PHYS 560 Nuclear Physics

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Part I
Preliminaries

1 Units

length	$1\text{fm} = 10^{-15}\text{m} = 10^{-13}\text{cm}$
energy	$1\text{MeV} = 10^{6}\text{eV}$
	$1\text{eV} = 1.602 \cdot 10^{-19}\text{J}$
mass	$\left\{ \begin{array}{l} \left. \begin{array}{l} m_p c^2 = 938.2592\text{MeV} \\ m_n c^2 = 939.5527\text{MeV} \\ m_e c^2 = 0.511\text{MeV} \end{array} \right\} \text{Wapstra \& Gove 1971} \\ \left. \begin{array}{l} 1\text{u(amu)} = 931.4812\text{MeV} \\ m_{C^{12}} = 12\text{u} \end{array} \right\} \end{array} \right.$
velocity of light	$c = 2.9979 \cdot 10^{8}\text{m/s} \approx 3 \cdot 10^{23}\text{fm/s}$
Planck's constant	$\hbar c = 197.32\text{MeV} \cdot \text{fm}$
fine structure constant	$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$
	$e^2 = 1.44\text{MeV} \cdot \text{fm}$
cross sections	$1\text{b(barn)} = 10^{-24}\text{cm}^2 = 100\text{fm}^2$
	$1\text{mb} = 10^{-27}\text{cm}^2 = 0.1\text{fm}^2$
radioactivity	$1\text{Ci(Curie)} = 3.7 \cdot 10^{10}\text{decays/sec}$
	$1\text{rad} = 10^{-5}\text{J/g}$

2 Definition of cross section

One measures the interaction probability P which is given by

$$P = \frac{\overbrace{\Delta N}^{\#\, of\, deflected\, particles}}{\underbrace{I \cdot S \cdot \tau}_{total\, \#\, of\, incident\, particles}} = \frac{I \cdot \sigma \cdot \tau}{I \cdot S \cdot \tau} = \frac{\sigma}{S}$$

with

I - beam intensity (# of particles per sec per cm^2)
S - cross section (transversal surface) of the beam
τ - iradiation time
σ - effective cross section.

The effective cross section σ is a measure of the interaction probability according to $P = \frac{\sigma}{S}$.

Note: One does not need to know the values of 'b' (impact parameters), i.e. the exact position of the target. Loosley speaking, measuring effective cross sections is analogous to some kind of FOURIER analysis.

In reality one scatters particles not off a single target, but off many of them (a small piece of material which contains many of them). Then

$$P = \frac{\sigma}{S} \cdot \overbrace{n \cdot \Delta V}^{\#\, of\, nuclei\, in\, the\, target} = \frac{\sigma \cdot n \cdot S \cdot \Delta x}{S} = \sigma \cdot n \cdot \Delta x$$

with

ΔV - volume of the target
n - # of nuclei per unit volume
S - transversal cross section of the piece of material (target)
Δx - length (thickness) of the target.

Note: The above formula $P = \sigma \cdot n \cdot \Delta x$ is true only iff:

1. A particle from the incident beam scatters only once, only on a single nucleus.
2. Consequently 'n' (# of nuclei in the target per unit volume) cannot be large. (Large in comparison with what? - requires a separate analysis)

3. The effective radius R of a nucleus, given by $\sigma = \pi R^2$, must be small in comparison with the mean distance among nuclei (i.e. 'small' effective σ).
4. The nuclei in the target are randomly distributed. One cannot apply such a formula for a case like the one shown in fig. A1.2, i.e. when the beam is parallel (or almost parallel) to a chain of nuclei. This is a very special and physically very interesting case, called *channeling*. This is a phenomenon similar to a BRAGG defraction.

3 Attenuation of the beam

How many particles in a beam survive when passing through a medium?

$$\Delta N = -P \cdot N = -\sigma \cdot n \cdot \Delta x \cdot N$$

with

ΔN - # of particles deflected from the incident beam
P - interaction probability per particle
N - total # of particles entering the medium
Δx - thickness of the target .

The same restrictions mentioned above apply. The minus sign arises since $\Delta N = N_{final} - N_{initial} < 0$.
We can rewrite the previous equation to find

$$\frac{dN}{dx} \approx \frac{\Delta N}{\Delta x} = -\sigma \cdot n \cdot N$$

hence with $\lambda \equiv \sigma \cdot n$

$$N(x) = N_0 \cdot e^{-\frac{x}{\lambda}}.$$

Note: Δx cannot be very large (in comparison with 'λ') to apply $\Delta N = -\sigma \cdot n \cdot \Delta x \cdot N$.

What is 'n'? Answer: From $\rho = n \cdot \frac{M}{N_A}$ with

N_A - Avogadro's number ($N_A = 6 \cdot 10^{23}$ particles/mol)
ρ - density (mass per unit volume)
M - molar mass

we find $n = \frac{\rho \cdot N_A}{M}$.
'λ' has the meaning of a mean free path, i.e.

$$\lambda = \frac{\int_0^\infty x N(x) dx}{\int_0^\infty N(x) dx}$$

with

$\frac{N(x)dx}{\int_0^\infty N(x)dx}$ - relative # of particles which survived at distance 'x'.

Example: Fast neutron beam in air

From $\rho_{air} = \frac{30\text{g}}{22.4\text{l}}$ (i.e. M=30g), $\sigma = 400\text{mb}$ and $n = 0.9 \cdot 10^{-4}\text{cm}^{-3} N_A$ we get

$$\lambda^{-1} = 400 \cdot 10^{-27}\text{cm}^2 \cdot 0.9 \cdot 10^{-4}\text{cm}^{-3} \cdot 6 \cdot 10^{23} = 2.16 \cdot 10^{-5}\text{cm}^{-1}$$

respectively

$$\lambda \approx 460\text{m}$$

and

$$N(x) = N_0 \cdot e^{-\frac{x}{460\text{m}}}$$

If x = λ = 460m the beam has bean attenuated almost to a third of its initial intensity N_0.

Part II
Nuclear sizes

You should know that $A = N + Z$ and $R = r_0 A^{1/3}$ with $r_0 = 1.2\text{fm}$.
Experimental methods:

- electron-scattering
- muonic-atoms

} precise

- Coulomb-energy in mirror-nuclei
- total effective σ

} rough

4 Electron-scattering

Scattering off protons and deuterons

First let us look at p(e,e')p and d(e,e')d experiments, i.e. elastic scattering of fast electrons off protons (hydrogen nuclei) and deuterons.
If the proton would be a point particle then the electron elastic cross section would be simply the Rutherford one, i.e.

$$\frac{d\sigma}{d\Omega} \sim \frac{1}{q^4}$$

with

$d\Omega = \sin\Theta d\Theta d\Phi$ - solid angle
$\vec{q} = \vec{p} - \vec{p\prime}$ - transferred momentum
$\vec{p}, \vec{p\prime}$ - initial and final electron momentum

In fig. 2.1 this corresponds to the curve labelled by $r_c = r_m = 0$. (Pay attention only to the solid curves.) Fig. 2.2 shows the charge (upper fig.) density distribution obtained from the analysis of p(e,e')p and d(e,e')d data.

Conclusion: The nucleons (protons and neutrons) are not point particles, but they have some finite dimensions - they are roughly (diffuse) spheres of radius $R_N = 0.9$fm - and consequently some structure.

Formal description

The scattering of a Dirac-particle is described by the Mott-formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{Ruth}\left[1 - \beta^2 \sin^2\frac{\theta}{2}\right]$$

with

$$\left(\frac{d\sigma}{d\Omega}\right)_{Ruth} = \frac{4Z^2\alpha^2(\hbar c)^2E^2}{(qc)^4}$$

Part III
Isospin

Part IV
Independent particle model

Part V
Magnetic dipole moments

Part VI
Pairing

For A=fixed one varies Z. Fig. 6.1 shows the binding energy

$$B(Z,N) = Zm_p + Nm_n - M(Z,N)$$

where $M(Z,N)$ is the mass of a $^{A}_{Z}X_N$ nucleus.

Note:

1. $B(A, A-Z)$ looks like a parabola (for A fixed).

2. The curve corresponding to odd Z and odd N is shifted upward, i.e. less binding.

The even-even nuclei are more bound (on average by about 1-2 MeV, this quantity varies over the periodic table smoothly) than even-odd or odd-odd nuclei. On the energy scale this effect is small but it leads to changes of orders of magnitude in transition rates!
To explain this effect one needs two ingredients:

1. Single particle orbitals in time-conjugated states (i.e. $j_z = m$ and $j_z = -m$) have a strong spatial overlap. In Fig. 6.3 you can see it for the case l=2 (spin neglected). Let us consider an orbit with high 'l'. The states with $m = \pm l$ have an angular part proportional to $sin^l\theta e^{\pm il\phi}$. The wave functions are concentrated in the equatorial

plane (x0y). $\vec{l}$ is 'parallel' to the 0z-axis in this case. For 'high' l the radial part of the wave function is pushed towards the surface by the centrifugal barrier. Consequently the $m = \pm l$ states have the shape of a doughnut.

2. The interaction between two nucleons is short-ranged and attractive. Consequently nucleons in such states ($\pm m$) interact all the time and one can expect that a configuration with two nucleons outside a closed shell with $m_1 = -m_2$ is favored in comparison with others.

If $|m_1| \neq |m_2|$ then the 'orbits' of the two nucleons lie in different planes. Remember that $\vec{l}$ is perpendicular to the plane orbit since $\vec{l} = \vec{r} \times \vec{p}$. For large '$l$' one can use the (semi-)classical notion of *orbit*. If $\vec{l_1}$ is antiparallel to $\vec{l_2}$ then $\vec{L} = \vec{l_1} + \vec{l_2} = 0$. If they form an angle then the total $\vec{L}$ is different from zero.
Let us couple $\vec{j_1}$ and $\vec{j_2}$ to different total $\vec{J}$.

$$\psi^{JM}_{j_1 j_2}(x_1, x_2) = \sum_{m_1, m_2} C^{JM}_{j_1 m_1 j_2 m_2} \phi_{n_1 l_1 j_1 m_1}(x_1) \phi_{n_2 l_2 j_2 m_2}(x_2)$$

and compute the interaction energy for a zero-range interaction, supposed to mimick the short range character of the real one:

$$V_I = \langle \psi^{JM} | - V_0 \delta(r_1 - r_2) | \psi^{JM} \rangle$$

The results of such a computation are shown in fig. 6.4 and 6.5. Here you have the angular part of the wavefunction of two nucleons coupled to $\vec{L} = 0$:

$$\begin{aligned} \langle \theta_1 \phi_1, \theta_2 \phi_2 | nljjI = 0, M = 0 \rangle \quad &\propto \quad \sum_{m=-l}^{l} (-1)^m Y_{lm}(\theta_1 \phi_1) Y_{l-m}(\theta_2 \phi_2) \\ &= \quad \frac{2l+1}{4\pi} P_l(\cos \theta_{12}) \end{aligned}$$

From the figures we also see that for $J^\pi = 0^+$ the attraction is very strong in comparison with $J \neq 0$. The states with $J \neq 0$ are almost degenerate.

Let us consider the so called pure pairing interaction

$$\langle j_2 m_2 j_2 - m_2 | V | j_1 m_1 j_1 - m_1 \rangle = -V_0$$

and all other matrix elements (involving different two particle configurations) vanishing. Instead of a slater determinant we shall consider the following wavefunction for the ground state of an even system. (In order not to complicate the matter only one kind of nucleon is explicitly considered here.)

$$\Phi_{gs} = \left(\begin{array}{c} \Omega \\ N/2 \end{array} \right)^{-1/2} \sum_{j_1 m_1, j_1 - m_1, \ldots, j_{N/2} m_{N/2}, j_{N/2} - m_{N/2}} \Phi(j_1 m_1, \ldots, j_{N/2} - m_{N/2})$$

where

- the sum runs over all possible configurations
- $\Phi(j_1 m_1, \ldots, j_{N/2} - m_{N/2})$ is a slater determinant with N particles in the single-particle states $j_1 m_1 \ldots j_{N/2} - m_{N/2}$
- 2Ω is the total number of single-particle levels (we shall consider levels within one shell only).

There are $C \equiv \left(\begin{array}{c} \Omega \\ N/2 \end{array} \right)$ terms in the sum. One typical configuration in the sum looks like the one shown in fig. A6.3.

For simplicity we shall suppose also that all single-particle levels are degenerate within one shell. Now let us compute $E_{pairing} = \langle \Phi | V | \Phi \rangle$ where V is the pure pairing interaction. We get

$$\Phi_{gs} = C^{-1/2} \sum \langle \Phi(j_1 m_1, \ldots, j_{N/2} - m_{N/2}) | V | \Phi(j_1' m_1', \ldots, j_{N/2}' - m_{N/2}') \rangle$$

where the sum runs over $j_1 m_1, \ldots, j_{N/2} - m_{N/2}$ and $j_1' m_1', \ldots, j_{N/2}' - m_{N/2}'$.

When acting on a Slater det. $\Phi(j_1 m_1, \ldots, j_{N/2} - m_{N/2})$ the interaction V can scatter one 'pair' $\{j_k m_k, j_k - m_k\}$ into another state $\{j_l m_l, j_l - m_l\}$, leaving the other particles unaffected. One can choose a pair $\{j_k m_k, j_k - m_k\}$ in 'N/2' different ways in one Slater determinant. The number of final states accessible for such a pair is

$$\Omega - (\frac{N}{2} - 1)$$

since Ω is the total number of states for a pair and $N/2 - 1$ is the number of states already occupied by other pairs. Consequently

$$E_{pairing} = -V_0 \frac{N}{2}(\Omega + 1 - \frac{N}{2}) = -\frac{1}{4} V_0 N (2\Omega + 2 - N)$$

The normalization $\begin{pmatrix} \Omega \\ N/2 \end{pmatrix}^{-1}$ is 'eaten up' by the total number of ways one can pick an initial Slater determinant.

Note:

1. $E_{pairing}$ is a parabola as a function of the total number of particles.
2. $\frac{d^2 E_{pairing}}{dN^2} > 0$, i.e. this parabola 'keeps water' (convex).

Note that $P_l(\cos\theta)$ for large 'l' is peaked at $\theta = 0, \pi$.

For an odd number of particles an appropriate wave-function will be

$$\Phi = C^{-1/2} \sum_{j_1 m_1, \ldots, j_{(N-1)/2} - m_{(N-1)/2}} \Phi(j_0 m_0, j_1 m_1, \ldots, j_{(N-1)/2} - m_{(N-1)/2})$$

where N is odd, (N-1)/2 is the number of 'pairs' and $j_0 m_0$ is the single particle state occupied by the odd particle. (Note that we do not sum over $j_0 m_0$.)
In a similar way one can establish that

$$E_{pairing} = -V_0 \frac{N-1}{2}[(\Omega - 1) - (\frac{N-1}{2} - 1)] = -\frac{1}{4} V_0 (2\Omega + 1 - N)$$

where (N-1)/2 is the number of ways one can pick a pair and $(\Omega - 1)$ is the total number of states for a pair.
Now let us compute the odd and even systems:

$$E_{even} = -\frac{1}{4} V_0 N [2\Omega + 2 - N] = \frac{1}{4} V_0 \{[N - (\Omega + 1)]^2 - (\Omega + 1)^2\}$$

with

$$E_{even}|_{min} = -\frac{1}{4} V_0 (\Omega + 1)^2$$

and

$$E_{odd} = \frac{1}{4} V_0 \{[N - (\Omega + 1)]^2 - (\Omega + 1)^2 + 2\Omega + 1\}$$

with

$$E_{odd}|_{min} = -\frac{1}{4} V_0 (\Omega + 1)^2 + \frac{1}{4} V_0 (2\Omega + 1)$$

Hence

$$E_{odd} = E_{even} + \frac{1}{4} V_0 (2\Omega + 1) > E_{even}$$

Consequently, the odd systems are less bound than the even systems. Since one can place the 'odd' particle $j_0 m_0$ on any level, the ground state for odd systems is multiply degenerate (namely 2Ω-degenerate). This fact agrees with the experimental evidence: odd nuclei have a dense low energy spectrum. Fig. 6.6 shows a typical example. (In our idealized model we considered all single particle levels in the odd system degenerate.)

I shall solve the same problem using the second quantization method. One introduces the Fock-Space

$$\mathcal{H}_{Fock} = \bigcup_{N=0}^{\infty} \mathcal{H}_N$$

where $\mathcal{H}_N$ is the Hilbert-space for an N-particle system, i.e. the usual Hilbert-space in quantum mechanics. $\mathcal{H}_0$ is the Hilbert-space for the vacuum, i.e. only one state denoted usually by $|0\rangle$. Besides usual operators used in quantum mechanics one needs operators which connect different Hilbert-spaces, corresponding to a different number of particles. Building blocks for such operators are the creation and annihilation operators. They link the $\mathcal{H}_N$ with $\mathcal{H}_{N\pm 1}$ spaces.

$a_k^\dagger$	creates a fermion in quantum state 'k'
a_k	annihilates a fermion in quantum state 'k'
$b_k^\dagger$	creates a boson in quantum state 'k'
b_k	annihilates a boson in quantum state 'k'

A wave-function for one nucleon is defined as the matrix element

$$\phi_k(x) = \langle x | a_k^\dagger | 0 \rangle$$

Properties:

Anticommutators

$$\{a_k, a_l^\dagger\} = a_k a_l^\dagger + a_l^\dagger a_k = \delta_{kl}$$
$$\{a_k, a_l\} = a_k a_l + a_l a_k = 0$$
$$\{a_k^\dagger, a_l^\dagger\} = a_k^\dagger a_l^\dagger + a_l^\dagger a_k^\dagger = 0$$

Commutators

$$[b_k, b_l^\dagger] = b_k b_l^\dagger + b_l^\dagger b_k = \delta_{kl}$$
$$[b_k, b_l] = b_k b_l + b_l b_k = 0$$
$$[b_k^\dagger, b_l^\dagger] = b_k^\dagger b_l^\dagger + b_l^\dagger b_k^\dagger = 0$$

Hermitian conjugation

$$
\begin{aligned}
(a_k^\dagger)^\dagger &= a_k \\
(b_k^\dagger)^\dagger &= b_k
\end{aligned}
$$

Vanishing bra's and ket's

$$
\begin{array}{r}
\left.\begin{array}{r} a_k|0\rangle = 0\ \forall\, k \\ \langle 0|a_k^\dagger = 0\ \forall\, k \end{array}\right\} a_k^2 = (a_k^\dagger)^2 = 0 \\
b_k|0\rangle = 0\ \forall\, k \qquad\qquad\qquad\quad \\
\langle 0|b_k^\dagger = 0\ \forall\, k \qquad\qquad\qquad\quad
\end{array}
$$

In second quantization our schematic interaction reads [1]

$$
\hat{V} = -V_0 \sum_{m,m'=1}^{\Omega} a_m^\dagger a_{-m}^\dagger a_{-m'} a_{m'}
$$

One can define the following operators

$$
\begin{aligned}
S_+ &= \sum_{m>0} a_m^\dagger a_{-m}^\dagger = S_x + iS_y \\
S_- &= \sum_{m>0} a_{-m} a_m = S_x - iS_y = S_+^\dagger \\
S_0 &= \frac{1}{2}\sum_{m>0} [a_m^\dagger a_m + a_{-m}^\dagger a_{-m} - 1] = S_z
\end{aligned}
$$

One can check that

$$
\begin{array}{c}
[S_+, S_-] = 2S_0 \\
[S_0, S_\pm] = \pm S_\pm
\end{array}
$$

Consequently these operators obey the same commutation relations as the usual angular momentum operators $J_\pm$, J_0. $S_\pm$, S_0 are called quasispin operators.

The operator

$$
\hat{N} = \sum_m [a_m^\dagger a_m + a_{-m}^\dagger a_{-m}]
$$

is the number operator, i.e. its average over a given state in the Fock space gives the average number of particles.

[1] I use m, m' as a shorthand notation for jm, jm'.

One can define

$$\vec{S} = (S_x, S_y, S_z) = (\frac{1}{2}(S_+ + S_-), \frac{1}{2}(S_+ - S_-), S_z)$$

Now establish that

$$\hat{V} = -V_0 S_+ S_- = -V_0[\vec{S}^2 - S_0^2 + S_0]$$

and

$$\begin{aligned} S_0|0\rangle &= -\tfrac{\Omega}{2}|0\rangle \\ \vec{S}^2|0\rangle &= \tfrac{\Omega}{2}\left(\tfrac{\Omega}{2}+1\right)|0\rangle \end{aligned}$$

i.e. the vacuum corresponds to a total quasispin $\Omega/2$ with $S_z = -\Omega/2$.
The spectrum of $\hat{V}$ is obviously characterized by S, S_0 and the eigenvalues are

$$E_{S,S_0} = \langle SS_0|\hat{V}|SS_0\rangle = -V_0[S(S+1) - S_0^2 + S_0]$$

Knowing that

$$|0\rangle = |S = \frac{\Omega}{2}, S_0 = -\frac{\Omega}{2}\rangle$$

one can construct all the states with the same total quasispin

$$|S = \frac{\Omega}{2}, S_0 = \frac{N-\Omega}{2}\rangle = \mathrm{const}(S_+)^{N/2}|S = \frac{\Omega}{2}, S_0 = -\frac{\Omega}{2}\rangle = \mathrm{const}(S_+)^{N/2}|0\rangle$$

S_+ creates two particles, therefore $(S_+)^{N/2}$ corresponds to a N-particle state. By inspection one can establish that this state corresponds to the state we used earlier for an even system. Consequently,

$$E_{even}(N) = -V_0[\frac{\Omega}{2}(\frac{\Omega}{2}+1) - (\frac{N-\Omega}{2})^2 + \frac{N-\Omega}{2}]$$

which reproduces our earlier result.
For an odd system one has to start with a one-particle state $a^\dagger_{m_0}|0\rangle = |m_0\rangle$. One can establish that

$$\begin{aligned} S_0|m_0\rangle &= -\tfrac{\Omega-1}{2}|m_0\rangle & \text{i.e. } S_0 = -\tfrac{\Omega-1}{2} \\ \vec{S}^2|m_0\rangle &= \tfrac{\Omega-1}{2}(\tfrac{\Omega-1}{2}+1)|m_0\rangle & \text{i.e. } S = \tfrac{\Omega-1}{2} \end{aligned}$$

and all other states will be $\mathrm{const}(S_+)^{(N-1)/2}|m_0\rangle$ and in this way one recovers our previous wave-functions and energies for an odd system.
Consequently the odd and even systems correspond to different values of the total quasispin (either $\frac{\Omega-1}{2}$ or $\frac{\Omega}{2}$). One can construct in this way also the

excited states of an even or odd system. They will correspond to a different quasispin. The so called Cooper-pair is the state

$$\mathrm{const} S_+|0\rangle = \mathrm{const} \sum_m a_m^\dagger a_{-m}^\dagger |0\rangle$$

and therefore the ground state of an even N-particle system is a 'Bose'-condensate of Cooper-pairs

$$\mathrm{const}(S_+)^{N/2}|0\rangle$$

If $N \ll 2\Omega$ then one can define

$$b_0^\dagger = \frac{1}{\sqrt{\Omega}} S_+, \ b_0 = \frac{1}{\sqrt{\Omega}} S_-$$

and obtain that

$$[b_0, b_0^\dagger] = -\frac{2}{\Omega} S_0 = -\frac{2}{\Omega}[\frac{\hat{N}}{2} - \frac{\Omega}{2}] = 1 - \frac{\hat{N}}{\Omega} \approx 1$$

i.e. b_0 and $b_0^\dagger$ are almost bosons and this exlains why people talk about a 'condensate' of Cooper-pairs and why superconductivity and superfluidity are 'similar'.

Part VII
Bulk properties of nuclei

Part VIII
Weizsäcker mass formula or liquid drop formula

5 Experimental facts

The experimental binding energies (cf. fig. 7.1) can be reproduced quite accurately with such a formula

$$E_{binding} = B(N, Z, A) = a_V A + a_S A^{2/3} + a_C \frac{z^2}{A^{1/3}} + a_I \frac{(N-Z)^2}{A} + \delta(A)$$

with

$$\begin{aligned} a_V &= -15.68\text{MeV} \\ a_S &= 18.56\text{MeV} \\ a_C &= 0.72\text{MeV} \\ a_I &= 28.1\text{MeV} \end{aligned}$$

$$\delta(A)^2 = \begin{cases} 34\,A^{-3/4} & \text{even-even} \\ 0 & \text{odd-even, even-odd} \\ -34\,A^{-3/4} & \text{odd-odd} \end{cases}$$

The first term is the volume energy, since for $A \to \infty$ the last three terms go to zero and $B(\text{nuclear matter}) = a_V < 0$ and also because volume= $\frac{4\pi}{3}R^3 = \frac{4\pi}{3}r_0^3 A$. The second term is the surface term since surface= $4\pi R^2 = 4\pi r_0^2 A^{2/3}$. One can extract the surface tension

$$E_{surface} = S \cdot \sigma = a_S A^{2/3} = 4\pi r_0^2 A^{2/3} \cdot \sigma$$

hence

$$\sigma = \frac{a_S}{4\pi r_0^2} \approx 1\text{MeV} \cdot \text{fm}^{-2}$$

Since this term is positive the tendency of all nuclei is to have a spherical shape. For a sphere the surface is minimal for a given volume and nuclei seem to be almost incompressible (remember $\rho_{interior} \approx$ const). [3] The third term is simply the Coulomb energy of a charged sphere

$$E_{Coulomb} = \frac{3}{5}\frac{Z^2 e^2}{R} = \frac{3}{5}\frac{e^2}{r_0}\frac{Z^2}{A^{1/3}} \approx 0.7\frac{Z^2}{A^{1/3}}\text{MeV}$$

since $e^2 = 1.4\text{MeV}\cdot\text{fm}$, $r_0 = 1.2\text{fm}$ and $\frac{3}{5}\frac{1.4}{1.2} \approx 0.7$. The forth term is the symmetry term. Since $a_I > 0$ nuclei tend to have an equal number of neutrons and protons. The last term describes pairing correlations (cf. part VII Pairing). For even-even nuclei (even Z and even N) the pairing energy is larger than for even-odd or odd-even nuclei, which in turn is larger than the corresponding pairing energy for odd-odd nuclei.

6 Volume term in HF approximation

One can try to estimate the volume term using the independent particle model $E = \langle T \rangle + \langle V \rangle$. For a rough estimate one can use the Fermi gas model,

[2] Instead of $34A^{-3/4}MeV$ some people use $12A^{-1/2}MeV$.
[3] However see later about the role of the Coulomb energy.

since we want to estimate a volume effect. We know that inside the nuclei the single-particle potential is canstant and therefore $F(\text{average}) \approx -\nabla V \approx 0$ (i.e. the average force acting on a nucleon generated by all the other nucleons vanishes).

$$\langle T \rangle = E_{kin} = \underbrace{2 \cdot 2}_{spin\, and\, isospin\, dof} \int_{k<k_F} \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} = \frac{3}{5} \epsilon_F A$$

where $\epsilon_F = (\hbar k_F)^2/(2m)$. Similarly

$$\begin{aligned} \langle V \rangle_{Hartree} &= \frac{1}{2} \int dx \int dy \rho(x) \rho(y) V(x,y) \\ &= \frac{1}{2} \int dx \rho(x) V_{Hartree}(x) \approx \frac{1}{2} A \qquad \underbrace{(-50\text{MeV})}_{depth\, of\, the\, s.p.\, potential} \end{aligned}$$

Consequently

$$a_V(\text{Fermi gas}) = \frac{3}{5}\epsilon_F + \frac{1}{2}(-50\text{MeV}) = \frac{3}{5} \cdot 38\text{MeV} - 25\text{MeV} \approx -2\text{MeV}$$

which is significantly 'smaller' than the 'experimental' value -16 MeV. The reason for this is that the two-body interaction between two nucleons in 'nuclear matter' is 'screened' like Coulomb-interaction in plasma and then the formula for the binding energy is more complicated than the HF one. The surface tension appears for reasons similar to those in case of a liquid. A nucleon at the 'surface' is attracted towards the interior by all nucleons inside the radius of nuclear forces and so an effective surface pressure appears.

7 Symmetry term

In the Fermi gas model we have

$$\begin{aligned} E_{kin} &= \underbrace{2 \int_{k<k_F} \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m}}_{\text{protons}} + \underbrace{2 \int_{k<k_F} \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m}}_{\text{neutrons}} \\ &= \frac{3}{5} \epsilon_F(\text{neutrons}) N + \frac{3}{5} \epsilon_F(\text{protons}) Z \end{aligned}$$

With $N+Z=A$ and $N\sim k_F^3(\text{neutrons})$ and $Z\sim k_F^3(\text{protons})$ this yields

$$E_{kin}=\text{const}\cdot[k_F^5(\text{neutrons})+k_F^5(\text{protons})].$$

For a given A

$$\frac{1}{3\pi^2}[k_F^3(\text{neutrons})+k_F^3(\text{protons})]=A$$

or

$$\begin{aligned}
E_{kin}(N,Z) &= \text{const}[N^{5/3}+Z^{5/3}]\\
&= \text{const}[N^{5/3}+(A-N)^{5/3}]\\
&\approx \underbrace{\frac{3}{5}\epsilon_F A}_{\text{this term is a volume one}} + \frac{(N-Z)^2}{2A}\frac{10}{9}(\epsilon_F\frac{3}{5})
\end{aligned}$$

where in the last step we did a Taylor expansion around the minimum $N=A/2$. Consequently

$$a_I(\text{kinetic})=\frac{1}{2}\frac{10}{9}\frac{3}{5}\epsilon_F\approx 12.5\text{MeV}$$

which is too small in comparison with the experimental value $a_I=28\text{MeV}$. The reason for this is that the nucleon-nucleon interavtion $V(x,y)$ depends on isospin (cf. part III Isospin)

$$V(x,y)=\underbrace{V_0(x,y)}_{isoscalar}+(\vec{\tau}_1\cdot\vec{\tau}_2)\underbrace{V_1(x,y)}_{isovector}$$

In such a case $V_{Hartree}$ is more complicated and there is a potential energy contribution to a_I. Let's compute $V_{Hartree}$ for such a N-N-interaction:

$$\begin{aligned}
V_{Hartree}(r,\sigma,\tau) &= \int dy\,V(x,y)\rho(y)\\
&= \sum_{\sigma'\tau'}\int V_0(r,\sigma;r',\sigma')\rho(r',\sigma',\tau')\,dr'\\
&+ \sum_{\sigma'\tau'}\int V_1(r,\sigma;r',\sigma')(\vec{\tau}\cdot\vec{\tau}')\rho(r',\sigma',\tau')\,dr'
\end{aligned}$$

Since

$$\rho(r',\sigma',\tau')=\begin{cases}\rho_{proton}(r',\sigma') & \text{for } \tau'=-\frac{1}{2}\\ \rho_{neutron}(r',\sigma') & \text{for } \tau'=+\frac{1}{2}\end{cases}$$

we get (dropping the spin)

$$V_{Hartree}(r,\tau) = \int dr'\, V_0(r,r')[\rho_p(r')+\rho_n(r')] \pm \int dr'\, V_1(r,r')[\rho_n(r')-\rho_p(r')]$$

where

+ for protons
- for neutrons.

Using [4]

$$\rho_n \approx \frac{N}{A}\rho, \;\; \rho_p \approx \frac{Z}{A}\rho$$

we find

$$V_{Hartree}(r) = \int dr'\, V_0(r,r')\rho(r') \pm (N-Z)\int dr'\, V_1(r,r')\rho(r').$$

Since the isospin of a nucleus is $T_z = (N-Z)/2$ we can rewrite that setting $\vec{t} = \vec{\tau}/2$

$$V_{Hartree}(r,\tau) = U_0(r) + (\vec{t}\cdot\vec{\tau})U_1(r)$$

Empirically one finds (by fitting single-particle energies for protons and neutrons in nuclei) the depth $U_1 \approx (70 \div 110\text{MeV})/A$. In such a case

$$a_I(\text{potential})A \approx \frac{1}{2}T_z^2 U_1$$

and therefore

$$a_I(\text{potential}) \approx \frac{U_1}{8} \approx 9 \div 14\text{MeV}.$$

Consequently

$$a_I(\text{kin}) + a_I(\text{potential}) \approx 22 \div 27\text{MeV}.$$

Note: I skipped some details and you should figure out every detail of such an estimate.

Let us consider now N and Z fixed and look at the surface- and Coulomb-energy. In such a case the volume- and symmetry-energies are constant (more or less) if one changes only the shape of the nucleus. The pairing energy is very small and we shall neglect it here. Let us consider an ellipsoidal shape for a nucleus, i.e.

$$R(\theta) = R_0[1 + \beta Y_{20}(\theta,\phi)]$$

[4] Note: $\rho_n + \rho_p = \rho$

with $\beta \ll 1$ and $Y_{20}(\theta,\phi) = \sqrt{\frac{15}{16\pi}}(3\cos^2\theta - 1)$. Up to terms of order β^2 the surface- and the Coulomb-energies become

$$
\begin{aligned}
E_S + E_C &= a_S A^{2/3}[1 + \frac{\beta^2}{2\pi}] + \frac{3}{5}\frac{Z^2e^2}{R_0}[1 - \frac{\beta^2}{4\pi}] + O(\beta^3) \\
&= \underbrace{E_S(\beta = 0)}_{=a_S A^{2/3}} + \underbrace{E_C(\beta = 0)}_{=a_C \frac{Z^2}{A^{1/3}}} + \frac{1}{2}\beta^2[4R^2\sigma - \frac{3}{10\pi}\frac{Z^2e^2}{R_0}] \\
&= E_{S0} + E_{C0} + \frac{1}{2}\beta^2 C_2.
\end{aligned}
$$

By deforming the nucleus the surface increases and so does the surface energy, but the Coulomb energy decreases (since the protons prefer to be at an infinite distance frome one another) because the average distance between two protons increases then. It is easy to estimate that $C_2 \approx 0$ for $Z^2/A \approx 50$ and nuclei then become 'soft'. Such nuclei actually fission, i.e. they spontaneously break into two fragments.

Part IX
β-decay

We shall be mainly interested in two types of decays: β-decay and α-decay. There are other modes of nuclear decay: proton and neutron decay, fission, heavy target decay (emission of nuclei like ^{14}C, Ne etc). Before turning to β-decay I shall study an abstract quantum mechanical problem, which has direct relevance to β-decay.

8 How a small perturbation can change completely the character of the spectrum

Let us consider the following one-dimensional (this is for the sake of simplicity only) two-channel problem

$$
\begin{pmatrix} h_1 & \Delta \\ \Delta^+ & h_2 \end{pmatrix} \begin{pmatrix} v_e \\ u_e \end{pmatrix} = E \begin{pmatrix} v_e \\ u_e \end{pmatrix}
$$

where, in the absence of the coupling term 'Δ', the spectrum is discrete in the first channel, i.e.

$$(h_1 - E_k)v_k = 0,\ \langle v_k|v_l\rangle = \delta_{kl}$$

and continuous in the second channel, i.e.

$$(h_2 - E)u_{0E} = 0,\ \langle u_{0E}|u_{0E'}\rangle = \delta(E - E')$$

One can 'solve' formally the first equation

$$(h_1 - E)v_E + \Delta u_E = 0$$

using the Green function method:

$$v_E = \frac{1}{E - h_1}\Delta u_E$$

If $E \neq E_k$ then

$$\frac{1}{E - h_1} = \sum_k \frac{|v_k\rangle\langle v_k|}{E - E_k}$$

(Remember $\sum_k |v_k\rangle\langle v_k| = 1$, i.e. the eigenstates v_k form a complete set of orthonormal vectors in the Hilbert space.)
If the coupling 'Δ' is 'small' and $E \approx E_0$ then

$$v_e(x) = v_0(x)\frac{\langle v_0|\Delta|u_E\rangle}{E - E_0} \equiv v_0(x)n^{1/2}(E) \qquad (1)$$

This means that in the first approximation, near E_0, the first component of the wave function has the same shape in space as $v_0(x)$, i.e. the wavefunction of the bound state - but for any energy E in the vicinity of E_0 not only for $E = E_0$. Even if $\Delta \to 0$, i.e. the state has become 'continuous'.
The second equation is a little bit tricky. Why? The spectrum is continuous for $(h_2 - E)u_{0E} = 0$ and one cannot simply invert $(E - h_2)$. Let us consider for simplicity that

$$h_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \quad \text{for} \quad 0 \leq x < \infty$$

and

$$u(x)|_{x=0} = 0$$

for physical solutions, like in the case of the radial Schrödinger equation. Then u_{0E}, the regular solution, is for large x approximately (supposed $\lim_{x\to\infty} V(x) = 0$)

$$u_{0E} \overset{x\to\infty}{\approx} \sqrt{\frac{2m}{\hbar^2 \pi k}} \sin(kx + \delta_{0E})$$

where $(\hbar^2 k^2)/(2m) = E$ and δ_{0E} is the phase shift.
The equation $(h_2 - E)u = 0$ has an additional solution, linearly independent from u_{0E}. We shall call it $\chi_E(x)$, the irregular one, which has the asymptotic behaviour

$$\chi_E(x) \overset{x\to\infty}{\approx} \sqrt{\frac{2m}{\hbar^2 \pi k}} \cos(kx + \delta_{0E})$$

Certainly $(h_2 - E)\chi_E = 0$ but $\chi_E(x)|_{x=0} \neq 0$.
One can define

$$G(x,y) = -\pi u_{0E}(x_<)\chi_E(x_>)$$

where $x_< \equiv \min(x,y)$ and $x_> \equiv \max(x,y)$,
and show that

$$-\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right) G(x,y) = \delta(x-y) \tag{2}$$

First both $u_{0E}(x)$ and $\chi_E(x)$ satisfy the homogeneous equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right) u_{0E}(x) = 0$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right) \chi_E(x) = 0$$

Second the Wronskian $W(x)$ is constant (Prove it!)

$$W(x) = u'_{0E}(x)\chi_E(x) - u_{0E}(x)\chi'_E(x) \equiv \frac{2m}{\hbar^2 \pi}$$

If $x < y$ then $\delta(x-y) = 0$ and equation (2) is obviously satisfied, since for $x < y$ we have $G(x,y) = -\pi u_{0E}(x)\chi_E(y)$. Similarly the equation holds if $x > y$, since then $G(x,y) = -\pi u_{0E}(y)\chi_E(x)$. Let us integrate this equation from $x = y - \epsilon$ to $x = y + \epsilon$ where ϵ is a very small positive quantity. The RHS is

$$\int_{y-\epsilon}^{y+\epsilon} \delta(x-y)dx = 1$$

while the LHS is

$$\int_{y-\epsilon}^{y+\epsilon} dx[E-V(x)]G(x,y) + \frac{\hbar^2}{2m}\left[\frac{\partial G(x,y)}{\partial x}|_{x=y+\epsilon} - \frac{\partial G(x,y)}{\partial x}|_{x=y-\epsilon}\right]$$

If $\epsilon \to 0$ then the integral term tends to zero [5] but the second term is exactly 1 if one remmbers that $W(x) = (2m)/(\hbar^2\pi)$. Consequently if one knows $G(x,y)$, this is called the Green function, the solution of

$$(E - V(x) + \frac{\hbar^2}{2m}\frac{d^2}{dx^2})u_E(x) = \Delta^*(x)v_E(x)$$

can be written as

$$u_E(x) = C(E)u_{0E}(x) + \int_0^\infty dyG(x,y)\Delta^*(x)v_E(y) \tag{3}$$

If $\Delta(x) \stackrel{x\to\infty}{\to} 0$ we know that

$$u_E \approx \sqrt{\frac{2m}{\hbar^2\pi k}}\sin(kx + \delta_E)$$

with N.B. $\delta_E \neq \delta_{0E}$. But

$$u_E \stackrel{x\to\infty}{\approx} \sqrt{\frac{2m}{\hbar^2\pi k}}\left\{\sin(kx+\delta_{0E}) + \pi\cdot n^{1/2}(E)\langle v_0|\Delta^+|u_{0E}\rangle\cos(kx+\delta_{0E})\right\}$$

$u_E(x)$ will have a right asymptotic behaviour if

$$C^2(E) + \pi^2 n(E)\left|\langle v_0|\Delta^+|u_{0E}\rangle\right|^2 = 1$$

At the same time (using eq. (1) and (3))

$$n^{1/2}(E) = \frac{\langle v_0|\Delta|u_E\rangle}{E-E_0} = \frac{1}{E-E_0}\left\{\langle v_0|\Delta|u_{0E}\rangle C(E) + \langle v_0|\Delta G\Delta^+|v_0\rangle\frac{n^{1/2}(E)}{E-E_0}\right\}$$

and finally one obtains

$$n(E) = \frac{1}{\pi}\frac{\frac{1}{2}\Gamma(E)}{(E-E_0-\delta E)^2 + \frac{1}{4}\Gamma(E)}$$

with

$$\Gamma(E) = 2\pi\left|\langle v_0|\Delta|u_{0E}\rangle\right|^2$$

[5] $\int_{y-\epsilon}^{y+\epsilon} f(x)dx \approx 2\epsilon\overline{f(x)} \to 0$

$$\delta E = -\langle v_0|\Delta G \Delta^+|v_0\rangle$$

The only approximation made so far was in eq. (3) when we retained only one term in the sum $\sum_k \frac{|v_k\rangle\langle v_k|}{E-E_k}$ which means that

$$\left|\frac{\langle v_k|\Delta|u_E\rangle}{E-E_k}\right| \ll \left|\frac{\langle v_0|\Delta|u_E\rangle}{E-E_0}\right|$$

In this sense the coupling Δ is weak.
One can usually consider then that $\Gamma(E) \approx \Gamma(E_0) = \text{const}$ and $\delta E \approx 0$. (Those of you interested in this can try an analysis of such an approximation.) It is easy to see then that $\int_{-\infty}^{\infty} n(E)dE = 1$ but $n(E) = \langle v_E|v_E\rangle$. Consequently the probability to find the system in the first channel after 'turning on' the coupling Δ is spread over the whole spectrum. If $\Delta \to 0$ then $n(E) \to \delta(E-E_0)$, i.e. one obtains again the discrete level.

Let us solve now the time dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} h_1 & \Delta \\ \Delta^+ & h_2 \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix}$$

with the initial condition

$$\left.\begin{pmatrix} v \\ u \end{pmatrix}\right|_{t=0} = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}, \quad (h_1 - E_0)v_0 = 0$$

Obviously,

$$\begin{pmatrix} v(t) \\ u(t) \end{pmatrix} = \int_{-\infty}^{\infty} dE\, A(E) e^{-\frac{i}{\hbar}Et}\begin{pmatrix} v_E \\ u_E \end{pmatrix}$$

where

$$A(E) = (v_E^*, u_E^*)\begin{pmatrix} v_0 \\ 0 \end{pmatrix} = \langle v_E|v_0\rangle = n^{1/2}(E)$$

therefore (using $v_E = n^{1/2}(E)v_0$)

$$v(t) = \int_{-\infty}^{\infty} dE\, e^{-\frac{i}{\hbar}Et} n^{1/2}(E) v_E = \int_{-\infty}^{\infty} dE\, e^{-\frac{i}{\hbar}Et} n(E) v_0$$

Using

$$n(E) = \frac{1}{2|pii} \left[\frac{1}{E - E_0 - \frac{i}{2}\Gamma} + \frac{1}{E - E_0 + \frac{i}{2}\Gamma} \right]$$

and integrating clockwise in the complex E-plane (i.e. picking up the pole $E_0 - \frac{i}{2}\Gamma$) yields

$$v(t)|_{t>0} = -v_0 e^{-\frac{i}{\hbar}E_0 t - \frac{1}{2}\frac{\Gamma t}{\hbar}}$$

Or, the probability to find the particle in the first channel

$$P_1(t)|_{t>0} = \langle v(t)|v(t)\rangle = e^{-\frac{\Gamma t}{\hbar}}$$

Γ gives both the 'width' of the state (cf. fig. A9.1) and the life time of this state $\tau = \hbar/\Gamma$.

9 Decay of the neutron

We know that nuclei are made up from protons and neutrons. The amazing thing is that the neutron is not a stable particle and at the same time many nuclei are stable. A free neutron decays

$$n \to p + e^- + \bar{\nu}$$

where e^- is an electron and $\bar{\nu}$ is an (anti-)neutrino. The masses are

baryons [6] $\left\{ \begin{array}{rcl} m_n c^2 &=& 939.55\,\mathrm{MeV} \\ m_p c^2 &=& 938.26\,\mathrm{MeV} \end{array} \right.$

leptons [7] $\left\{ \begin{array}{rcl} m_e c^2 &=& 0.511\,\mathrm{MeV} \\ m_{\bar{\nu}} c^2 &=& 0(?)\,\mathrm{MeV} \end{array} \right.$

The neutrino is a massless fermion of spin 1/2. The life time of the neutron is about 11 minutes. The interaction responsible for such a process is called *weak interaction*. If we 'turn off' the weak interaction the neutron is stable as the proton is. At the formal level this situation resembles the problem we analyzed before (cf. ch. 8).

[6] 'heavy' in Greek
[7] 'light' in Greek

Energy balance

$$m_n c^2 = m_p c^2 + E_p + m_{e^-} c^2 + E_{e^-} + E_{\bar{\nu}}$$

where E_p, E_{e^-} and $E_{\bar{\nu}}$ are the kinetic energies of the final particles (proton, electron and antineutrino respectively). The decay is possible since

$$Q \equiv E_p + E_{e^-} + E_{\bar{\nu}} = m_n c^2 - m_p c^2 - m_{e^-} c^2 = 0.78\,\mathrm{MeV} > 0$$

Momentum balance

$$\vec{p}_p + \vec{p}_e + \vec{p}_{\bar{\nu}} = 0$$

Spin balance

$$\text{neutron} \quad 1/2 \quad \rightarrow \left\{ \begin{array}{ll} \text{proton} & 1/2 \\ \text{electron} & 1/2 \\ \text{antineutrino} & 1/2 \end{array} \right\} \text{couple to } 1/2 \text{ or } 3/2$$

All four particles must have *half-integer spin.*

From the momentum balance it is clear that 'on average' all momenta are of the same order:

$$|\vec{p}_p| \sim |\vec{p}_{e^-}| \sim |\vec{p}_{\bar{\nu}}|$$

If one momentum is much smaller than the other two, than the latter are almost equal - but such a configuration has a small statistical weight. Any triangle is allowed and there are much more triangles with comparable sides than such shown in fig. A9.4'.

But since $m_e \sim \frac{m_p}{2000} \ll m_p$ we find $E_e \gg E_p$ and therefore

$$Q \approx E_{e^-} + E_{\bar{\nu}}$$

i.e. the recoil energy of the proton is very small and all released energy is 'eaten up' by e^- and $\bar{\nu}$. Since proton- and electron-momentum are of the same order

$$p_p \sim p_e$$

we find that

$$v_p \sim \frac{v_e}{2000} \ll v_e$$

i.e. the proton 'remains' in the same place where the neutron was. Since $Q \sim 1$ MeV we find for the e^-- and $\bar{\nu}$-wavevectors (supposed that both particles are massless)

$$k_e \sim k_{\bar{\nu}} \sim \frac{Q}{\hbar c} \sim \frac{1}{200}\mathrm{fm}^{-1}$$

Any other more exact estimation will give a smaller k_e or $k_{\bar{\nu}}$. Since the electron is very fast (even taking into account the Coulomb-interaction between proton and electron) a good estimate for the wave functions will be plane waves for all particles. But since

$$\lambda_e = \frac{2\pi}{k_e} \sim 1200\,\mathrm{fm} \sim \lambda_{\bar{\nu}}$$

the wavefunctions can be considered constant within the neutron radius and the weak interaction with a high accuracy is equal to

$$V_{weak}(r_n, r_p, r_e, r_\nu) = g\delta^3(\vec{r}_p - \vec{r}_n)\delta^3(\vec{r}_p - \vec{r}_e)\delta^3(\vec{r}_p - \vec{r}_\nu)$$

where $\delta^3(\vec{r}_p - \vec{r}_n)$ means that proton and neutron will be in the same place. If instead of δ-functions one considers any other function, which is different from zero when all coordinates are within the nucleon-volume, the effective result will be the same. Within the nucleon volume all wavefunctions are 'constant'. The life time $\tau = \hbar/\Gamma$ of the neutron can be obtained using Fermi's Golden Rule

$$\Gamma = 2\pi |V_{fi}|^2 N_{final\,states}$$

where $N_{final\,states}$ is the number of final states (the phase-space density). This is a generalization of the formula on pg. 20. V_{fi} is an 'average' matrix element between the initial state and a 'representative' final state. In our case all these matrix elements are equal since the wavefunctions are 'always' one inside the neutron.

$$N_{final\,states} = V^3 2^3 \int \frac{d^3p_p}{(2\pi)^3}\frac{d^3p_e}{(2\pi)^3}\frac{d^3p_{\bar{\nu}}}{(2\pi)^3}\delta(Q - E_p - E_e - E_{\bar{\nu}})\delta(\vec{p}_p + \vec{p}_e + \vec{p}_{\bar{\nu}})$$

where the factor of 2 arises from the spin degrees of freedom and V is the volume [8] of the 'laboratory', one for each particle. Since one can 'neglect' E_p we have $\delta(Q - E_p - E_e - E_{\bar{\nu}}) \approx \delta(Q - E_e - E_{\bar{\nu}})$ and the δ-function

[8] It drops from any formula since $\phi(r) = V^{-1/2} e^{i\vec{k}\cdot\vec{r}}$

$\delta(\vec{p}_p + \vec{p}_e + \vec{p}_{\bar{\nu}})$ can be integrated out. Since $Q = E_e + cp_{\bar{\nu}}$ we get after integration over angles

$$N_{final\,states} = V^2 2^2 \int_0^{p_e^{max}} \frac{4\pi p_e^2 dp_e}{(2\pi\hbar)^3} \frac{4\pi p_{\bar{\nu}}^2}{(2\pi\hbar)^3}\Bigg|_{cp_{\bar{\nu}}=Q-E_e}$$

Consequently

$$\Gamma = \text{const} \int_0^{p_e^{max}} dp_e\, p_e^2 (Q - E_e)^2$$

Since the electron is relativistic its kinetic energy is $E_e = \sqrt{m_e^2c^2 + p_e^2c^2} - m_e^2$. The probability of having an electron with momentum p_e is

$$\text{const}\, p_e^2 (Q - E_e)^2$$

From $\tau \approx$11 min we know Γ, which is connected to the coupling constant g by $\Gamma = \text{const}\, g^2$. Knowing $Q = 0.78$ MeV we can work out this constant and determine g this way:

$$g \approx 10^{-4}\,\text{MeV}\,\text{fm}^3$$

The interaction assumed on pg. 24 does not involve spins. It is called Fermi interaction. In such a case the spin of the nucleon remains unaffected. In a similar way as neutron and proton form an isospin doublet the electron and (anti-)neutrino also form an isospin doublet.

$$\begin{pmatrix} e^- \\ \bar{\nu} \end{pmatrix} - \text{lepton isospin doublet}$$

Taking the isospin into account the Fermi interaction reads

$$V_{Fermi} = g_F\, \vec{\tau}_{baryons} \cdot \vec{\tau}_{leptons}\, \delta(\ldots)$$

It is possible (and it has been observed experimentally) to have a weak coupling which involves spin.

$$V_{Gamow-Teller} = g_{GT}\, \vec{\sigma}_{baryons} \cdot \vec{\sigma}_{leptons}\, \vec{\tau}_{baryons} \cdot \vec{\tau}_{leptons}\, \delta(\ldots)$$

In such a case a neutron flips the spin during the decay. For the neutron we have

$$\frac{\langle V_{Gamow-Teller} \rangle}{\langle V_{Fermi} \rangle} \approx 2$$

Effect of the Coulomb field

I will mention one simple correction to the formula for Γ given on pg. 25. I argued that one can use a plane wave for the electron (or positron). In heavy nuclei the influence of the Coulomb field of the nucleus on the electron wave function is not anymore a negligible effect. A more precise expression for Γ is

$$\Gamma = \text{const} \int_0^{p_e^{max}} dp_e \, p_e^2 (Q - E_e)^2 F(\pm Z, E_e)$$

where

$$F(\pm Z, E_e) = \frac{|\psi_{e^\mp}(0)|^2_{Coulomb}}{|\psi_{e^\mp}(0)|^2_{free\,wave}}$$

and

$$\psi_{e^\mp}(0) = \frac{2\pi\eta}{1 - e^{-2\pi\eta}}$$

with

$$\eta = \pm \frac{Ze^2}{\hbar r}$$

$\psi_{e^\mp}(0)$ is the amplitude of the electron ($e^\mp$) wave function in a Coulomb field at the origin.
In β-decay we have $v \approx c$ and therefore $\eta = \pm Z\alpha \approx \pm Z/137$. If $Z \ll 137$ then

$$\psi_{e^\mp}(0) = \frac{2\pi\eta}{1 - e^{-2\pi\eta}} \approx \frac{2\pi\eta}{1 - (1 - 2\pi\eta + ...)} \approx 1$$

For	electrons	$\eta > 0$	and	$\psi_{e^-}(0) > 1$,
for	positrons	$\eta < 0$	and	$\psi_{e^+}(0) < 1$.

10 β-decay of nuclei

In most of the nuclei half or more of there nucleon content are neutrons. Why do they exist? Answer: Pauli principle. If the lowest unoccupied proton level is 0.5 MeV (=electron rest mass) or more higher than the highest occupied neutron level then the process $n \to p + e^- + \bar{\nu}$ is forbidden. There are no final states available for the proton. In the case of a free neutron the number of available final states is 'infinite'. This explains also why the reaction $n \to p + e^- + \bar{\nu}$ goes only one way. The total probability for $n \to p + e^- + \bar{\nu}$ is proportional to the number of final states. The inverse reaction has a vanishing relative probability. This situation is completely similar to what happens with a sugar cube in a cup of tea or a drop of ink ia a glass of water. The neutron is 'dissolved'. In nuclei it is possible to have also β^+(=positron)-decay, namley if the proton level is higher than the neutron level. The energy of the initial state is the proton rest energy minus its binding energy, i.e.

$$E_{initial} = m_p c^2 - B_p$$

The energy of the final state is analogously

$$E_{final} = m_n c^2 - B_n + m_e c^2 + E_{e^+} + E_\nu$$

Energy conservation means $E_{initial} = E_{final}$ and thus the reaction

$$p \to n + e^+ + \nu$$

goes if

$$E_{e^+} + E_\nu = m_p c^2 - B_p - m_n c^2 + B_n - m_e c^2 > 0$$

respectively if

$$B_n - B_p > m_e c^2 + (m_n - m_p)c^2 = 0.5\,\text{MeV} + 1.3\,\text{MeV} = 1.8\,\text{MeV}$$

Another possible reaction is the *K-capture*

$$p + e^- \to n + \nu$$

where the electron is captured from the lowest atomic orbit (i.e. the K-orbit). In such a case one does not have to create a positron or electron and the only requirement is

$$B_p - B_n > 1.3\,\text{MeV}.$$

Since nucleons in a nucleus are not exactly independent the β-stability of a given nucleus must be decided by considering the total mass of the initial and final nuclei. For β^--decay one finds

$$M(A,Z)c^2 = M(A,Z+1)c^2 + m_e c^2 + E_{e^-} + E_{\bar{\nu}}$$

and consequently

$$\begin{array}{lr} M(a,Z)c^2 - M(A,Z+1)c^2 - m_{e^-}c^2 > 0 & (\text{for } \beta^-\text{-decay}) \\ M(a,Z)c^2 - M(A,Z-1)c^2 - m_{e^+}c^2 > 0 & (\text{for } \beta^+\text{-decay}) \\ M(a,Z)c^2 - M(A,Z-1)c^2 + m_{e^-}c^2 > 0 & (\text{for K-capture}) \end{array}$$

(I have neglected the recoil energy of the residual nucleons.)

The distribution of β-stable nuclei in the N-Z plane can be found from the Weizsäcker mass formula

$$B(N,Z) = a_V A + a_S A^{2/3} + a_C \frac{Z^2}{A^{1/3}} + a_I \frac{(N-Z)^2}{A}$$

In a weak process the total number of baryons (i.e. neutrons and protons) is conserved, i.e. A=const. Consequently only the Coulomb- and symmetry-energies change. (I neglect pairing, though it is important.) The curve $N(Z)$, which is a solution of $\left.\frac{\partial B(N,Z)}{\partial Z}\right|_{N+Z=A} = 0$ defines the so-called β-stability valley:

$$\left.\frac{\partial B(N,Z)}{\partial Z}\right|_{N+Z=A} = a_C \frac{2Z}{A^{1/3}} + 4a_I \frac{2Z-A}{A} \stackrel{!}{=} 0$$

which implies

$$Z\left[\frac{2a_C}{A^{1/3}} + \frac{8a_I}{A}\right] = 4a_I$$

and finally yields

$$Z = A\frac{4a_I}{8a_I + 2a_C A^{2/3}} = \frac{A}{2}\frac{1}{1+4\frac{a_C}{a_I}A^{2/3}}$$

For every A this formula gives the charge of the β-stable isotope. If $a_C = 0$ (i.e. no Coulomb-interaction) then $Z = A/2 = N$, otherwise

$$Z < \frac{A}{2} < N$$

Pairing gives certain corrections. You can analyze them yourself.

11 Selection rules for β-decay

Fermi interaction

The nuclear part is simply $\vec{\tau}$, consequently

$$\begin{array}{rcr} J_{in} & = & J_{fin} \\ T_{in} & = & T_{fin} \\ T^z_{in} & = & T^z_{fin} \pm 1 \\ \text{parity} & - & \text{unchanged} \end{array}$$

These kind of transitions are called *superallowed transitions.*

Gamow-Teller interaction

This operator is $\sim \vec{\tau} \otimes \vec{\sigma} = (\vec{\tau}_{nucleons} \cdot \vec{\tau}_{leptons})(\vec{\sigma}_{nucleons} \cdot \vec{\sigma}_{leptons})$ and implies

$$\begin{array}{lcr} J_{in} & = & J_{fin} \text{ or } J_{fin} - 1 \\ T_{in} & = & T_{fin} \\ T^z_{in} & = & T^z_{fin} \pm 1 \\ \text{parity} & - & \text{unchanged} \end{array}$$

These are called *allowed transitions*.

Part X

ˆ-decay

12 Qualitative analysis of the behaviour of the wave function

One dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\phi''(x) + (V(x) - E)\phi(x) = 0$$

or

$$\phi''(x) + k^2(x)\phi(x) = 0$$

with

$$k^2(x) = \frac{2m}{\hbar^2}(E - V(x)) = \begin{cases} > 0 & \text{classically allowed region} \\ < 0 & \text{classically forbidden region} \\ = 0 & \text{classical turning points} \end{cases}$$

I will assume that $V(x)$ is not singular. Singular points $x = \pm\infty$ and $x = 0$ if there is a centrifugal barrier.

1. All zeros of $\phi(x)$ are simple. If $\phi(x_0) = \phi'(x_0) = 0$ then $\phi''(x_0) = 0$ from the Schrödinger equation and therefore all derivatives vanish and $\phi(x) \equiv 0$.

2. For $k^2(x) > 0$ we have

$$\begin{array}{llll} \text{if} & \phi(x) > 0 & \text{then} & \phi''(x) < 0 \\ \text{if} & \phi(x) < 0 & \text{then} & \phi''(x) > 0 \end{array}$$

   Consequently the profile of the wave function looks like the one shown in fig. A9.6.

3. For $k^2(x) < 0$ we have

$$\begin{array}{llll} \text{if} & \phi(x) > 0 & \text{then} & \phi''(x) > 0 \\ \text{if} & \phi(x) < 0 & \text{then} & \phi''(x) < 0 \end{array}$$

   Possible profiles are shown in fig. A9.7 and A9.8. The wave function can have at most one simple zero in the region where $k^2(x) < 0$.

Square well

We consider the potential

$$V(x) = \begin{cases} -V_0 & (x < x_0) \\ 0 & (x > x_0) \end{cases}$$

and assume $\phi(0) = 0$ as in the case of the radial Schrödinger equation. Since the phase of the wave function can be chosen arbitrarily, I further assume that $\phi'(0) > 0$. Let $0 > E_0 > E$ be the energy of the lowest bound state. We consider the three cases

$$\begin{array}{l} V_0 < E < E_0 \\ E = E_0 \\ E > E_0 \end{array}$$

which are shown in fig. A9.10 to A9.12.
How does one solve numerically this equation? For h 'small' enough we have

$$\phi'(x) = \lim_{h\to 0} \frac{\phi(x+h/2) - \phi(x-h/2)}{h} \approx \frac{\phi(x+h/2) - \phi(x-h/2)}{h}$$

$$\phi''(x) = \lim_{h\to 0} \frac{\phi'(x+h/2) - \phi'(x-h/2)}{h} \approx \frac{\phi(x+h) - 2\phi(x) + \phi(x-h)}{h^2}$$

One defines a mesh h and computes the wave function only at the points $x_i = ih = 0, h, 2h, \ldots$. That yields $\phi(ih) = \phi_i$ and $k^2(ih) = k_i^2$. Then the Schrödinger equation becomes

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} + k_i^2\phi_i = 0$$

or

$$\phi_{i+1} = 2\phi_i - \phi i - 1 - h^2 k_i^2 \phi_i$$

At $x = 0$ we have $\phi_0 = 0$. At $x = h$ we choose $\phi_1 = 1$. (That is possible since the equation is homogeneous.) Then

$$\phi_2 = 2\phi_1 - \phi_0 - h^2 k_1^2 \phi_1$$

and so on. One 'integrates' in this way the equation up to 'infinity', i.e. to a point $x \gg x_0$ (the radius of the potential well). Then one counts the number of modes (zeros) and makes sure that the wave function does not grow exponentially. If it does then depending on the behaviour of the wave function at 'infinity' one increases or decreases the energy until one finds the bound state. A good approximation is to require that the wave function has a zero at a very large distance $x \approx 3 \div 5x_0$ which for all practical purposes can be identified with 'infinity' (i.e. $\infty \approx 3 \div 5x_0$). This is not the best method but it works.

Potential barrier

Now we consider the potential

$$V(x) = \begin{cases} -V_0 & (x < x_0) \\ V_1 & (x_0 < x < x_1) \\ 0 & (x > x_1) \end{cases}$$

Let $E = E_0 > 0$ be the energy of a bound state in the case when $x_1 \to \infty$ and $\phi_0(x)$ its wave function (cf. fig. A9.14). Let us analyze the three cases

$$\begin{array}{c} E < E_0 \\ E = E_0 \\ E > E_0 \end{array}$$

for the above potential. The results are shown in the fig. A9.15 to A9.17.

Lesson: If $E = E_0$ the amplitude inside of $V(x)$ is much greater than the amplitude outside. If $E > E_0$ or $E < E_0$ the reverse holds.

Qualitatively: The amplitude (probability) of finding the particle inside the potential well is a strongly energy dependent function of energy when there is a potential barrier (cf. fig. A9.18).

It can be shown that the probability of finding the particle in the well has the following energy dependence:

$$P(E) = \frac{1}{\pi} \frac{\frac{1}{2}\Gamma}{(E - E_0)^2 + \frac{1}{4}\Gamma^2}$$

where

$$P(E) = \int_0^{x_0} |\phi_E(x)|^2 dx, \text{ with } \int_{-\infty}^{\infty} P(E) dE = 1$$

and

$$-\frac{\hbar^2}{2m}\phi_E''(x) + (V(x) - E)\phi_E(x) = 0$$

If $E = E_0$ then we get for $x < x_1$

$$\phi_{E_0}(x) \approx \sqrt{\frac{2}{\pi\Gamma}}\phi_0(x)$$

since

$$\int_0^{x_0} |\phi_0(x)|^2 dx \approx \int_0^{\infty} |\phi_0(x)|^2 dx = 1$$

I shall estimate roughly Γ using the WKB-approximation-method.

1. For $x < x_0$ we have

   $$\phi_0(x) \approx \frac{A_0 \sin(k_{in}x)}{\sqrt{k_{in}}}$$

   with

   $$\int_0^{x_0} |\phi_0(x)|^2 dx \approx 1$$

   and

   $$\frac{\hbar^2 k_{in}^2}{2m} = E_0 - V_0$$

   On the other hand

   $$\int_0^{x_0} \sin^2(k_{in}x) dx \approx \frac{1}{2}x_0$$

   and therefore

   $$A_0^2 = \frac{2k_{in}}{x_0}$$

   We already know that

   $$\phi_{E_0}(x) = \sqrt{\frac{2}{\pi\Gamma}}\phi_0(x)$$

2. For $x_0 < x < x_1$ we find

   $$\phi_{E_0}(x) \approx \frac{B}{\sqrt{k_B}} e^{-\int_{x_0}^{x} k_B(x)dx}$$

   with

   $$\frac{\hbar^2 k_B^2}{2m} = V_1 - E_0$$

3. For $x > x_1$ we finally get

$$\phi_{E_0}(x) \approx \frac{1}{\sqrt{k_0}} \sin(k_0 x + \delta_0)$$

with

$$\frac{\hbar^2 k_0^2}{2m} = E_0$$

Now matching the wave function at x_0 and x_1 (all this must be done in the complex x-plane): Since

$$e^{-\int_{x_0}^{x} k_B(x)dx} = 1$$

we get

$$B = \sqrt{\frac{2}{\pi\Gamma}} A_0$$

or

$$\Gamma = \frac{2A_0^2}{\pi B^2} = \frac{4k_{in}}{\pi x_0} e^{-2\int_{x_0}^{x_1} k_B(x)dx}$$

Now x_0/k_{in} is practically x_0/v_{in}, i.e. the time the particle needs to reach the barrier starting at the origin. The preexponential factor is proportional to the frequency of 'hits', the exponent gives the probability to penetrate the barrier.
Now let us 'create' a particle inside the potential well at $t = t_0$ with the wave function $\phi(x, t = 0) = \phi_0(x)$. The time evolution is given by

$$i\hbar\dot{\phi}(x,t) = \hat{H}\phi(x,t)$$

with the general solution

$$\phi(x,t) = \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}Et} C(E)\phi_E(x)dE$$

where $\phi_E(x)$ is a solution to the stationary equation

$$(H - E)\phi_E(x) = 0$$

and the weight function $C(E)$ is given by

$$C(E) = \langle\phi_E|\phi_0\rangle = \sqrt{P(E)}$$

As in the situation analyzed previously (cf. pgs. 21-22) one obtains finally

$$\int_0^{x_0} |\phi(x,t)|^2 dx \approx e^{-\frac{\Gamma t}{\hbar}}$$

i.e. the particle leaves eventually the 'nucleus'.

13 Penetration of the Coulomb barrier by α-particle (Gamow)

For many nuclei one has that

$$\underbrace{M(N,Z)c^2}_{\text{parent nucleus}} - \underbrace{M(N-2,Z-2)c^2}_{\text{daughter nucleus}} - \underbrace{M(2,2)c^2}_{\text{mass of } ^4He \ (\alpha\text{ -particle})} > 0$$

If this condition is fullfilled, the nucleus (N,Z) can emit an α-particle. Such a process is slowed down due to two reasons:

1. The emitted α-particle has to penetrate a strong Coulomb-barrier.
2. Two protons and two neutrons inside the parent nucleus have to come together in a relatively small volume (α-particle) and leave the nucleus.

Our potential is shown in fig. A9.19 with

$$Q = M(N,Z)c^2 - M(N-2,Z-2)c^2 - M(2,2)c^2$$

The penetration factor is

$$P = \exp\left(-2\int_{R_0}^{R_1}\sqrt{\frac{2m_\alpha}{\hbar^2}(V_C(r)-Q)}dr\right)$$

where m_α is the mass of the α-particle. [9]
We have

$$\frac{(Z-2)2e^2}{R_1} = Q \quad \text{and} \quad \frac{(Z-2)2e^2}{R_0} = B$$

It is easy to do the integral. With

$$x = \frac{Q}{B} = \frac{QR_0}{(Z-2)2e^2}$$

we find

$$P = \exp\left(-2\sqrt{\frac{2m_\alpha}{\hbar^2 Q}}(Z-2)2e^2(\arccos\sqrt{x} - \sqrt{x(1-x)}\right)$$

[9] To be exact one has to use the reduced mass m_r, but

$$m_r = \frac{m_\alpha M(N-2,Z-2)}{m_\alpha + M(N-2,Z-2)} \approx m_\alpha$$

i.e.

$$P = \exp(-\frac{\text{const}}{\sqrt{Q}})$$

which yields the Geiger-Nuttal law

$$\Gamma \sim e^{-\text{const}/\sqrt{Q}}$$

For $Z = 82$ the exponent is roughly

$$2\sqrt{\frac{2 \cdot 4 \cdot 10^3}{4 \cdot 10^4 Q}} 80 \cdot 2 \cdot 1 \cdot 4 \cdot (1) \approx \frac{200}{\sqrt{Q}}$$

which yields

$$\begin{array}{ll} \text{for } Q = 1\text{MeV} & \Rightarrow P \approx e^{-200} \approx 10^{-70} \\ \text{for } Q = 10\text{MeV} & \Rightarrow P \approx e^{-64} \approx 10^{-20} \end{array}$$

i.e. a very strong variation with Q.

Formation of α-particle

Denoting

$$\begin{array}{ll} V_\alpha & -\text{volume of } \alpha\text{-particle} \\ V & -\text{volume of the parent nucleus} \end{array}$$

we find the relative probability to find one nucleon inside the volume V_α

$$P_1 = \frac{V_\alpha}{V}$$

and thus the relative probability for formation of an α-particle

$$P_4 = \left(\frac{V_\alpha}{V}\right)^4$$

With

$$\begin{array}{l} V_\alpha = \frac{4\pi}{3} r_0^3 \cdot 4 \\ V = \frac{4\pi}{3} r_0^3 \cdot A \end{array}$$

that yields for $A \approx 200$

$$P_4 = \left(\frac{4}{A}\right)^4 \approx \frac{1}{60^4} \approx 10^{-7}$$

Role of pairing

If there is pairing nucleons are strongly correlated. One has to put together then only a neutron pair with a proton pair. Let V_0 denote the volume of pairing correlations, i.e. with $r_0 \approx 1$fm

$$V_0 = \frac{4\pi}{3} r_0^3 \approx V_\alpha$$

Then with

$$P_2 = \frac{V_0}{V}$$

we get

$$P_4 \approx P_2^2 \approx \frac{1}{60^2} \approx 3 \cdot 10^{-4}$$

The relative enhancement is about 2÷3000.

Role of deformation

If a nucleus is not spherical but elipsoidal, then the Coulomb barrier is thinner at the 'nose' and this results in increased penetrability (cf. fig. A9.21).

14 Proton and neutron decay

If

$$Q_n = M(N, Z)c^2 - M(N-1, Z)c^2 - m_n c^2 > 0$$

then the nucleus (N, Z) can emit a neutron.
Similarly if

$$Q_p = M(N, Z)c^2 - M(N, Z-1)c^2 - m_p c^2 > 0$$

the nucleus (N, Z) can emit a proton.
In fig. A9.22 the resulting area of stable nuclei in the N-Z-plane is shown. The lines $Q_n = 0$ and $Q_p = 0$ are determined from the Weizsäcker mass formula.