V_{11}(\vec r)\psi_1(\vec r) + V_{12}(\vec r)\psi_2(\vec r) = E\psi_1(\vec r) \\ -\dfrac{\hbar^2}{m}\Delta\psi_2(\vec r) + V_{21}(\vec r)\psi_1(\vec r) + V_{22}(\vec r)\psi_2(\vec r) = E\psi_2(\vec r) \end{cases}$$

$$V_{aa} \Rightarrow V_{11} + V_{12}\frac{1}{E - T_k - V_{22} + i\eta}V_{21} \approx V_{11} + V_{12}\,|\phi_0\rangle\frac{1}{E - E_0 + i\eta}\langle\phi_0|V_{21}$$

$$(T_k + V_{22})\phi_0 = E_0\phi_0$$

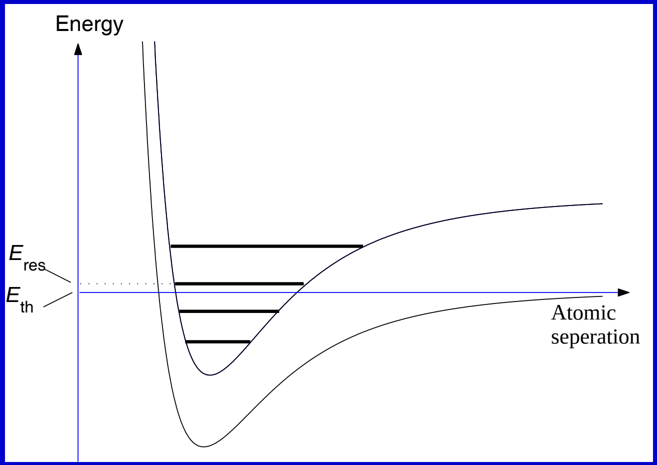
$$\frac{m}{4\pi\hbar^2 a} = \frac{1}{\lambda_{aa} - \dfrac{\alpha^2}{\varepsilon}} + \frac{mk_c}{2\pi^2\hbar^2} \quad \text{(cut-off reg.)} \qquad \frac{m}{4\pi\hbar^2 a} = \frac{1}{\lambda_{aa} - \dfrac{\alpha^2}{\varepsilon}} \quad \text{(DR)}$$

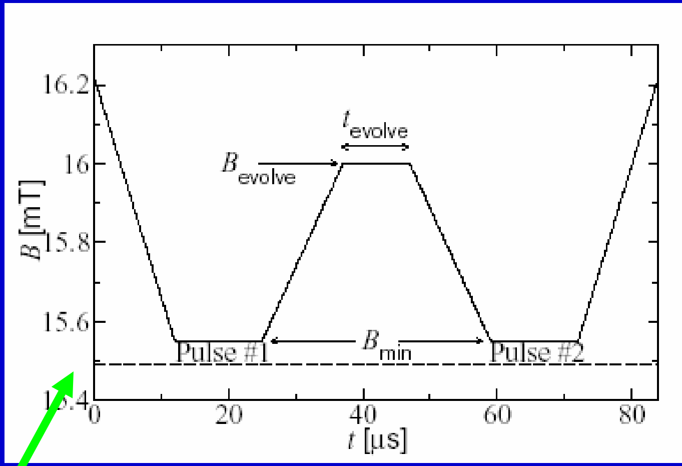
Lesson:

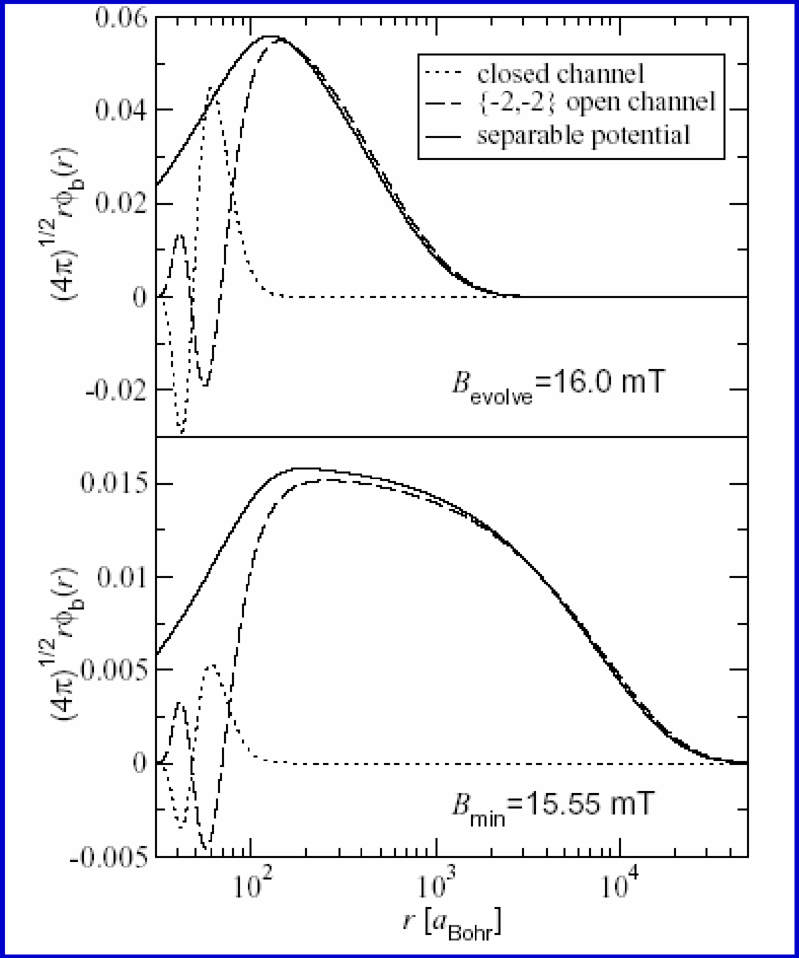
$$\lambda_{aa} \Rightarrow \lambda_{aa}^{eff} = \lambda_{aa} - \frac{\alpha^2}{\varepsilon}$$

After introducing contact couplings

Köhler, Gasenzer, Jullienne and Burnett, cond-mat/0305028.







Feshbach resonance

NB The size of the "Feshbach molecule" (closed channel state) is largely B-independent and smaller than the interparticle separation.

Some simple estimates in case a > 0 and $a \gg r_0$

wf in open channel at $r > r_0$

$$r\psi_1(r) = Ar_0\left[1 + O\left(\frac{r_0}{a}\right)\right]\exp\left(-\frac{r}{a}\right)$$

wfs in region $r < r_0$

$$r\psi_1(r) \approx r\psi_2(r) \approx r_0 A$$

Probability to find two atoms:

$$P(r < r_0) = \int_0^{r_0} r^2 dr\left[\psi_1(r)^2 + \psi_2(r)^2\right] \approx \frac{2A^2 r_0^3}{3} \quad \left(\text{or } \frac{A^2 r_0^3}{3} \text{ if oscillate}\right)$$

$$P(r > r_0) = \int_{r_0}^{\infty} r^2 dr \psi_1(r)^2 \approx \frac{A^2 a r_0^2}{2}$$

$$\frac{P(r > r_0)}{P(r < r_0)} \approx \frac{3a}{4r_0} >> 1 \quad \left(\text{or } \frac{3a}{2r_0}\right)$$

Most of the time the two atoms spend at large separations,

y1(r) — open channel (dimer), y2(r) — closed channel (Feshbach molecule)

- So far we discussed only interaction between atoms and one needs to include molecules.

- If atoms and molecules coexist, it makes sense to introduce molecules as independent degrees of freedom.

- Previously various authors, starting with Timmermans *et al. (1998)* introduced explicitely the "Feshbach molecules" for several reasons:

❖ There was hope to overcome the restriction na^3 á 1 close to a Feshbach resonance, when $|a| \gg r_0$, and replace it hopefully with the milder condition nr_0^3 á 1 and thus still be able to use the many-body tools developed for dilute systems.

❖ Develop a formalism for a mixture of atoms and molecules.

- It is relatively easy to convince oneself that corrections to the energy of a system of either Bose atoms and molecules, or Fermi atoms and Bose molecules (bound state of two Fermi atoms) are <u>always</u> controlled by the parameter na^3 and <u>never</u> by the parameter nr_0^3.

  (Essentially one has to repeat the old Lee, Young and Huang 1957 calculations and compute the correlation energy.)

- In order to decide whether a given program is feasible one has to construct the ground state properties of the system under consideration within the framework of the formalism of choice and then consider higher order corrections. This aspect was largely ignored in previous works.

- We shall develop a theoretical framework to describe atoms and dimers (not Feshbach molecules) for the case $a>0$, $a \gg r_0$ and na^3 á 1. We shall show that in this regime a mixture of atoms and dimers can be described by one coupling constant, the scattering length a.

- The regime $a > 0$ and $na^3 \gg 1$ (strong coupling) can be studied as well, but using different methods (*ab initio).*

- The regime $a<0$ and $|a| \gg r_0,$ $n|a|^3$ á 1 is also universal and a new class of truly quantum liquids (not gases) appears, see A.B. Phys. Rev. Lett. 89, 050402 (2002)

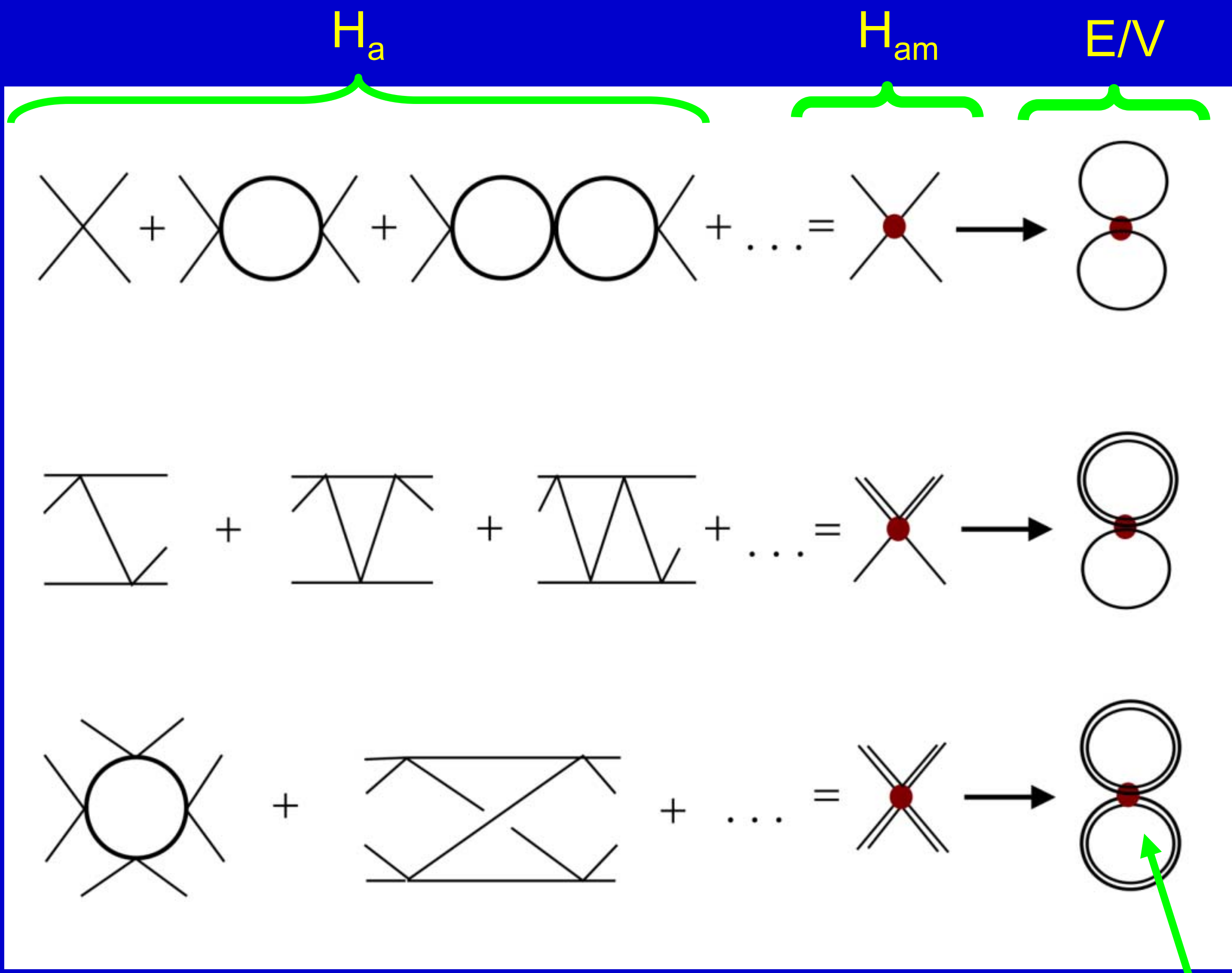
In order to develop our program we have at first to have a well defined procedure for constructing an effective Hamiltonian for interacting atoms and dimers starting from the "fundamental" Hamiltonian describing bare interacting atoms.

$$H_a = -\psi_a^+ \frac{\hbar^2 \nabla^2}{2m} \psi_a + \frac{1}{2} \lambda_2 \psi_a^+ \psi_a^+ \psi_a \psi_a + \frac{1}{3} \lambda_3 \psi_a^+ \psi_a^+ \psi_a^+ \psi_a \psi_a \psi_a$$

$$H_{am} = \psi_a^+ \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \psi_a + \psi_m^+ \left(-\frac{\hbar^2 \nabla^2}{4m} + \varepsilon_2 \right) \psi_m$$
$$+ \frac{1}{2} \lambda_{aa} \psi_a^+ \psi_a^+ \psi_a \psi_a + \lambda_{am} \psi_a^+ \psi_m^+ \psi_m \psi_a + \frac{1}{2} \lambda_{mm} \psi_m^+ \psi_m^+ \psi_m \psi_m$$

H_a is a low energy reduction of the "fundamental" Hamiltonian, l_2 and l_3 are determined by the scattering length a and a three-body characteristic (denoted below by a_3'). Interaction terms with derivatives are small as long as kr_0 á 1.

H_{am} is determined by the matching" to be briefly described below.



atom-atom vertex
(Lippmann-Schwinger eq.)

atom-dimer vertex
(Faddeev eqs.)

dimer-dimer vertex
(Yakubovsky eqs.)

Matching between the 2--, 3-- and 4--particle amplitudes computed with H_a and H_{am}.
Only diagrams containing l_2--vertices are shown.

The effective vertices thus defined (right side) can then be used to compute the ground state interaction energy in the leading order terms in an na^3 expansion, which is given by the diagrams after the arrows.

✓ In medium the magnitude of the relevant momenta are determined by estimating The quantum fluctuations of the mean-field. One thus easily can show that $p = \hbar k \approx \hbar(na)^{1/2}/m$. As long as kr_0 µ ka µ $(na^3)^{1/2}$ Ü 1 one can use contact couplings.

✓ The accuracy of the mean-field approximation can be ascertained by estimating the magnitude of the quantum fluctuations to the energy density µ$n\,(na^3)^{3/2}\hbar^2/ma^2$.

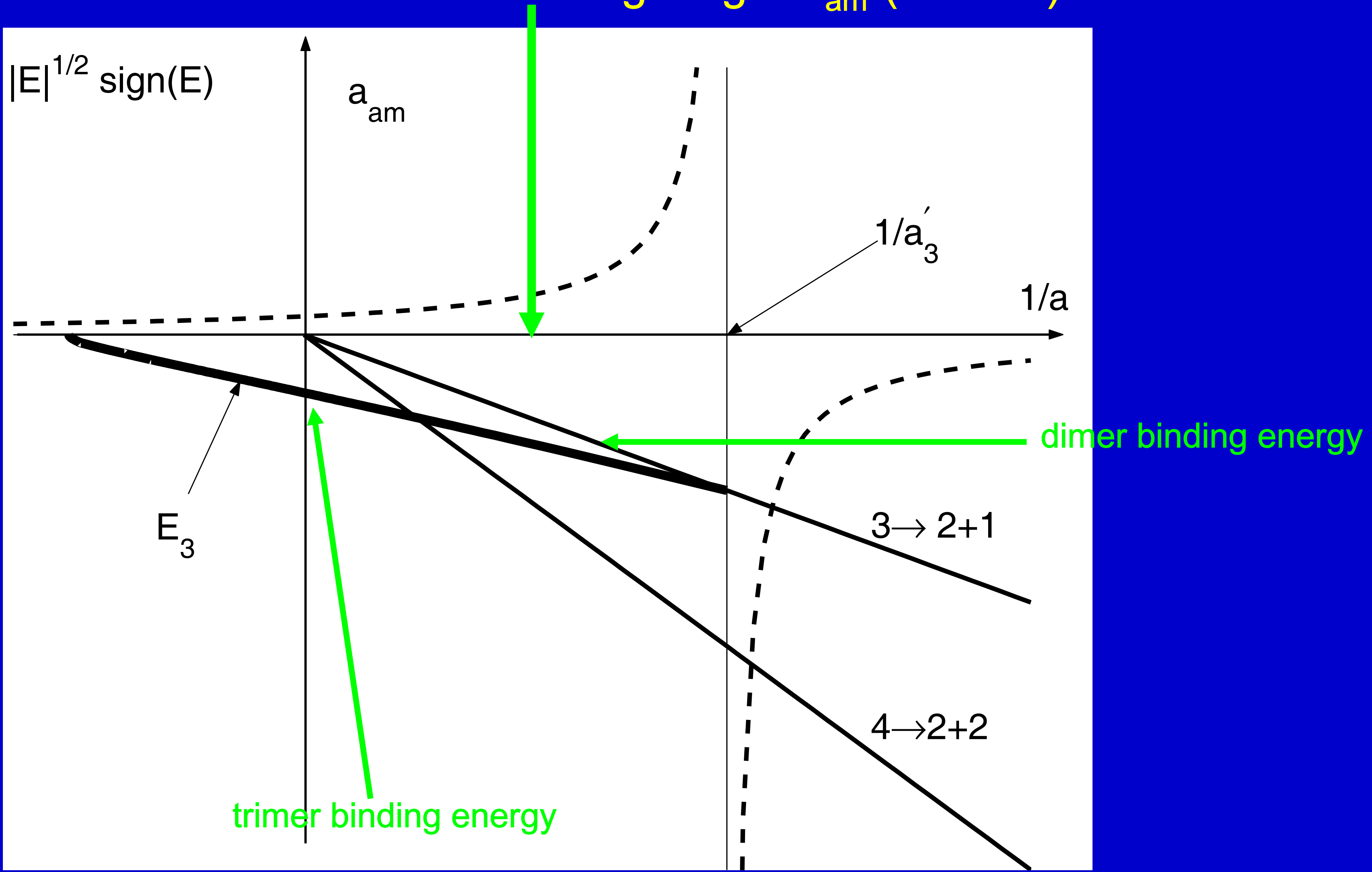
✓ We shall consider the regime when $a \gg r_0$ when the relevant momenta satisfy $p = \hbar k \approx \hbar/a$ ¥ $(na^3)^{1/2}$ Ü $\hbar/a$ Ü $\hbar/r_0$.

✓ Note that both Hamiltonians H_a and H_{am} are appropriate for ka Ü 1 and kr_0 Ü 1.

✓ However, while perturbation theory is not valid for H_a when $p \approx \hbar/a$, all the non-perturbative physics at this scale (dimers of size $\approx a$ and the Efimov effect) Have been encapsulatted in the couplings of the Hamiltonian H_{am}.

✓ The “matching” described here was performed in vaccum, at length scales of order $O(a)$ and this matching is not modified by the many-body physics, which occurs at scales $O(a/(na^3)^{1/2}) \gg O(a)$.

The atom-molecule scattering length a_{am} (bosons)



Efimov plot

Bose atoms

$$\lambda_{aa} = \frac{4\pi\hbar^2 a}{m}, \qquad \varepsilon_2 = -\frac{\hbar^2}{ma^2}$$

$$\lambda_{am} = \frac{3\pi\hbar^2 a_{am}}{m} = \frac{3\pi\hbar^2 a}{m}\left[c_1 + c_2 \cot\left(s_0 \ln\frac{a}{a_3'}\right)\right],$$

$$c_1 \cong 1.46, \quad c_2 \cong 2.15$$

$$\lambda_{mm} = \frac{2\pi\hbar^2 a_{mm}}{m} = \frac{4\pi\hbar^2 a}{m} c_3,$$

$$c_3 \cong 1 \quad (?)$$

Efimov derived the analytical form (1979). Simenog and Sytnichenko (1981) and Braaten, Hammer and Kusunoki (2003) computed the numerical constants c_1 and c_2.

Fermi atoms

$$\lambda_{aa} = \frac{4\pi\hbar^2 a}{m}, \qquad \varepsilon_2 = -\frac{\hbar^2}{ma^2}$$

$$\lambda_{am} = \frac{3\pi\hbar^2 a_{am}}{m} = \frac{3.537\pi\hbar^2 a}{m}, \qquad a_{am} = 1.179a$$

$$\lambda_{mm} = \frac{2\pi\hbar^2 a_{mm}}{m} = \frac{4\pi\hbar^2 a}{m} c_3, \qquad a_{mm} \overset{?}{=} 2a, \quad c_3 \overset{?}{=} 1$$

This amplitude was first computed first by Skornyakov and Ter-Martirosian (1957) who studied neutron-deuteron scattering. Randeria (and others) estimated c_3 (1993).

Role of effective range corrections on fermion-boson scattering length

$$k \cot \delta(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + \dots$$

if $\frac{r_0}{a}$ is varied from 0.0 to 0.3 then $\frac{a_{fb}}{a} = 1.179$ changes by less than 1%.

BCS →BEC crossover

Leggett (1980), Nozieres and Schmitt-Rink (1985), Randeria *et al.* (1993)

If a<0 at T=0 a Fermi system is a BCS superfluid

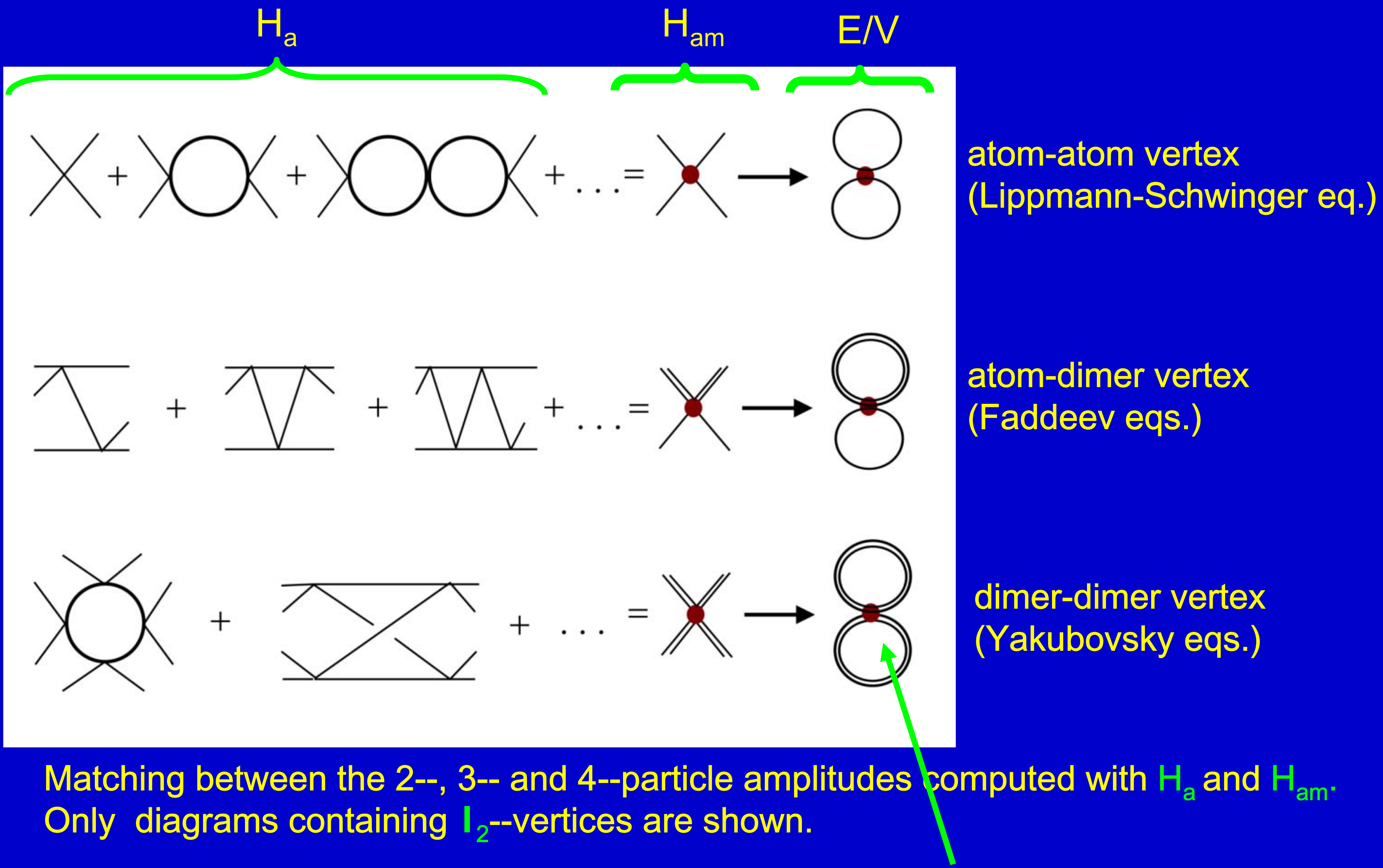
$$\Delta \approx \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left(\frac{\pi}{2k_F a}\right), \quad \text{iff} \quad k_F\,|a| << 1 \text{ and } \xi = \frac{1}{k_F}\frac{\varepsilon_F}{\Delta} >> \frac{1}{k_F}$$

If |a|=∞ and nr_0^3 á 1 a Fermi system is strongly coupled and its properties are universal. Carlson *et al.* nucl-th/0302041 and Carlson *et al.* physics/0303094

$$\frac{E_{\text{normal}}}{N} \approx 0.54\frac{3}{5}\varepsilon_F, \qquad \frac{E_{\text{superfluid}}}{N} \approx 0.44\frac{3}{5}\varepsilon_F \quad \text{and } \xi = O(\lambda_F)$$

If a>0 ($a \gg r_0$) and na^3 á 1 the system is a dilute BEC of tightly bound dimers

$$\varepsilon_2 = -\frac{\hbar^2}{ma^2} \quad \text{and} \quad n_b a^3 << 1, \quad \text{where} \quad n_b = \frac{n_f}{2} \text{ and } a_{bb} \stackrel{?}{=} 2a > 0$$



Matching between the 2--, 3-- and 4--particle amplitudes computed with H_a and H_{am}. Only diagrams containing I_2--vertices are shown.

The effective vertices thus defined (right side) can then be used to compute the ground state interaction energy in the leading order terms in an na^3 expansion, which is given by the diagrams after the arrows.

Consider now a dilute mixture of fermionic atoms and (bosonic) dimers at temperatures smaller than the dimer binding energy ($a>0$ and $a \gg r_0$)

$$\frac{E}{V} = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m} n_f + \frac{\pi\hbar^2 a}{m} n_f^2 + \frac{3.537\pi\hbar^2 a}{m} n_f n_b + \frac{2\pi\hbar^2 a}{m} n_b^2 + \varepsilon_2 n_b + \text{corrections}$$

$$n_f = \frac{k_F^3}{3\pi^2}, \qquad \varepsilon_2 = -\frac{\hbar^2}{ma^2}$$

We shall show that pairing is weak!

$$U_{fbf}(q,\omega) = U_{fb}^2 \frac{2n_b\varepsilon_q}{\hbar^2\omega^2 - \varepsilon_q(\varepsilon_q + 2n_b U_{bb})}$$

$$U_{bb} = \frac{4\pi\hbar^2 a_{bb}}{m_b}, \qquad \varepsilon_q = \frac{\hbar^2 q^2}{2m_b}$$

in coordinate representation at $\omega = 0$

$$U_{fbf}(r) = -\frac{U_{fb}^2}{U_{bb}}\frac{1}{4\pi\xi_b^2 r}\exp\left(-\frac{r}{\xi_b}\right)$$

$$\xi_b = \frac{\hbar}{2m_b s_s} = \frac{a_{bb}}{\sqrt{16\pi n_b a_{bb}^3}} >> a_{bb}, \qquad s_b^2 = \frac{n_b U_{bb}}{m_b}$$

Induced fermion-fermion interaction

Bardeen *et al.* (1967),
Heiselberg *et al.* (2000),
Bijlsma *et al.* (2000)
Viverit (2000),
Viverit and Giorgini (2000)

← coherence/healing length and speed of sound

The overall fermion-fermion intercation is a sum of a short range repulsion + weak long range attraction

repulsion

weak attraction

$$U_{ff}(q) = \frac{\pi\hbar^2 a}{m}\left[2 - \frac{3.128}{1+q^2\xi_b^2}\right]$$

Calogero's criterion for existense of a bound state

$$\frac{2}{\pi}\int_0^\infty dr\sqrt{\frac{-m_f U_{fbf}(r)}{\hbar^2}} = \sqrt{\frac{2U_{fb}^2 m_f}{\pi^2\hbar^2 U_{bb}\xi_b}} \overset{?}{\geq} 1$$

cannot be satisfied for dilute systems, when $\xi_b >> a$

Consequently, weak coupling BCS pairing is thus expected.

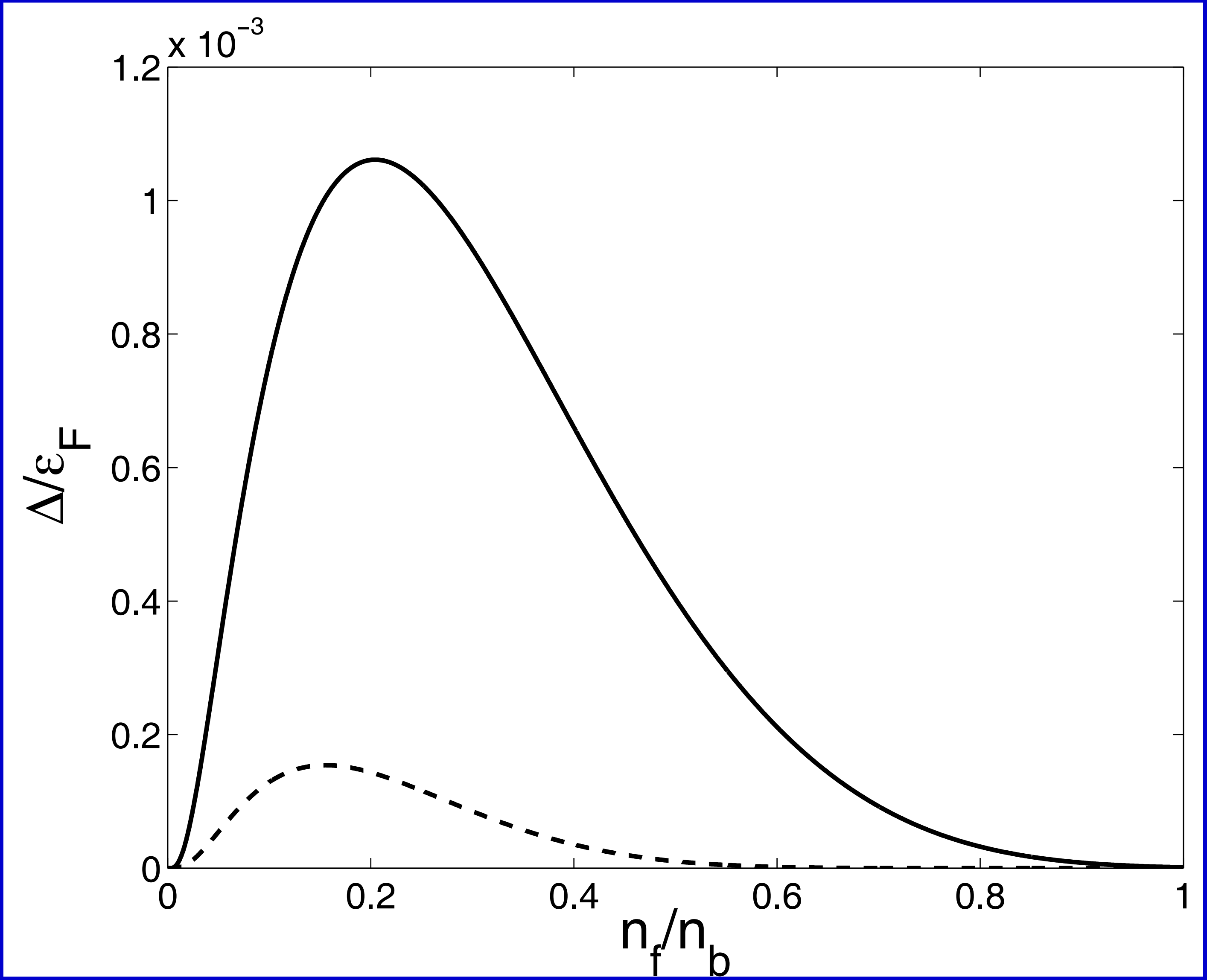
If $a_{bb} = 2a$!

$$\Delta = \left(\frac{2}{e}\right)^{7/3} \frac{\hbar^2 k_F^2}{2m} \exp\left[\frac{2}{\pi k_F a}\left(1 - 1.564\,\frac{\ln\left(1+4k_F^2\xi_b^2\right)}{4k_F^2\xi_b^2}\right)^{-1}\right]$$

$$2k_F\xi_b = 0.62\left(\frac{n_f}{n_b}\right)^{1/3} \frac{1}{\left(n_b a^3\right)^{1/6}} \propto \frac{s_f}{s_b}$$

$$s_f = \frac{\mathrm{v}_F}{\sqrt{3}} = \frac{\hbar\left(3\pi^2 n_f\right)^{1/3}}{\sqrt{3}m}, \qquad s_b = \sqrt{\frac{4\pi\hbar^2 a n_b}{m}}$$

If fermions are superfluid,
otherwise s_f is greater than the Fermi velocity (Landau's zero sound).

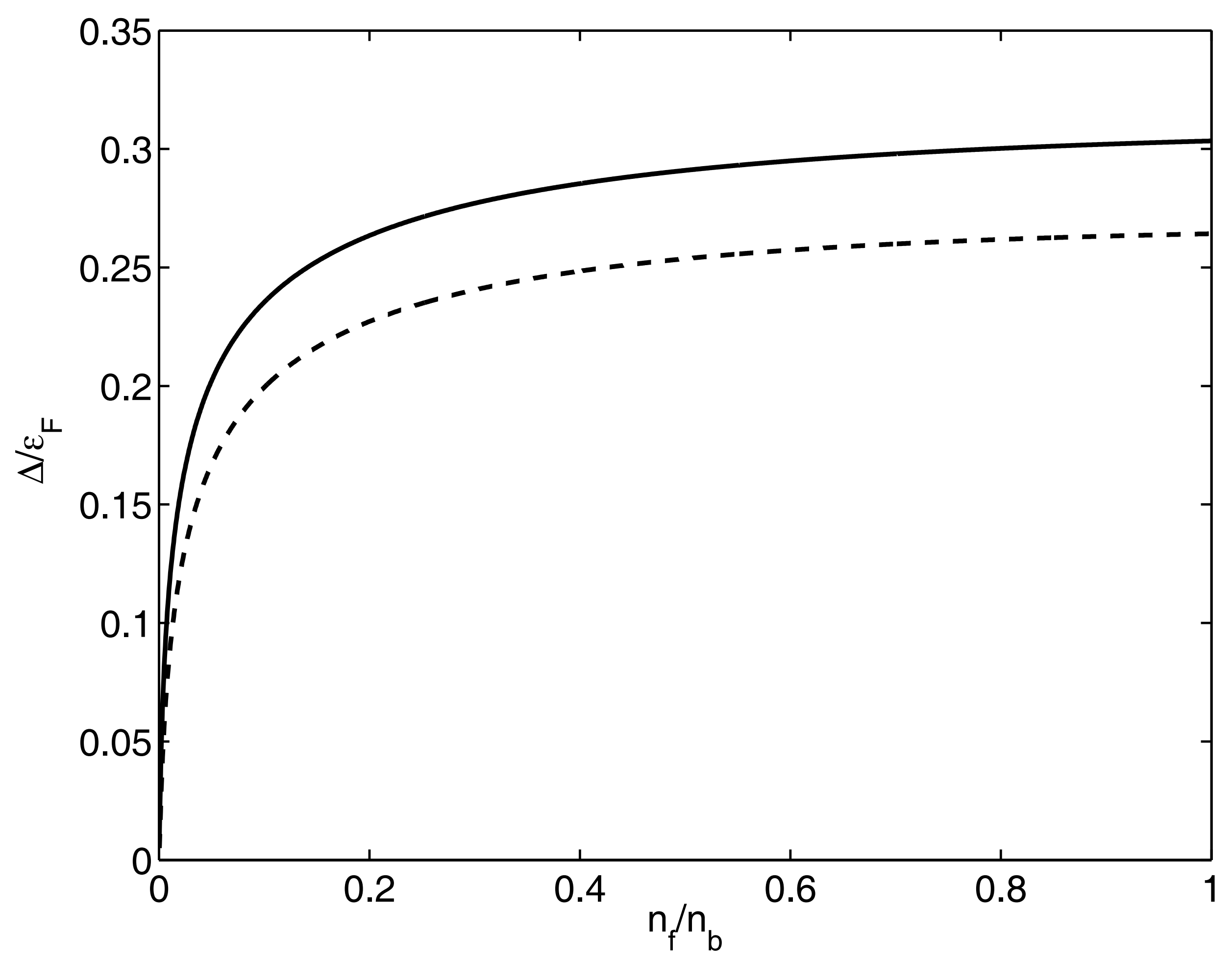


$a = n_b^{-1/3}/2.5$ (solid line)

$a = n_b^{-1/3}/3$ (dashed line)

The value of the gap depends strongly on a_{bb}!

The same thing using an unpublished result from Dmitry Petrov and G. Shlyapnikov for the dimer-dimer scattering length 0.6a instead of the approximate old value of 2a. Pieri and Strinati, Phys. Rev. B 61, 15730 (2000) quote a value of 0.75a for this quantity.



$a = n_b^{-1/3}/2.5$ (solid line)

$a = n_b^{-1/3}/3$ (dashed line)

The value of the gap depends strongly on a_{bb} !

The density distribution in a trap can be determined rather accurately in the Thomas-Fermi approximation and since the pairing field is rather weak one can neglect the influence of the pairing field.

Trapping potentials for fermions and bosons respectively

$$\frac{\hbar^2 k_F^2(\vec{r})}{2m} + \left(U_{ff} - \frac{U_{fb}^2}{U_{bb}}\right) n_f(\vec{r}) = \mu_f - V_f(\vec{r}) - \left[\mu_b - \varepsilon_2 - V_b(\vec{r})\right]$$

$$\left(U_{ff} - \frac{U_{fb}^2}{U_{bb}}\right) = \frac{\pi\hbar^2 a}{m}\left(2 - 3.\overset{?}{1}28\right) \overset{?}{=} -1.128\frac{\pi\hbar^2 a}{m},$$

$$n_b(\vec{r}) = \frac{\mu_b - \varepsilon_2 - V_b(\vec{r})}{U_{bb}} - n_f(\vec{r})\frac{U_{fb}}{U_{bb}}, \qquad \frac{U_{fb}}{U_{bb}} \overset{?}{=} \frac{3.537}{4}, \quad U_{bb} \overset{?}{=} \frac{4\pi\hbar^2 a}{m}$$

Atomic-Molecular BEC (amBEC)

$$\frac{E}{V} = \frac{2\pi\hbar^2}{m} n_a^2 + \frac{3\pi\hbar^2}{m}\left[c_1 + c_2 \cot\left(s_0 \ln\frac{a}{a_3'}\right)\right] n_a n_b + \frac{2\pi\hbar^2}{m} c_3 n_b^2 + \varepsilon_2 n_b$$

$$s_0 = 1.00624, \quad c_1 = 1.46, \quad c_2 = 2.15, \quad c_3 \overset{?}{=} 1, \quad \varepsilon_2 = -\frac{\hbar^2}{ma^2}, \quad a >> r_0$$

$$P = \frac{2\pi\hbar^2}{m} n_a^2 + \frac{3\pi\hbar^2}{m}\left[c_1 + c_2 \cot\left(s_0 \ln\frac{a}{a_3'}\right)\right] n_a n_b + \frac{2\pi\hbar^2}{m} c_3 n_b^2$$

If

$$T < T_c \propto \left(na^3\right)^{2/3} \frac{\hbar^2}{ma^2} << \frac{\hbar^2}{ma^2} \quad \text{and} \quad na^3 << 1$$

$$\mu_a = \lambda_{aa} n_a(\vec{r}) + \lambda_{ab} n_b(\vec{r}) + V_a(\vec{r})$$

$$\mu_b = \lambda_{ab} n_a(\vec{r}) + \lambda_{bb} n_b(\vec{r}) + V_b(\vec{r})$$

Reactions:

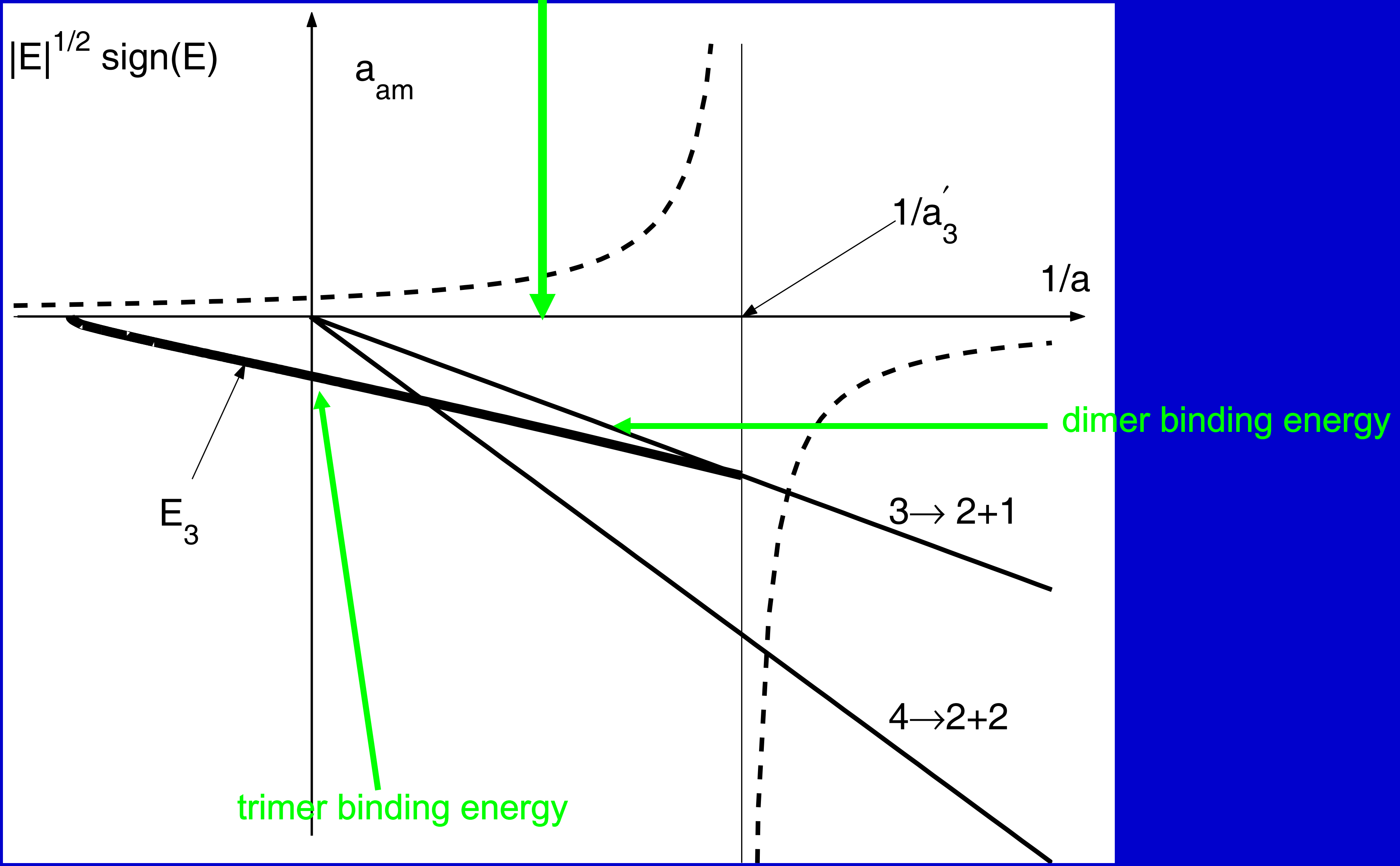
- ✓ $A+A+A \to A+A_2$ — rate ∂ $\hbar a^4 n_a^2/m$
- ✓ $A+A_2 \to A+A+A$ — suppressed (large activation energy)
  $A_2+A_2 \to A_2+A+A$

These reactions lead to slow heating and slow chemical equilibration

- ✓ $A+A_2 \to A^*+A_2^*$ — A^* and A_2^* are fast atom and dimer
  $A_2+A_2 \to A^*+A^*+A_2^*$

- The kinetic energies of these fast particles are $O(\hbar^2/mr_0^2) \gg (\hbar^2/ma^2) \gg O(3T/2)$ and momenta $O(\hbar/r_0) \gg O(\hbar/a)$ and have an interaction cross section $O(r_0^2)$ á $O(a^2)$.
- Similarly to neutrinos in the Sun, the products of these reactiosn react weakly with the medium and can in principle be used to monitor the amBEC
- The rates of these reaction are also controlled by the parameter ∂ r_0/a

The atom-molecule scattering length a_{am} (bosons)



Efimov plot

Another interesting reaction: $A_2+A_2 \to A^*+A_3^*$

if $a_{am}>0$ (if trimer exists)

- If $a_{am} \gg a$ the trimer is loosely bound and not very stable

$$e_3 \approx e_2 - 3\hbar^2/4ma_{am}^2$$

- Phase separation is most likely also in this case.
- If $0< a_{am}< a$ this reaction leads to a significant heating.
- If $a_{am}< 0$ and $|a_{am}|< a$ there are no shallow trimers and this reaction does not occur.

↑

Apparently this is the best regime for an amBEC

Conclusions

✓ There is a new universal regime in which one can describe atom-dimer mixtures (both Bose and Fermi constituents) in terms of a single parameter, the atom-atom scattering length, if this is relatively large and positive. The properties of these systems are widely tunable.

The systems one can study under these conditions are:

- ✓ Normal Fermi gas + Bose superfluid
- ✓ Superfluid Fermi gas + Bose superfluid
- ✓ Atomic Bose superfluid + molecular Bose superfluid