Separation of Variables in Cylindrical Coordinates

We consider two dimensional problems with cylindrical symmetry (no dependence on z). Our variables are s in the radial direction and ϕ in the azimuthal direction. The LaPlace equation in cylindrical coordinates is:

$$\frac{1}{s}\frac{\partial}{\partial s}\Big(s\frac{\partial V(s,\phi)}{\partial s}\Big) + \frac{1}{s^2}\frac{\partial^2 V(s,\phi)}{\partial \phi^2} \;=\; 0$$

We try to find a solution of the form $V(s,\phi) = F(s)G(\phi)$. Then

$$\frac{G(\phi)}{s}\frac{\partial}{\partial s}\Big(s\frac{\partial F(s)}{\partial s}\Big) + \frac{F(s)}{s^2}\frac{\partial^2 G(\phi)}{\partial \phi^2} \;=\; 0$$

Dividing by $F(s)G(\phi)$ and multiplying by s^2 gives us:

$$\frac{s}{F}\frac{\partial}{\partial s}\Big(s\frac{\partial F(s)}{\partial s}\Big) + \frac{1}{G}\frac{\partial^2 G(\phi)}{\partial \phi^2} \;=\; 0$$

The first term depends upon s alone while the second term depends upon ϕ alone. The only way the equation can be true for all s and ϕ is if each term equals a constant:

$$\frac{s}{F}\frac{\partial}{\partial s}\Big(s\frac{\partial F(s)}{\partial s}\Big) \;=\; -C \qquad \text{and} \qquad \frac{1}{G}\frac{\partial^2 G(\phi)}{\partial \phi^2} \;=\; C$$

Just like the two dimensional case in Cartesian coordinates, there are 3 possibilities: $C > 0$, $C < 0$, and $C = 0$. But unlike Cartesian coordinates s and ϕ are not interchangeable. Consider the ϕ equation:

$$\frac{1}{G}\frac{\partial^2 G(\phi)}{\partial \phi^2} \;=\; C \quad => \quad \frac{\partial^2 G(\phi)}{\partial \phi^2} \;=\; C\; G(\phi)$$

If C is positive, the solution is $G(\phi) = Ae^{\sqrt{C}\phi} + Be^{-\sqrt{C}\phi}$. But G is periodic in 2π, that is, we need $G(\phi) = G(\phi + 2\pi)$ and the exponential solutions don't have this property. Therefore, C must be negative or zero solet's define it as $C = -k^2$. Then

$$\frac{\partial^2 G(\phi)}{\partial \phi^2} \;=\; -k^2\; G(\phi) \quad => \quad G(\phi) \;=\; A\cos k\phi + B\sin k\phi$$

$$\text{Furthermore} \quad G(\phi + 2\pi) \;=\; G(\phi) \quad => \quad 2\pi k \;=\; 2\pi n$$

where n is an integer. Therefore, k must be a non-negative integer: $k = 0, 1, 2, ...$ (negative integers give the same solutions as positive integers). The solution for $k = 0$ is $G(\phi) = A\cos k\phi + B\sin k\phi = A =$ constant, which is clearly a needed solution for problems with azimuthal symmetry.

Note, unlike the Cartesian case, the condition that $\phi + 2\pi$ describes the same position in the plane as ϕ forces the separation constant to be an integer, leaving us with the radial equation:

$$s\frac{\partial}{\partial s}\Big(s\frac{\partial F(s))}{\partial s}\Big) \;=\; k^2 F(s)$$

This is easy: we take two derivatives, but multiply by s^2, so we should try a power law solution, $F = s^p$:

$$s\frac{\partial}{\partial s}\Big(s\frac{\partial s^p}{\partial s}\Big) \;=\; s\frac{\partial}{\partial s}\Big(ps^p\Big) \;=\; p^2 s^p \;=\; k^2 F(s) \;=\; k^2 s^p \quad => \quad p \;=\; \pm k$$

$$=> \quad F(s) \;=\; Ds^k + Es^{-k} \quad \text{for} \quad k = 1, 2, 3, ...$$

For $k = 0$ our result gives us only one solution, $F(s) =$ constant, but we started with a second order differential equation which requires two solutions, so we consider the equation again with $k = 0$:

$$s\frac{\partial}{\partial s}\left(s\frac{\partial F(s)}{\partial s}\right) \;=\; 0 \quad => \; s\frac{\partial F(s)}{\partial s} \;=\; \text{constant} \;=\; Q \quad => \quad \frac{\partial F(s)}{\partial s} \;=\; \frac{Q}{s}$$

$$=> \quad F(s) \;=\; Q\ln s + P$$

with Q and P constants.

(For $k = 0$, we only had one solution for $G(\phi)$. The entire solution for $G(\phi)$ for $k = 0$ was $G(\phi) = A + B\phi$, but the requirement that $G(\phi + 2\pi) = G(\phi)$ made $B = 0$.)

Therefore, the general solution for two dimensional cylindrical coordinates is:

$$V(s,\phi) \;=\; a_0 + b_0 \ln s + \sum_{k=1}^{\infty}\left[s^k(a_k\cos k\phi + b_k\sin k\phi) + s^{-k}(c_k\cos k\phi + d_k\sin k\phi)\right]$$

Note, an infinite line charge has cylindrical symmetry. Our solution should provide the potential for the infinite line charge. By symmetry, the solution cannot depend upon ϕ, so $a_k = b_k = c_k = d_k = 0$, leaving us with $V(s,\phi) = a_0 + b_0 \ln s$, which is the solution we found by using Gauss's law.