Physics 322 Solution to Homework Set #6 Spring 2008

1. Problem 7.1 in your textbook. Note, part c should read $b >> a$.

(a) Let q be the charge on the inner shell of radius a. Then in the space between the shells, $\vec{E} = q/(4\pi\epsilon_0 r^2)\hat{r}$, and

$$V = V_a - V_b = -\int_b^a \vec{E}\cdot d\vec{r} = -\frac{q}{4\pi\epsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$I = \int_S \vec{J}\cdot d\vec{a} = \sigma_c \int_S \vec{E}\cdot d\vec{a} = \sigma_c \frac{q}{\epsilon_0}$$

where we used Gauss's law. Therefore, substituting for q,

$$I = \frac{\sigma_c}{\epsilon_0}\frac{4\pi\epsilon_0 V}{(1/a - 1/b)} = \frac{4\pi\sigma_c V}{(1/a - 1/b)}$$

(b)

$$V = IR \quad => \quad R = \frac{V}{I} = \frac{1}{4\pi\sigma_c}\left(\frac{1}{a} - \frac{1}{b}\right)$$

(c) For $b >> a$, $1/b << 1/a$ => $R \approx 1/(4\pi\sigma_c a)$. The resistance is dominated by the region near the inner shell: spherical shells further out have a larger surface area (and hence lower current density) and contribute less and less to R as you move further out. For two submerged spheres, each has a resistance, $R \approx 1/(4\pi\sigma_c a)$ to a point a distance $>> a$ from either sphere and therefore a resistance $R_T = 2R = 2/(4\pi\sigma_c a)$ between the two spheres. By measuring the current, I, between the submerged spheres when a voltage, V, is maintained between them, one determines R_T from Ohm's law and hence the conductivity, σ_c.

2. Problem 7.7 in your textbook.

(a) We find the current from Ohm's law, $V = IR$ where the voltage is the induced EMF in the loop formed by the resistor, rails, and sliding metal bar. Let x be the distance of the metal bar from the resistor:

$$V = \mathcal{E} = -\frac{d\Phi}{dt} = -B\frac{d}{dt}(lx) = -Bl\frac{dx}{dt} = -Blv = IR$$

$$=> \quad I = \frac{Blv}{R}$$

The flux into the page is increasing so the current flows counter-clockwise to produce a field that points out of the page ($\vec{v}\times\vec{B}$ points parallel to the metal bar in the upward direction, forcing current to flow counter-clockwise.)

(b) Recall that the force that a magnetic field exerts on a current segment is given by:

$$\vec{F} = I\int d\vec{l}\times\vec{B} = IB\int dl\,(-\hat{x}) = -IBl\,\hat{x} = -\frac{B^2l^2v}{R}\,\hat{x}$$

where the direction of the force is opposite to $\vec{v}$.

(c)

$$F = ma = m\frac{dv}{dt} = -\frac{B^2l^2v}{R} \quad => \quad \frac{dv}{dt} = -\left(\frac{B^2l^2}{mR}\right)v \quad => \quad v(t) = v_0 e^{-\frac{B^2l^2}{mR}t}$$

(d) The energy is dissipated in the resistor as heat. The power dissipated is $P = I^2 R$. Therefore,

$$P = \frac{dU}{dt} = I^2 R = \frac{B^2 l^2 v^2}{R^2} R = \frac{B^2 l^2}{R} v_0^2 e^{-2\beta t} \quad \text{where} \quad \beta = \frac{B^2 l^2}{mR}$$

$$U = \int_0^\infty \frac{dU}{dt}\, dt = \beta m v_0^2 \int_0^\infty e^{-2\beta t}\, dt = \beta m v_0^2 \left.\frac{e^{-2\beta t}}{-2\beta}\right|_0^\infty = \frac{1}{2} m v_0^2$$

3. Problem 7.8 in your textbook.

(a) The field from the long wire is $\vec{B} = \mu_0 I/(2\pi r)\hat{\phi}$. Therefore,

$$\Phi = \int \vec{B} \cdot d\vec{a} = \frac{\mu_0 I}{2\pi} \int_s^{s+a} \frac{1}{r}(a dr) = \frac{\mu_0 I a}{2\pi} \ln\left(\frac{s+a}{s}\right)$$

(b) $v = ds/dt$ =>

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I a}{2\pi} \frac{d}{dt}[\ln(s+a) - \ln(s)] = -\frac{\mu_0 I a}{2\pi} \left(\frac{1}{s+a}\frac{ds}{dt} - \frac{1}{s}\frac{ds}{dt}\right) = \frac{\mu_0 I a^2 v}{2\pi s(s+a)}$$

The field points out of the page and the flux decreases as the loop is pulled. Therefore, the current flows counter-clockwise to increase the flux pointing out of the page ($\vec{v} \times \vec{B}$ points to the right on the segment closest to the wire where the field is largest, pushing the current counter-clockwise).

(c) The flux does not change if the loop is pulled parallel to the wire so $\mathcal{E} = 0$.

4. Problem 7.10 in your textbook. The flux through the loop is $\Phi = \vec{B} \cdot \vec{a} = Ba^2 \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{B}$ ($a \cos\theta$ is the projection of the loop perpendicular to $\vec{B}$ as seen from above). At constant angular speed, $\theta = \omega t$, giving us:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Ba^2(-\omega \sin\omega t) = B\omega a^2 \sin\omega t$$

5. Problem 7.11 in your textbook. As in problem 7.7, $\mathcal{E} = Blv = IR$ => $I = Blv/R$. The magnetic field produces an upward magnetic force on the current, $F_{mag} = IlB = B^2 l^2 v/R$ which opposes the gravitational force, $F_{grav} = mg$. Therefore,

$$F_T = F_{grav} - F_{mag} = mg - \frac{B^2 l^2 v}{R} = m\frac{dv}{dt} \quad => \quad \frac{dv}{dt} = g - \beta v \quad \text{where} \quad \beta = \frac{B^2 l^2}{mR}$$

At terminal velocity, $dv/dt = 0$ => $g - \beta v_t = 0$

$$=> \quad v_t = \frac{g}{\beta} = \frac{mgR}{B^2 l^2}$$

We find the speed as a function of time (assuming the loop starts at rest) from:

$$\frac{dv}{g-\beta v} \;=\; dt \quad => \quad -\frac{1}{\beta}\ln(g-\beta v) \;=\; t + const \quad => \quad g - \beta v \;=\; Ae^{-\beta t}$$

At $t = 0$, $v = 0$ => $A = g$. Therefore,

$$\beta v \;=\; g(1-e^{-\beta t}) \quad => \quad v(t) \;=\; \frac{g}{\beta}(1-e^{-\beta t}) \;=\; v_t(1-e^{-\beta t})$$

At 90% of terminal velocity, $v/v_t = 0.9 = 1 - e^{-\beta t}$ => $e^{-\beta t} = 0.1$

$$=> \quad -\beta t \;=\; \ln(0.1) \quad => \quad \beta t \;=\; \ln(10) \quad => \quad t \;=\; \frac{1}{\beta}\ln(10) \;=\; \frac{v_t}{g}\ln(10)$$

Putting in numbers, for $A =$ the cross sectional area of the conducting loop, $l =$ its side length, and $\eta = 2700 kg/m^3 =$ the density of aluminum, we have $m = 4\eta A l$.

$R = 4l/(\sigma_c A) = 4l\rho_c/A$ where $\rho_c = 2.8\times10^{-8} ohm - m$ is the resistivity of aluminum. With $B = 1T$, we have:

$$v_t \;=\; \frac{4\eta A l g 4 l \rho_c}{AB^2l^2} \;=\; \frac{16 g\eta\rho_c}{B^2} \;=\; \frac{(16)(9.8)(2700)(2.8\times10^{-8})}{1} \;=\; 1.2\ cm/s$$

The time to reach 90% of termonal velocity is:

$$t \;=\; \frac{1.2\times10^{-2}}{9.8}\ln(10) \;=\; 2.8\ ms$$

If the loop were cut, it would fall freely with an acceleration of g.