NAME (print legibly)

Physics 322 First Midterm Exam Spring 2008

This is a closed book exam. You are allowed one sheet of notes. Show all of your work to receive full credit. The exam is worth 50 pts (1 pt/min). Below are some useful results.

Lorentz force law: $\vec{F} = \vec{F}_E + \vec{F}_M = q[\vec{E} + (\vec{v} \times \vec{B})]$.

Force on a section of wire carrying current, I: $\vec{F}_M = I \int (d\vec{l} \times \vec{B}) = \int (\vec{J} \times \vec{B}) d\tau$

$$\text{Continuity equation :} \quad \vec{\nabla} \cdot \vec{J} \;=\; -\frac{d\rho}{dt}; \qquad I \;=\; \int_S \vec{J} \cdot d\vec{a} \qquad \vec{J} = \rho\vec{v}$$

$$\text{Biot} - \text{Savart law :} \quad d\vec{B}(\vec{r}) \;=\; \frac{\mu_0 I}{4\pi} \frac{d\vec{l'} \times (\vec{r} - \vec{r}')}{(\vec{r} - \vec{r}')^3} \qquad \vec{B}(\vec{r}) \;=\; \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l'} \times (\vec{r} - \vec{r}')}{(\vec{r} - \vec{r}')^3}$$

Field of an infinite wire: $\vec{B} = \mu_0 I/(2\pi s)\ \vec{\phi}$

Field on axis from a circular loop:

$$B(z) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}$$

$$\text{Ampere's law :} \quad \vec{\nabla} \times \vec{B} \;=\; \mu_0 \vec{J} \qquad \int_C \vec{B} \cdot d\vec{l} \;=\; \mu_0 I_{enc}$$

Field of an infinite solenoid: $\vec{B}_{in} = \mu_0 n I\ \hat{z}$, $\quad \vec{B}_{out} = 0$

Vector potential: $\vec{B} = \vec{\nabla} \times \vec{A}$ Coulomb Gauge: $\vec{\nabla} \cdot \vec{A} = 0$ $\quad \int_C \vec{A} \cdot d\vec{l} = \int_S \vec{B} \cdot d\vec{a}$

Vector potential in Coulomb gauge:

$$\vec{A}(\vec{r}) \;=\; \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{\vec{r} - \vec{r}'}\, d\tau'$$

Vector potential of a dipole:

$$\vec{A}_{dip}(\vec{r}) \;=\; \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \qquad \vec{m} \;=\; I \int d\vec{a}$$

Magnetic field of a dipole:

$$\vec{B}_{dip}(\vec{r}) \;=\; \frac{\mu_0}{4\pi r^3}[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}]$$

Curl in cylindrical coordinates:

$$\vec{\nabla} \times \vec{v} \;=\; \Big[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\Big]\hat{s} + \Big[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\Big]\hat{\phi} + \frac{1}{s}\Big[\frac{\partial (s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial \phi}\Big]\hat{z}$$

Part I. (16 pts) A long cylindrical wire of radius a whose axis is along the z axis carries a current I in the positive $\hat{z}$ direction and has a current density $\vec{j}(s) = j_z(s)\hat{z}$ given by (cylindrical coordinates):

$$j_z(s) \;=\; j_0\Big(\frac{s}{a}\Big)^2 \quad \text{for} \quad s < a \qquad \text{and} \qquad j_z(s) \;=\; 0 \quad \text{for} \quad s > a$$

1. (4 pts) Calculate the current, I, in terms of j_0, s, and a.

$$I \;=\; \int j_z\; da_\perp \;=\; \int_0^a j_0\Big(\frac{s}{a}\Big)^2\; 2\pi s\; ds \;=\; \frac{2\pi j_0}{a^2}\int_0^a s^3\; ds \;=\; \frac{2\pi j_0}{a^2}\,\frac{a^4}{4} \;=\; \frac{\pi j_0 a^2}{2}$$

2. (7 pts) Calculate the magnetic field, $\vec{B}(s)$, (magnitude and direction) for all s.

We use Ampere's law because by symmetry we know the field is in the $\hat{\phi}$ direction. Take a circular path centered on the z axis:

$$\text{For} \quad s > a, \quad \int \vec{B}\cdot d\vec{l} \;=\; 2\pi s B(s) \;=\; \mu_0 I_{enc} \;=\; \mu_0 I \quad => \quad \vec{B}(s) \;=\; \frac{\mu_0 I}{2\pi s}\;\hat{\phi}$$

$$\text{For} \quad s < a, \quad \int \vec{B}{\cdot}d\vec{l} = 2\pi s B(s) \;=\; \mu_0 I_{enc} \;=\; \mu_0\int_0^s j_z(s')da' \;=\; \frac{\mu_0 2\pi j_0}{a^2}\int_0^s (s')^3\; ds' \;=\; \frac{\mu_0 \pi j_0 s^4}{2a^2} \;=\; \mu_0 I\frac{s^4}{a^4}$$

$$=> \quad \vec{B}(s) \;=\; \frac{\mu_0 I s^3}{2\pi a^4}\;\hat{\phi} \;=\; \frac{\mu_0 j_0 s^3}{4a^2}\;\hat{\phi}$$

3. (5 pts) Calculate the vector potential, $\vec{A}(s)$, in the Coulomb Gauge for $s < a$. Take $\vec{A}(0) = 0$.

There are several ways to do this. We could use $\int \vec{B}\cdot d\vec{a} = \int \vec{A}\cdot d\vec{l}$ or $\vec{\nabla}\times\vec{A} = \vec{B}$. Let's use the latter:

$$\vec{B} \;=\; B(s)\;\hat{\phi} \;=\; \vec{\nabla}\times\vec{A} \quad => \quad B(s) \;=\; (\vec{\nabla}\times\vec{A})_\phi \;=\; \Big[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\Big]$$

But $\vec{A}$ is parallel to $\vec{I}$ and can only depend on the radial coordinate s, $\vec{A} = A_z(s)\hat{z}$,

$$=> \quad B(s) \;=\; -\frac{\partial A_z}{\partial s} \quad => \quad A_z(s) \;=\; -\int_0^s B(s')\; ds'$$

There is no integration constant because $A(0) = 0$. Using the result for $B(s)$ from above,

$$A_z(s) \;=\; -\frac{\mu_0 I}{2\pi a^4}\int_0^s (s')^3 ds' \;=\; -\frac{\mu_0 I s^4}{8\pi a^4}$$

Part II. (17 pts) A uniform current, I, flows to the right, along $+\hat{x}$, through a rectangular bar of conducting material in the presence of a uniform external magnetic field, $\vec{B} = B\ \hat{z}$. The bar is of width w in the $\hat{z}$ direction and of thickness t in the $\hat{y}$ direction.

1. (2 pts) What is the current density, $\vec{J}$, within the bar?

$$\vec{J} \;=\; \frac{\vec{I}}{Area} \;=\; \frac{\vec{I}}{wt} \;=\; \frac{I\hat{x}}{wt}$$

2. (3 pts) If the moving charges are positive, in which direction are they deflected by the external magnetic field?

$$\vec{F} \;=\; \vec{I} \times \vec{B} \;\propto\; \hat{x} \times \hat{z} \;=\; -\hat{y} \quad => \quad \text{Downward}$$

3. (5 pts) This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric field, $\vec{E}$, to counteract the magnetic force. Equilibrium occurs when the electric and magnetic forces cancel, leaving a potential difference, $V_{Hall} = Et$, between the top and bottom of the bar. This phenomenon is known as the Hall effect and is used to make magnetometers. Calculate the Hall voltage, V_{Hall}, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.

We need the total force on the charges to cancel. The vertical component of the force is:

$$F_y \;=\; q[E_y + (v\hat{x} \times B\hat{z})_y] \;=\; 0 \;=\; q[E_y - vB] \quad => \quad E_y \;=\; vB$$

$$V_{Hall} \;=\; E_y t \;=\; vBt$$

4. (4 pts) If there are n charges, q, per unit volume, ($\rho = nq$), derive an expression for the Hall voltage in terms of B, I, n, q, and any relevant dimensions of the bar.

$$\vec{J} \;=\; \rho\vec{v} \;=\; nq\vec{v} \quad \text{and} \quad I \;=\; Jwt \;=\; nqvwt \quad \text{from above} \quad => \quad vt \;=\; \frac{I}{nqw}$$

Substituting for vt in the result from # 3,

$$V_{Hall} \;=\; \frac{IB}{nqw}$$

5. (3 pts) Can the sign of the Hall voltage in the presence of a known magnetic field be used to determine the sign of charge carriers, q, in the bar? Explain.

Yes, if the charge carriers were negative in the example above, for the same current flowing in the $+\hat{x}$ direction, the electrons would be moving in the $-\hat{x}$ direction and would feel a force downward as did the positive charge carriers above ($\vec{I} \times \vec{B}$ points down for both signs of charge carriers). Therefore, negative charge would accumulate on the bottom of the slab, leaving positive charge on the top of the slab, creating an opposite sign for V_{Hall}. The sign of the Hall voltage depends on the sign of the charge carriers.

Part III. (17 pts) Two circular coils of radius a have their axes along the z axis and are separated by a distance $2d$, symmetric with respect to the origin. The coils carry opposite currents, $\pm I$.

1. (3 pts) What is the magnetic field at the origin, $\vec{B}(0)$?

$\vec{B}(0) = 0$ because the top coil makes a field pointing along $+\hat{z}$ at the origin and the bottom coil, with opposite current, makes a field of the same magnitude but pointing along $-\hat{z}$. The two fields cancel.

2. (2 pts) What is the magnetic dipole moment of the upper ring?

$$\vec{m} \ = \ I \cdot Area\ \hat{n} \ = \ \pi a^2 I\ \hat{z}$$

3. (4 pts) For a point, z, on the z axis with $z >> d$, with what power of z does the magnetic field vary as z varies ($e.g\ 1/z^n$). Show your work.

By superposition and the equation for the field on the axis of a current loop on the front page,

$$\vec{B}(z) \ = \ \frac{\mu_0 I a^2 \hat{z}}{2}\left(\frac{1}{[a^2+(z-d)^2]^{3/2}} - \frac{1}{[a^2+(z+d)^2]^{3/2}}\right)$$

But $$\frac{1}{[a^2+(z+\epsilon)^2]^{3/2}} - \frac{1}{[a^2+(z-\epsilon)^2]^{3/2}} \ = \ 2\epsilon\frac{d}{dz}\left(\frac{1}{[a^2+z^2]^{3/2}}\right) \ \propto \ \frac{z}{[a^2+z^2]^{5/2}}$$

where the expression above is the definition of a derivative for small ϵ. Therefore,

$$B(z) \ \propto \ \frac{z}{[a^2+z^2]^{5/2}} \quad \rightarrow \quad \frac{1}{z^4} \quad \text{for} \quad a << z$$

4. (4 pts) For a point, P, in the xy plane a distance s from the origin, with $s >> d$, derive an expression for the magnetic field at P (magnitude and direction) in terms of I, a, d, s.

The field at s is the sum of the two dipole fields. Call the upper dipole 1 and the lower 2 and use the result from the front page:

$$\vec{B}(s) \ = \ \vec{B}_1(s) + \vec{B}_2(s) \ = \ \frac{\mu_0}{4\pi r_1^3}[3(\vec{m}_1\cdot\hat{r}_1)\hat{r}_1 - \vec{m}_1] + \frac{\mu_0}{4\pi r_2^3}[3(\vec{m}_2\cdot\hat{r}_2)\hat{r}_1 - \vec{m}_2]$$

where $\vec{m}_1 = \pi a^2 I\hat{z} = -\vec{m}_2 = \vec{m}$ and $r_1^2 = r_2^2 = s^2 + d^2 = r^2$.

$$=> \quad \vec{B}(s) \ = \ \frac{\mu_0}{4\pi r^3}[3(\vec{m}_1\cdot\hat{r}_1)\hat{r}_1 - \vec{m}_1 + 3(\vec{m}_2\cdot\hat{r}_2)\hat{r}_2 - \vec{m}_2] \ = \ \frac{3\mu_0}{4\pi r^3}[(\vec{m}\cdot\hat{r}_1)\hat{r}_1 - (\vec{m}\cdot\hat{r}_2)\hat{r}_2]$$

But, $\vec{m}\cdot\hat{r}_1 = -\vec{m}\cdot\hat{r}_2 = -md/r$ and $\hat{r}_1 + \hat{r}_2 = (2s/r)\hat{s}$

$$=> \quad \vec{B}(s) \ = \ \frac{3\mu_0}{4\pi r^3}\ \frac{-md}{r}\ \frac{2s}{r}\hat{s} \ = \ -\frac{3\mu_0 m d s}{2\pi r^5}\ \hat{s}$$

5. (4 pts) For the point P in the question above, what is the vector potential, $\vec{A}$, at P?

Like above, $\vec{A}$ is the sum of the dipole vector potentials from each loop:

$$\vec{A}(s) \ = \ \frac{\mu_0}{4\pi}\left[\frac{\vec{m}_1\times\hat{r}_1}{r_1^2} + \frac{\vec{m}_2\times\hat{r}_2}{r_2^2}\right] \ = \ 0 \quad \text{because} \quad r_1^2 = r_2^2 \quad \text{and} \quad \vec{m}_1\times\hat{r}_1 = -\vec{m}_2\times\hat{r}_2$$